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What is a Fuzzy Bi-implication?

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For my mother and my sister Gabriela.

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Abstract

In order to make this document self-contained, we first present all the necessary theory as a background. Then we study several definitions that extended the classic bi-implication in to the domain of well stablished fuzzy logics, namely, into the $[0, 1]$ interval. Those approaches of the fuzzy bi-implication can be summarized as follows: two axiomatized definitions, which we proved that represent the same class of functions, four defining standard (two of them proposed by us), which varied by the number of different compound operators and what restrictions they had to satisfy. We proved that those defining standard represent only two classes of functions, having one as a proper subclass of the other, yet being both a subclass of the class represented by the axiomatized definitions. Since those three clases satisfy some contraints that we judge unnecessary, we proposed a new defining standard free of those restrictions and that represents a class of functions that intersects with the class represented by the axiomatized definitions.

By this dissertation we are aiming to settle the groundwork for future research on this operator.

Keywords: fuzzy bi-implication, fuzzy equivalence, fuzzy bi-residuation, fuzzy logic

Resumo

A fim de tornar este documento auto-suficiente, nós apresentamos toda a teoria necessária como arcabouço teórico. Em seguida, estudamos várias definições que estenderam a bi-implicação clássica para o domínio da bem estabelecida lógica difusa, ou seja, no intervalo $[0, 1]$. Essas abordagens da bi-implicação difusa podem ser resumidas da seguinte forma: duas definições axiomatizadas, que demonstramos que representam a mesma classe de funções, quatro padrões definitórios (dois deles proposto por nós), que variam com o número de diferentes operadores que as compõem e quais restrições que tinham para satisfazer. Nós demonstramos que esses padrões definitórios representam apenas duas classes de funções, tendo uma como uma subclasse própria da outra, mas sendo ambas subclasses da classe representada pelas definições axiomatizadas. Uma vez que esses três clases satisfazer algumas restrições que julgamos desnecessárias, propusemos um novo padrão definitório sem essas restrições e que representa uma classe de funções que se interseta com a classe representada pelas definições axiomatizadas.

Nesta dissertação estamos pretendendo estabelecer as bases para futuras pesquisas sobre este operador.

Palavras-chave: bi-implicação difusa, equivalência difusa, bi-residuation difusa, logica difusa

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Chapter 1

Introduction

There are several reasons by which we conducted this research. That is why we dedicated this first chapter to exhibit in detail what motivated us, how we justify this work and which are the boundaries that this dissertation embraces.

1.1 Motivation

After the introduction of fuzzy set theory by Lotfi Zadeh [34], it began the research of fuzzy logic in a narrow sense, which is an extension of multivalued logic (page 2 of [19]). Thanks to this there have been made extensions to the usual logic connectives, as it is sumarized in the following items:

1. classic conjunction, which may have several extensions to the unit square, being the triangurlar norm operator [21, 22, 23, 24] the most accepted in the scientific community, because it does not only satisfy the border condition, it also presents a reasonable behavior in $(0,1)^2$,

2. classic disjunction, analogously than the case of the above connective, the classic disjunction can be extended by many operators, but the most stablished one is the triangular conorm [21, 22, 23, 24],

3. classic negation, which has a reasonable well studied fuzzy operator [9] and

4. classic implication, although there are several non equivalent extensions of this classic connective to the fuzzy logic, we decided to work with [1], because it is being well accepted.

The classic bi-implication has been extended in fuzzy logic theory under the names of T-indistinguishability operator [31], fuzzy bi-implication [6, 4], fuzzy equality [28], fuzzy bi-residuation [25], fuzzy equivalence [17, 10], T-equivalence [26], fuzzy similarity [19] and restricted equivalence function [7]. In all these approaches the operators where constrained to at least one of the following restrictions:

1. Satisfy the fuzzy equivalence properties (reflexivity, symmetry and T-transitivity);

2. Be compatible with the notion of distance on $[0, 1]$,

3. Define the fuzzy bi-implication in terms of the conjunction (t-norm) and implication connectives.

In [28] a class of algebras named EQ-algebras was introduced in which one of its basic primitive connectives is a fuzzy equality. That class of algebras was applied in [15] to create a fuzzy logic in which the basic connective was fuzzy equality, and was also used in [27] to develop a theory of higher-order fuzzy logic called as Fuzzy Type Theory (FTT). Our work does not go in the direction of this research, because our fuzzy bi-implications are constructed from other connectives and are not part of a specific algebra. Moreover, the fuzzy equality $\sim$ in the EQ-algebra is compatible with the usual distance on $[0, 1]$ in the sense that if $x \leq y \leq z$ then $z \sim x \leq z \sim y$ and $x \sim z \leq x \sim y$, whereas our fuzzy bi-implication has no concern with a distance notion. So despite the fact that it is a good idea to define a fuzzy equality as a primitive connective, the fuzzy equality has some of the same limitations than the other papers that deal with fuzzy bi-implication.

In [13, 14] it is proposed a resemblance relation that does not guarantee the transitivity. Our work goes in a different direction than that theory, because the resemblance relation it is based on a distance concept, and thus it also satisfied the identity principle.

1.2 Justification

The Fuzzy bi-implication may be usefull in aplications where the implementation of some kind of similarity evaluation is required, e.g., it has been used in the comparison of images [8] and in the state reduction problem for fuzzy automata [11].

The study of a new fuzzy bi-implication, that does not demand the satisfaction of some properties, may offer some benefits to the areas where this connective is needed, but where the contrains of stronger notions of fuzzy bi-implication can be very hard. This research can also lay the groundwork for constructing a logic or an algebra which include our fuzzy bi-implications as operator.

Three objects are transitive if the relations between two of them to a third is the same as the relation between those two.

The identity principle was first explicitly formulated by Wilhelm Gottfried Leibniz in his Discourse on Metaphysics, Section 9, in which he says "it is not true that two substances may be exactly alike and differ only numerically". We can interpret this by saying that no two objects have exactly the same properties.

Our proposed operator may also be usefull in some quantum logics, as in the Schrödinger logics that where proposed in [12], because in the Schrödinger logics it is not possible to compare an element with himself, so the bi-implication does not satisfy the identity principle.

Along this work, we made a few relevant decisions, which we believe to be important to justify. That is the reason that we are going to explain why our proposed operator does not guarantee transitivity neither the identity principle.

1. Non-transitivity

   - We find the following analysis of the book "The Republic" of Plato in page 108 of [18]: "When a carpenter makes a bed, he makes it after a model he has inside his head. According to Plato, this model, intelligible as it is, is an idea: it is the idea of a bed. When a painter paints a bed, he does it after a model which is the carpenter's bed. The carpenter's bed is to th painter's what the bed idea is to the carpenter's bed. ...To Plato, an object, something like the bed on which we sleep, is the image of the idea, just like the painting of the bed is the image of that object, of that thing. ...There are, therefore, three beds: the ideal bed, the sensorial bed and the painted bed. ...The sensorial bed is an image of the intelligible bed, like the painted bed is an image of the sensorial bed. The painted bed is then an image of an image." From that we may conclude that the relation between the painted bed and the sensorial bed is not the same relation that exists between the painted bed and the ideal bed.
   - The Poincaré paradox says that two physical objects can be distinguished from one another, but both may be indistinguishable from a third. This paradox can be summarized to the following equations (page 34-35 of [29] and page 69 of [30]):

     $$A = B, \quad B = C, \quad A < C$$

     where $=$ stands for indistinguishability and $<$ for distinguishability that the left-side element is less than the right-side one.

2. Non-identity principle

   - Using the same bed example we can obtain another conclusion: Two beds made by the carpenter, having as a model the same ideal bed, are not identical.

- "Heraclitus thought that at no time does one thing remain identical to itself: its identity consists, thus, on being always different" (page 31 of [18]). So when comparing an object it will not be identitical to itself.

1.3 Objectives

The general objectives of this dissertation are:

1. Present concepts and there relations that allow a well understanding of what a fuzzy bi-implication is.
2. Enrich the knowledge of this operator in the same direction of the related theory that already exists.
3. Satisfy the necessity of the scenarios that need a fuzzy bi-implication operator with less contrains.

In order to acomplish those global objectives, the following more specific ones are satisfied:

1. Know the existing definitions of fuzzy bi-implication and the relations between them.
2. Propose new definitions of fuzzy bi-implication that go in the same direction of the existing ones.
3. Understand how those new proposed definitions are related to the existing ones.
4. Stablish what are the limitations that the existing definitions impose.
5. Propose a more relaxed definition of fuzzy bi-implication.

1.4 Organization of the dissertation

- Chapter 2 - Background: In this chapter we present a backgroung of fuzzy connectives in order to understand the subsequent chapters.
- Captulo 3 - Fuzzy Bi-implication: Here we first make an introduction to fuzzy bi-implication, we analyze and compare several existing definitions, and proposed a few others.
- Captulo 4 - Conclusions and future works: Through this chapter we first detail some conclusions and then we propose several future directions of investigation, in order to study more deeply the subject of this dissertation..

1.5 Clarifying note

All the proofs included in this work were made by the author. All the definitions, propositions, lemmas, theorems and remarks that do not cite a reference, were also proposed by the author of this dissertation.

Chapter 2

Background

2.1 Triangular Norms

There are infinite ways in which classical conjunction $\wedge$ may be extended for the $[0,1]$ interval and fuzzy logics that can satisfy the boundary condition but not all of them behave as what is intuitively expected from a generalization of the Boolean conjuction in the unit square. That is why the fuzzy logic community has adopted t-norms as the canonical extension of classical conjunction.

All the material contained in this section is based on [33] and [22].

Definition 2.1.1 ([33]). *A* uninorm *is a binary operator F on the unit interval $[0,1]$ which is commutative, associative, monotone and has a neutral element $e \in [0,1]$, i.e., it is an operator $F : [0,1]^2 \to [0,1]$ such that, for all $x, y, z \in [0,1]$:*

$(T1)$ $F(x,y) = F(y,x)$

$(T2)$ $F(x, F(y,z)) = F(F(x,y), z)$

$(T3)$ $F(x,y) \leq F(x,z)$ *whenever* $y \leq z$

$(U4)$ $F(x,e) = x$

Definition 2.1.2 ([33]). *A* triangular norm *(in short* t-norm*) T is a uninorm with 1 as neutral element, i.e. it satisfies for all $x \in [0,1]$*

$(T4)$ $T(x,1) = x$.

Definition 2.1.3 ([22]). *A t-norm T is* left-continuous *if for each $y \in [0,1]$ and for all non-decreasing sequences $(x_n)_{n\in\mathbb{N}}$ it holds that $\lim_{n\to\infty} T(x_n, y) = T(\lim_{n\to\infty} x_n, y)$.*

There are uncountably many t-norms, but in the following example we present some of the most usual ones.

Example 2.1.1 ([22]). *The following are the four basic t-norms, namely, the minimum T_M, the product T_P, the Łukasiewicz t-norm T_L, and the drastic product T_D:*

1. $T_M(x, y) = min(x, y)$,
2. $T_P(x, y) = x \cdot y$,
3. $T_L(x, y) = max(x + y - 1, 0)$,
4. $T_D(x, y) = \begin{cases} 0 & \text{if } (x, y) \in [0, 1)^2, \\ min(x, y) & \text{otherwise.} \end{cases}$

Let's notice that the three first t-norms mentioned in the Definition 2.1.1 are continuous in the usual notion of continuity of $\mathbb{R}^2$.

Since t-norms are just operators from the unit square into the unit interval, the comparison of t-norms is done in the usual way, i.e., pointwise.

Definition 2.1.4 ([22]). *If, for two t-norms T_1 and T_2, we have $T_1(x, y) \leq T_2(x, y)$ for all $(x, y) \in [0, 1]^2$, then we say that T_1 is weaker than T_2 or, equivalently, that T_2 is stronger than T_1, and we write in this case $T_1 \leq T_2$. We shall write $T_1 < T_2$ if $T_1 \leq T_2$ and $T_1 \neq T_2$, i.e., if $T_1 \leq T_2$ and if $T_1(x_0, y_0) < T_2(x_0, y_0)$ for some $(x_0, y_0) \in [0, 1]^2$.*

As an immediate consequence of $(T1)$, $(T3)$ and $(T4)$, the drastic product T_D is the weakest, and the minimum T_M is the strongest t-norm, i.e., for each t-norm T we have

$$T_D \leq T \leq T_M \tag{2.1}$$

Notice that the t-norms with the order presented in Definition 2.1.4 do not build a lattice, because the join of two t-norms do not satisfy the associativity axiom of t-norms.

With a t-norm T, $x \in [0, 1]$ and $n \in \mathbb{N}$ we stablish the following notation:

$$x_T^n = \begin{cases} x_T^1 = x, \\ x_T^{n+1} = T(x_T^n, x). \end{cases} \tag{2.2}$$

2.2 Triangular Conorms

Analogously to the case of the classic conjunction $\wedge$, there can be created infinite generalizations of the classic disjunction $\vee$ in $[0, 1]$-valued and fuzzy logics that can satisfy the boundary condition but not all of them behave as what is intuitively expected from an extension of the Boolean disjunction in the unit square. That is why the fuzzy logic community has adopted the t-conorms as a usual generalization of that classic operator.

All the material contained in this section is based on [33] and [22].

Definition 2.2.1 ([33]). *A triangular conorm (in short* t-conorm*) is a uninorm S with 0 as neutral element, i.e. it satisfies for all $x \in [0,1]$*

$(S4)$ $S(x,0) = x$

Example 2.2.1 ([22]). *The following are the four basic t-conorms, namely, the maximum S_M, the probabilistic sum S_P, the Łukasiewicz t-conorm or (bounded sum) S_L, and the drastic sum S_D:*

1. $S_M(x,y) = max(x,y)$,
2. $S_P(x,y) = x + y - x \cdot y$,
3. $S_L(x,y) = min(x+y, 1)$,
4. $S_D(x,y) = \begin{cases} 1 & \text{if } (x,y) \in (0,1]^2, \\ max(x,y) & \text{otherwise.} \end{cases}$

The t-conorms are ordered in a similar way as the t-norms, i.e., they are ordered as it was explained in Definition 2.1.4. In fact the weakest t-conorm is S_M and the strongest t-conorm is S_D, so for all t-conorms the following order holds

$$S_M \leq S \leq S_D \tag{2.3}$$

In a dually manner as with t-norms, the t-conorms fitted with the an order, do not build a lattice.

2.3 Fuzzy Implication

The notion of fuzzy implication has several non-equivalent definitions (se for example [9, 20, 32]). Since it is the most commonly used in the literature, in our work we consider the axiomatization in [1], which is equivalent to the one presented in [17, 20].

The information present in this section is based on [1] with the exception of the Definition 2.3.3 which is from [4].

Definition 2.3.1 ([1]). *An operator $I : [0,1]^2 \to [0,1]$ is called a* fuzzy implication *if it satisfies, for all $x, x_1, x_2, y, y_1, y_2 \in [0,1]$, the following conditions:*

$(I1)$ *if $x_1 \leq x_2$, then $I(x_1, y) \geq I(x_2, y)$, i.e., $I(\cdot, y)$ is decreasing*

$(I2)$ *if $y_1 \leq y_2$, then $I(x, y_1) \leq I(x, y_2)$, i.e., $I(x, \cdot)$ is increasing*

$(I3)$ $I(0,0) = 1$

$(I4)$ $I(1,1) = 1$

$(I5)$ $I(1,0) = 0$

Remark 2.3.1 ([1]). *Directly from Definition 2.3.1 we have that each fuzzy implication I satisfies the following properties, respectively called left and right boundary conditions:*

(LB) $I(0,y) = 1, \quad y \in [0,1],$

(RB) $I(x,1) = 1, \quad x \in [0,1].$

Indeed, (LB) follows from $(I2)$ and $(I3)$, since $I(0,y) \geq I(0,0) = 1$. Similarly, because of $(I1)$ and $(I4)$ we get $I(x,1) \geq I(1,1) = 1$, i.e., I satisfies (RB). Note that (LB) means that falsity implies anything, while (RB) expresses that a tautology is implied by anything.

Example 2.3.1 ([1]). *The following are four basic implications, namely, the Gödel implication I_M, the Goguen implication I_P, the Łukasiewicz implication I_L, and the Weber implication I_D:*

1. $I_M(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise,} \end{cases}$

2. $I_P(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ y/x & \text{otherwise,} \end{cases}$

3. $I_L(x,y) = min(1 - x + y, 1),$

4. $I_D(x,y) = \begin{cases} y & \text{if } x = 1, \\ 1 & \text{otherwise.} \end{cases}$

Proposition 2.3.1 ([1]). *The set of all fuzzy implications in the sense of Definition 2.3.1 has a least and great element with respect to the order induced by the usual order on $[0,1]$, in fact*

$$I_0(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 1, \\ 0, & \text{if } x > 0 \text{ and } y < 1, \end{cases} \qquad x, y \in [0,1],$$

and

$$I_1(x,y) = \begin{cases} 1, & \text{if } x < 1 \text{ or } y > 0, \\ 0, & \text{if } x = 1 \text{ and } y = 0, \end{cases} \qquad x, y \in [0,1].$$

are fuzzy implications such that for any fuzzy implication I

$I_0 \leq I \leq I_1$

In the unpublished chapter of a book [2] it is prooved that an extension of the implication of Definition 2.3.1 valued on a bounded lattice and having a similar order to the one above is a bounded lattice.

Definition 2.3.2. *A fuzzy implication I is said to satisfy:*

(IP) *the* identity principle, *if* $I(x,x) = 1, \quad x \in [0,1],$

(OP) *the* ordering property*, if* $x \leq y \Longleftrightarrow I(x,y) \geq 1$ [1]*,* $\quad x,y \in [0,1]$.

The following property was mentioned in [4], but it did not had a name, so that is why we name it as (LOP).

Definition 2.3.3 ([4])**.** *A fuzzy implication I is said to satisfy:*

(LOP) *the* left ordering property*, if $x \leq y$, then $I(x,y) = 1$ for all $x,y \in [0,1]$.*

The identity principle (IP) states that the overall truth value should be 1 when the truth values of the antecedent and the consequent are equal and can be seen as the generalization of the tautology $p \to p$ from the classical logic.

The ordering property (OP), imposes an ordering on the underlying set and generalizes the Meta Theorem of Deduction in the case of tautologies.

2.4 R-Implication

The material that in this section is explained is based on [1], with the exception of Theorem 2.4.3 and Proposition 2.4.2, which where based on [17] and [4] respectively.

Definition 2.4.1 ([1])**.** *An operator $I : [0,1]^2 \to [0,1]$ is called an* R-implication *if there exists a t-norm T such that*

$$I(x,y) = sup\{z \in [0,1] \mid T(x,z) \leq y\}, \quad x,y \in [0,1] \tag{2.4}$$

If I is an R-Implication based on a t-norm T, it will be often denoted by I_T.

Since for every t-norm T and all $x,y \in [0,1]$ we have $T(x,0) = 0$, one can easily observe that the set $\{z \in [0,1] \mid T(x,z) \leq y\}$ in (2.4) is non-empty. The name R-implication is a short version of residual implication, and I_T is also called the residuum of T. This class of implications is related to a residuation concept from the intuitionistic logic [16]. Although the operator I_T obtained as in Definition 2.4.1 is a fuzzy implication for any t-norm T (see Theorem 2.4.1), when the t-norm is left-continuous it guarantees some interesting properties, as for example, the residuation one.

Definition 2.4.2 ([1])**.** *A t-norm T and a R-implication I_T are said to satisfy:*

(RP) *the* residual principle*, if* $T(x,z) \leq y \Leftrightarrow I_T(x,y) \geq z, \quad x,y,z \in [0,1]$.

Notice that (RP) generalizes the Meta Theorem of Deduction.

Proposition 2.4.1 ([1])**.** *For a t-norm T and an R-implication I_T the following statements are equivalent:*

[1] If we consider the fact that I is defined on $[0,1]$, then we can rewrite (OP) in the following way

$$x \leq y \Longleftrightarrow I(x,y) = 1, \quad x,y \in [0,1]$$

(i) *T is left-continuous.*

(ii) *T and I_T form an adjoint pair, i.e., they satisfy (RP).*

It is easy to see that if T_1 and T_2 are comparable t-norms such that $T_1 \leq T_2$, then the R-implications I_{T_1} and I_{T_2} satisfy $I_{T_1} \geq I_{T_2}$.

Theorem 2.4.1 (Theorem 2.5.4, page 70 of [1])**.** *If I is an R-implication of a t-norm T, then I is a fuzzy implication (in the sense of Definition) 2.3.1 and it satisfies (IP).*

Proposition 2.4.2 ([4])**.** *If I is an R-implication of a t-norm T then I satisfies (LOP).*

Proof. Let $x, y \in [0,1]$ if $x \leq y$ then

$$\begin{array}{lll} T(x,1) & = x & \text{Because of } (T4). \\ & \leq y & \text{By hypothesis.} \end{array}$$

So $1 \in \{z \in [0,1] \mid T(x,z) \leq y\}$ then

$$\begin{array}{lll} 1 & = sup\{z \in [0,1] \mid T(x,z) \leq y\} & \text{By the ordered induced by } [0,1]. \\ & = I(x,y) & \text{Because of Definition 2.4.1.} \end{array}$$

□

Theorem 2.4.2 (Theorem 2.5.7, page 73 of [1])**.** *If I is an R-implication of a left-continuous t-norm T, then I is a fuzzy implication (in the sense of Definition 2.3.1) and it satisfies (IP) and (OP).*

Definition 2.4.3 (Page 53 of [17])**.** *A binary fuzzy operator F on $[0,1]$ satisfies the* T-transitivity property *if for a left-continuous t-norm T and any $x, y, z \in [0,1]$ the following equation is satisfied*

$$T(F(x,y), F(y,z)) \leq F(x,z) \tag{2.5}$$

Theorem 2.4.3 (Proposition 1.10, page 27 of [17])**.** *If I is an R-implication of a left-continuous t-norm T then for all $x, y, z \in [0,1]$ it satisfies the T-transitivity property.*

2.5 Archimedean Triangular Norms

Definition 2.5.1 (Definition 2.9, page 26 of [21])**.** *A t-norm T has the* Archimedean property *if for all $(x,y) \in (0,1)^2$ there exists an $n \in \mathbb{N}$ such that $x_T^n < y$.*

Definition 2.5.2 (Continuous additive generator)**.** *A* continuous additive generator *is a strictly decreasing and continuous function $f : [0,1] \to [0,\infty]$ with $f(1) = 0$.*

Proposition 2.5.1 ([24])**.** *A t-norm T has a continuous additive generator such that it satisfies*

$$T(x,y) = f^{(-1)}(f(x) + f(y)), \tag{2.6}$$

where $f^{(-1)}$ is the pseudoinverse of f defined by

$$f^{(-1)}(x) = \begin{cases} f^{-1}(x), & \text{if } x \leq f(0), \\ 0, & \text{otherwise} \end{cases} \tag{2.7}$$

if and only if T is a continuous Archimedean t-norm.

2.6 A few properties of minimum

Proposition 2.6.1. *$min(x, y) \leq x$ and $min(x, y) \leq y$ for all $x, y \in [0, 1]$*

Proof. Case $x \leq y$ then

$$\begin{aligned} min(x,y) &= x && \text{By hypothesis.} \\ &\leq x && \\ &\leq y && \text{By hypothesis.} \end{aligned}$$

Case $y < x$ then

$$\begin{aligned} min(x,y) &= y && \text{By hypothesis.} \\ &\leq y && \\ &< x && \text{By hypothesis.} \end{aligned}$$

□

Proposition 2.6.2. *If $z \leq x$ and $z \leq y$ then $z \leq min(x, y)$*

Proof. Suppose $z \leq x$ and $z \leq y$.

Case $x \leq y$ then

$$\begin{aligned} min(x,y) &= x && \text{By hypothesis.} \\ &\geq z && \text{By hypothesis.} \end{aligned}$$

Case $y < x$ then

$$\begin{aligned} min(x,y) &= y && \text{By hypothesis.} \\ &\geq z && \text{By hypothesis.} \end{aligned}$$

□

Chapter 3

Fuzzy Bi-implication

As it was seen in the previos chapter, a reasonable requirement that an extension on the unit square of a classic operator is expected to satisfy is the boundary condition. But beyond that, in the case of a fuzzy bi-implication operator it is desirable that it relates to a notion of similarity.

3.1 Introduction to Fuzzy Bi-implication

In this section we present six definitions of fuzzy bi-implication, two of them based on axioms and the other four constructed from t-norms. The two axiomatic definitions were proposed independently in [17, 4] and we prove here that they are equivalent. In the case of the other four (half of them proposed in [21, 4] whereas the other two are a proposal of the present work) we prove that they are pairwise equivalent, two of them being more general than the other two definitions, yet all of them are bi-implications in the sense of [17]. We finish this section by presenting a recipe for constructing a fuzzy bi-implication by additive generators as proposed in [21].

Definition 3.1.1 ([17])**.** *A binary operator B on the unit interval $[0, 1]$ is called a* Fodor-Roubens fuzzy bi-implication *if it respects the following axioms for all $w, x, y, z \in [0, 1]$:*

($B1$) $B(x, y) = B(y, x)$

($B2$) $B(0, 1) = 0$

($B3$) $B(x, x) = 1$

($B4$) *If $w \leq x \leq y \leq z$, then $B(w, z) \leq B(x, y)$*

Notice that if B is a Fodor-Roubens fuzzy bi-implication then Definition 3.1.1 allows $B(x, y) = 1$ for some $x, y \in [0, 1]$ with $x \neq y$.

Definition 3.1.2 ([4])**.** *Let $B^\diamond$ be a binary operator on $[0,1]$. $B^\diamond$ is called a* Bedregal-Cruz fuzzy bi-implication *if it satisfies the following axioms for all $x, y, z \in [0,1]$:*

($B1^\diamond$) $B^\diamond(x,y) = B^\diamond(y,x)$

($B2^\diamond$) $B^\diamond(0,1) = 0$

($B3^\diamond$) *If $x = y$, then $B^\diamond(x,y) = 1$*

($B4^\diamond$) *If $x \le y \le z$ then $B^\diamond(x,y) \ge B^\diamond(x,z)$ and $B^\diamond(y,z) \ge B^\diamond(x,z)$*

Proposition 3.1.1. *An operator B is a Fodor-Roubens fuzzy bi-implication if and only if it is a Bedregal-Cruz fuzzy bi-implication.*

Proof. Trivially it can be seen that $(B1), (B2)$ and $(B3)$ are identical to $(B1^\diamond)$, $(B2^\diamond)$ and $(B3^\diamond)$. Now we are going to prove that $(B4)$ and $(B4^\diamond)$ are equivalent.

Suppose B is a Fodor-Roubens fuzzy bi-implication and that $w, x, y, z, k \in [0,1]$.

If $x = w \le x \le y \le z \le k = z$ then

$$\begin{aligned} B(x,y) &\ge B(w,z) && \text{Because of } (B4). \\ &= B(x,z) && \text{By hypothesis.} \end{aligned}$$

and

$$\begin{aligned} B(y,z) &\ge B(x,k) && \text{Because of } (B4). \\ &= B(x,z) && \text{By hypothesis.} \end{aligned}$$

Therefore B satisfies $(B4^\diamond)$.

Now suppose B is a Bedregal-Cruz fuzzy bi-implication and that $w, x, y, z \in [0,1]$.

If $w \le x \le y \le z$ then

$$\begin{aligned} B(x,y) &\ge B(x,z) && \text{Given } x \le y \le z \text{ and part 1 of } (B4^\diamond). \\ &\ge B(w,z) && \text{Given } w \le x \le z \text{ and part 2 of } (B4^\diamond). \end{aligned}$$

Therefore B satisfies $(B4)$. □

Definition 3.1.3 (Page 235 of [21])**.** *Let T be a left-continuous t-norm and I_T be an R-implication (Definition 2.4.1). The binary operator B on $[0,1]$ is called a* KMP fuzzy bi-implication *if for each $x, y \in [0,1]$, $B(x,y) = T(I_T(x,y), I_T(y,x))$.*

In the next definition we propose an apparent generalization of the Definition of KMP fuzzy bi-implication.

Definition 3.1.4. *If T is a t-norm, T' is a left-continuous t-norm and $I_{T'}$ is an R-implication (Definition 2.4.1), the binary operator B on $[0,1]$, defined as $B(x,y) = T(I_{T'}(x,y), I_{T'}(y,x))$ is called an* LT fuzzy bi-implication*.*

Lemma 3.1.1. *If T' is a left-continuous t-norm and $I_{T'}$ is an R-implication (Definition 2.4.1) then $min(I_{T'}(x,y), I_{T'}(y,x)) = I_{T'}(max(x,y), min(x,y))$ for all $x, y \in [0,1]$*

Proof. Let T' be a left-continuous t-norm and $I_{T'}$ be an R-implication. Assume w.l.o.g. that $x \leq y$. Then

$$\begin{aligned} min(I_{T'}(x,y), I_{T'}(y,x)) &= min(1, I_{T'}(y,x)) && \text{By Prop. 2.4.2.} \\ &= I_{T'}(y,x) \\ &= I_{T'}(max(x,y), min(x,y)) && \text{Because } x \leq y. \end{aligned}$$

In the case that $y \leq x$ the proof is analogous. □

In the following proposition we make an apparent generalization of the properties proposed in [5] by setting the operator as a LT fuzzy bi-implication instead of an KMP fuzzy bi-implication.

Proposition 3.1.2. *If T is a t-norm, T' is a left-continuous t-norm, $I_{T'}$ is an R-implication (Definition 2.4.1) and B is an LT fuzzy bi-implication based on T and $I_{T'}$, then B satisfies the following properties, for all $x, y, z \in [0, 1]$:*

$(B1')$ $B(x,y) = 1$ *if and only if* $x = y$

$(B2')$ $B(x,y) = B(y,x)$

$(B3')$ $B(x,y) = min(I_{T'}(x,y), I_{T'}(y,x))$

$(B4')$ $T'(B(x,y), B(y,z)) \leq B(x,z)$

$(B5')$ $B(x,y) = I_{T'}(max(x,y), min(x,y))$

Proof. $(B3')$ Suppose that $x \leq y$. Then

$$\begin{aligned} B(x,y) &= T(I_{T'}(x,y), I_{T'}(y,x)) && \text{By def. of LT fuzzy bi-implication.} \\ &= T(1, I_{T'}(y,x)) && \text{By } (LOP). \\ &= I_{T'}(y,x) && \text{By } (T1) \text{ and } (T4). \\ &= min(I_{T'}(y,x), 1) \\ &= min(1, I_{T'}(y,x)) && \text{By commutativity of } min. \\ &= min(I_{T'}(x,y), I_{T'}(y,x)) && \text{By } (LOP). \end{aligned}$$

The proof is analogous in the case $y \leq x$. Thus B satisfies $(B3')$.

$(B1')$ Suppose $x = y$. Then

$$\begin{aligned} B(x,y) &= min(I_{T'}(x,y), I_{T'}(y,x)) && \text{Since } B \text{ satisfies } (B3'). \\ &= min(1,1) && \text{By Theorem 2.4.1 } I_{T'} \text{ satisfies } (IP). \\ &= 1 \end{aligned}$$

Therefore if $x = y$ then $B(x,y) = 1$.

Now we must prove that if $B(x,y) = 1$ then $x = y$.

Let $B(x,y) = 1$ and suppose that $x \neq y$. So if $x < y$, then

$$\begin{aligned} B(x,y) &= min(I_{T'}(x,y), I_{T'}(y,x)) && \text{Since } B \text{ satisfies } (B3'). \\ &= min(1, I_{T'}(y,x)) && \text{By } (LOP). \\ &= I_{T'}(y,x) \\ &\neq 1 && \text{Since } x < y \text{ and } I_{T'} \text{ satisfies } (OP). \end{aligned}$$

This is an absurd. In the case that $y < x$ the proof is analogous. So by contradiction if $B(x,y) = 1$ then $x = y$. Therefore B satisfies $(B1')$.

$(B2')$

$$\begin{aligned} B(x,y) &= min(I_{T'}(x,y), I_{T'}(y,x)) && \text{By } (B3'). \\ &= min(I_{T'}(y,x), I_{T'}(x,y)) && \text{By the commutativity of } min. \\ &= B(y,x) && \text{By } (B3'). \end{aligned}$$

Therefore B satisfies $(B2')$.

$(B4')$

Now we are going to prove that $T'(B(x,y), B(y,z)) \leq I_{T'}(x,z)$.

$T'(B(x,y), B(y,z))$

$= T'(min(I_{T'}(x,y), I_{T'}(y,x)), min(I_{T'}(y,z), I_{T'}(z,y)))$ By $(B3')$.

$\leq T'(I_{T'}(x,y), min(I_{T'}(y,z), I_{T'}(z,y)))$ By Prop. 2.6.1, $(T3)$ and $(T1)$.

$\leq T'(I_{T'}(x,y), I_{T'}(y,z))$ Because of Prop. 2.6.1 and $(T3)$.

$\leq I_{T'}(x,z)$ By Theorem 2.4.3.

Now we are going to prove that $T'(B(x,y), B(y,z)) \leq I_{T'}(z,x)$.

$T'(B(x,y), B(y,z))$

$= T'(min(I_{T'}(x,y), I_{T'}(y,x)), min(I_{T'}(y,z), I_{T'}(z,y)))$ By $(B3')$.

$\leq T'(I_{T'}(y,x), min(I_{T'}(y,z), I_{T'}(z,y)))$ By Prop. 2.6.1, $(T3)$ and $(T1)$.

$\leq T'(I_{T'}(y,x), I_{T'}(z,y))$ Because of Prop. 2.6.1 and $(T3)$.

$\leq I_{T'}(z,x)$ By $(T1)$ and Theorem 2.4.3.

Therefore

$$\begin{aligned} T'(B(x,y), B(y,z)) &\leq min(I_{T'}(x,z), I_{T'}(z,x)) && \text{By Prop. 2.6.2.} \\ &= B(x,z) && \text{By } (B3'). \end{aligned}$$

Thus B satisfies $(B4')$.

$(B5')$

By Lemma 3.1.1 we know that $(B3')$ is equal to $(B5')$, therefore B also satisfies $(B5')$. □

Remark 3.1.1. *The properties $(B3')$ and $(B5')$ of the Proposition 3.1.2 show that the specific t-norms over which a given LT fuzzy bi-implications are based are inconsequential, i.e. all the LT fuzzy bi-implications that are based on the same R-implication are equivalent.*

In the following theorem we show the relationship between the KMP fuzzy bi-implications and the LT fuzzy bi-implications.

Theorem 3.1.1. *An operator B is a KMP fuzzy bi-implication if and only if it is an LT fuzzy bi-implication.*

Proof. Let T be a t-norm, T' be a left-continuous t-norm and $I_{T'}$ be an R-implication (Definition 2.4.1). Let also an operator B be a KMP fuzzy bi-implication based on T' and $I_{T'}$. Because B is commutative, without loss of generality, we can suppose that $x \leq y$, then

$$\begin{aligned} B(x,y) &= T'(I_{T'}(x,y), I_{T'}(y,x)) && \text{By def. of KMP fuzzy bi-implication.} \\ &= T'(1, I_{T'}(y,x)) && \text{By } (LOP). \\ &= I_{T'}(y,x) && \text{By } (T1) \text{ and } (T4). \\ &= min(I_{T'}(y,x), 1) && \\ &= min(1, I_{T'}(y,x)) && \text{By commutativity of min.} \\ &= min(I_{T'}(x,y), I_{T'}(y,x)) && \text{By } (LOP). \end{aligned}$$

Now let an operator B be an LT fuzzy bi-implication based on T and $I_{T'}$.

$$\begin{aligned} B(x,y) &= T(I_{T'}(x,y), I_{T'}(y,x)) && \text{By def. of LT fuzzy bi-implication.}\\ &= min(I_{T'}(x,y), I_{T'}(y,x)) && \text{By Prop. 3.1.2 it satisfies } (B3'). \end{aligned}$$

By transitivity of equality the operator B constructed as a KMP fuzzy bi-implication is the same than when it is constructed as an LT fuzzy bi-implication.

□

In the next proposition we prove that any KMP fuzzy bi-implication is also a Fodor-Roubens fuzzy bi-implication.

Proposition 3.1.3. *Any KMP fuzzy bi-implication is also a Fodor-Roubens fuzzy bi-implication.*

Proof. Let T' be a left-continuous t-norm and $I_{T'}$ be an R-implication (Definition 2.4.1). Suppose B is a KMP fuzzy bi-implication based on T' and $I_{T'}$, then by Theorem 3.1.1 it is a LT fuzzy bi-implication and by Proposition 3.1.2 it satisfies properties $(B1'), (B2'), (B3'), (B4')$ and $(B5')$:

$(B1)$

Since B satisfies $(B2')$ it is trivial that it satisfies $(B1)$.

$(B2)$

$$\begin{aligned} B(0,1) &= min(I_{T'}(0,1), I_{T'}(1,0)) && \text{By } (B3').\\ &= min(1,0) && \text{By } (LB) \text{ and } (I5).\\ &= 0 \end{aligned}$$

Hence B also satisfies $(B2)$.

$(B3)$

Because of $(B1')$ B also satisfies $(B3)$.

$(B4)$

Suppose $w \leq x \leq y \leq z$ then

$$\begin{aligned} B(w,z) &= min(I_{T'}(w,z), I_{T'}(z,w)) && \text{By } (B3').\\ &= min(1, I_{T'}(z,w)) && \text{By } (LOP).\\ &= I_{T'}(z,w)\\ &\leq I_{T'}(y,w) && \text{Because of } (I1) \text{ and } y \leq z.\\ &\leq I_{T'}(y,x) && \text{By } (I2) \text{ and } w \leq x.\\ &= min(1, I_{T'}(y,x))\\ &= min(I_{T'}(x,y), I_{T'}(y,x)) && \text{By } (LOP) \text{ and } x \leq y.\\ &= B(x,y) && \text{By } (B3'). \end{aligned}$$

Therefore B satisfies $(B4)$. Thus B is also a Fodor-Roubens fuzzy bi-implication. □

Thanks to the last proposition we know that the class of all operators that are KMP fuzzy bi-implications is a subclass of all operators that are Fodor-Roubens fuzzy bi-implications. In the following we show an operator that is a Fodor-Roubens fuzzy bi-implication, but not a KMP fuzzy bi-implication.

Proposition 3.1.4. *Not every Fodor-Roubens fuzzy bi-implication is a KMP fuzzy bi-implication. For an example, consider*

$$B'_G(x,y) = \begin{cases} min(x,y), & \textit{if } max(x,y) = 1,\\ 1, & \textit{otherwise,} \end{cases} \tag{3.1}$$

that was a operator proposed in [4].

Proof. Because max and min are commutative, so B'_G satisfies $(B1)$.
$B'_G(0,1) = min(0,1) = 0$, thus B'_G satisfies axiom $(B2)$.
Suppose $x = 1$ then $B'_G(x,x) = min(x,x) = x = 1$, otherwise $B'_G(x,x) = 1$. Therefore B'_G satisfies $(B3)$.
Suppose $w \leq x \leq y \leq z$, if $z < 1$ then $B'_G(w,z) = 1 \leq 1 = B'_G(x,y)$, if $z = 1$ then $B'_G(w,z) = min(w,1) = w \leq x$. In case $y = 1$ then $x = min(x,1) = B'_G(x,y)$, in case $y < 1$ then $x < 1 = B'_G(x,y)$. Therefore by transitivity of the inequality $B'_G(w,z) \leq B'_G(x,y)$. So B'_G also satisfies $(B4)$.
Suppose $x < y < 1$ then $B'_G(x,y) = 1$. Since we have $B'_G(x,y) = 1$ and $x \neq y$ then B'_G does not satisfy $(B1')$.

Therefore B'_G is a Fodor-Roubens fuzzy bi-implication, but since all operators that are a KMP fuzzy bi-implication satisfy $(B1')$ and B'_G does not satisfy that property then B'_G is not a KMP fuzzy bi-implication. □

Given propositions 3.1.3 and 3.1.4 we can conclude that the class of all operators that are KMP fuzzy bi-implications, or LT fuzzy bi-implications, is a proper subclass of all operators that are Fodor-Roubens fuzzy bi-implication.

Untill now we know that KMP fuzzy bi-implications and LT fuzzy bi-implications satisfy property $(B4')$, in other words, that they satisfy the T-transitivity property. So we believe that it is important to verify if that property is satisfied by the Fodor-Roubens fuzzy bi-implication. Through the following proposition we are going to answer this question.

Proposition 3.1.5. *The Fodor-Roubens fuzzy bi-implication does not guarantee the T-transitivity property.*

Proof. By a counterexample we are going to show that for a left-continuous t-norm T the operator B'_G, which is a Fodor-Roubens fuzzy bi-implication, is not T-transitive.

Let $x = 1, y = .9$ and $z = .8$. Then we have that

$$\begin{aligned} T(B'_G(1,.9), B'_G(.9,.8)) &= T(.9,1), \\ &= .9 \qquad\qquad \text{By } (T4), \\ &\nleq .8, \\ &= B'_G(1,.8) \end{aligned}$$

□

Since the KMP fuzzy bi-implications satisfy $(B4')$ and $(B4)$ in the case that an operator of that class is based on T_M (Definition 2.1.1) we observe a particular property.

Proposition 3.1.6. *If $x \leq y \leq z$ and B is a KMP fuzzy bi-implication based on T_M (Definition 2.1.1) then $min(B(x,y), B(y,z)) = B(x,z)$, for all $x, y, z \in [0,1]$.*

Proof. Let an operator B be a KMP fuzzy bi-implication based on T_M and suppose $x = w \leq x \leq y \leq z \leq k = z$ then

$$\begin{aligned} B(x,y) &\geq B(w,z), && \text{By } (B4). \\ &= B(x,z), && \text{By hypothesis.} \end{aligned}$$

and we also have that

$$\begin{aligned} B(y,z) &\geq B(x,k), && \text{By } (B4). \\ &= B(x,z), && \text{By hypothesis.} \end{aligned}$$

Therefore by Proposition 2.6.2 we know that the following equation holds

$$min(B(x,y), B(y,z)) \geq B(x,z) \tag{3.2}$$

Because of the property $(B4')$ we have the following

$$min(B(x,y), B(y,z)) \leq B(x,z) \tag{3.3}$$

So by the inequalities (3.2) and (3.3) the following equation holds whenever $x \leq y \leq z$

$$min(B(x,y), B(y,z)) = B(x,z) \tag{3.4}$$

□

As one of the side effects of the T-transitivity, we can see that the last result is counterintuitive, because the similarity between two outer elements of an ascending three-element chain is similar in the same grade than one of those outer elements and the element that is between them.

In the following definition we present a generalization of the class of LT fuzzy bi-implications, because the R-implications are constructed with arbitrary t-norms, not necessarie left-continuous ones. Recall that an R-implication of a non-left-continuous t-norm never satisfies (RP) (Definition 2.4.2) neither (OP) (Definition 2.3.2) nor the T-transitivity property (Definition 2.4.3).

Definition 3.1.5 ([4])**.** *If T and T^* are t-norms and I_{T^*} is an R-implication (Definition 2.4.1), the binary operator B on $[0,1]$ defined as $B(x,y) = T(I_{T^*}(x,y), I_{T^*}(y,x))$ is called a* BC fuzzy bi-implication.

Proposition 3.1.7. *BC fuzzy bi-implications satisfy properties $(B2'), (B3')$ and $(B5')$.*

Proof. Let T and T^* be t-norms and I_{T^*} be an R-implication (Definition 2.4.1). Let also B be a BC fuzzy bi-implication based on T and I_{T^*}.

$(B2')$

$$\begin{aligned} B(x,y) &= T(I_{T^*}(x,y), I_{T^*}(y,x)) && \text{By def. of BC fuzzy bi-implication.} \\ &= T(I_{T^*}(y,x), I_{T^*}(x,y)) && \text{Because of } (T1). \\ &= B(y,x) && \text{By def. of BC fuzzy bi-implication.} \end{aligned}$$

Thus B satisfies $(B2')$.

$(B3')$

Suppose $x \leq y$ then

$$\begin{aligned} B(x,y) &= T(I_{T^*}(x,y), I_{T^*}(y,x)) && \text{By def. of BC fuzzy bi-implication.} \\ &= T(1, I_{T^*}(y,x)) && \text{By } (LOP) \text{ and hypothesis.} \\ &= I_{T^*}(y,x) && \text{Because of } (T1) \text{ and } (T4). \\ &= min(1, I_{T^*}(y,x)) && \text{Because } I_{T^*}(y,x) \leq 1. \\ &= min(I_{T^*}(x,y), I_{T^*}(y,x)) && \text{By } (LOP) \text{ and hypothesis.} \end{aligned}$$

The case when $y \leq x$ is analogous. Thus B satisfies $(B3')$.
$(B5')$
Suppose $x \leq y$ then

$$\begin{array}{lll} B(x,y) & = T(I_{T^*}(x,y), I_{T^*}(y,x)) & \text{By def. of BC fuzzy bi-implication.} \\ & = T(1, I_{T^*}(y,x)) & \text{By } (LOP) \text{ and hypothesis.} \\ & = I_{T^*}(y,x) & \text{Because of } (T1) \text{ and } (T4). \\ & = I_{T^*}(max(x,y), min(x,y)) & \text{By hypothesis.} \end{array}$$

The case where $y \leq x$ is analogous. Thus B satisfies $(B5')$.

□

Through the following proposition we prove that some BC fuzzy bi-implications are not LT fuzzy bi-implications, because they do not satisfy $(B1')$ or $(B4')$.

Proposition 3.1.8. *There are BC fuzzy bi-implications that fail properties $(B1')$, e.g. $B_{1'} = T_P(I_D(x,y), I_D(y,x))$ and $(B4')$, e.g.
$B_{4'} = T_D(I_D(x,y), I_D(y,x))$.*

Proof. $(B1')$
Let B be a BC fuzzy bi-implication based on T_P and I_D. For $x = .3$ and $y = .8$ we have that

$$\begin{array}{lll} I_D(.3,.8) & = 1 & = I_D(.8,.3) \\ T_P(1,1) & = 1 \times 1 & = 1, \\ B(.3,.8) & = T_P(I_D(.3,.8), I_D(.8,.3)) & \\ & = T_P(1,1) & \\ & = 1 & \end{array}$$

Since $B(.3,.8) = 1$ and $.3 \neq .8$ then B does not satisfy $(B1')$.
$(B4')$
Let B be a BC fuzzy bi-implication based on T_D and I_D. For $x = .4, y = .5$ and $z = 1$ we have that

$$\begin{array}{l} T_D(T_D(I_D(.4,.5), I_D(.5,.4)), T_D(I_D(.5,1), I_D(1,.5))) \\ = T_D(T_D(1,1), T_D(1,.5)) \\ = T_D(1,.5) \\ = .5 \\ \not\leq .4 \\ = T_D(1,.4) \\ = T_D(I_D(.4,1), I_D(1,.4)) \end{array}$$

Since $T_D(B(.4,.5), B(.5,1)) \not\leq B(.4,1)$ then B does not satisfy $(B4')$.

□

Since in Proposition 3.1.2 it was proved that LT fuzzy bi-implications satisfy $(B1'), (B2'), (B3'), (B4')$ and $(B5')$ and in Proposition 3.1.8 it was shown that there are BC fuzzy bi-implications that do not satisfy $(B1')$ or $(B4')$ we know that the class of operators that are BC fuzzy bi-implications is not identical to the class of operators that are LT fuzzy bi-implications. Now we are going to analyze if one of the classes contains the other. Since the class of all operators that are LT fuzzy bi-implications restricts the t-norms in which the R-implications are based to be left-continuous and the class of all operators that are BC fuzzy bi-implications allows any t-norm, then we can infer that the former is a subclass of the later. Since we also know that these classes are not identical, we can conclude that the class of all operators that are LT fuzzy bi-implications is a proper subclass of all the operators that are BC fuzzy bi-implications.

In the following proposition we are going to prove that an operator that is a BC fuzzy bi-implication is also a Fodor-Roubens fuzzy bi-implication.

Proposition 3.1.9. *If B is a BC fuzzy bi-implication, then B is also a Fodor-Roubens fuzzy bi-implication.*

Proof. Let T and T^* be t-norms and I_{T^*} be an R-implication (Definition 2.4.1). Let also B be a BC fuzzy bi-implication based on T and I_{T^*}.

$(B1)$

$B(x,y)$	$= T(I_{T^*}(x,y), I_{T^*}(y,x))$	By def. of BC fuzzy bi-implication.
	$= T(I_{T^*}(y,x), I_{T^*}(x,y))$	Because of $(T1)$.
	$= B(y,x)$	By def. of BC fuzzy bi-implication.

Therefore B satisfies $(B1)$.

$(B2)$

$B(0,1)$	$= T(I_{T^*}(0,1), I_{T^*}(1,0))$	By def. of BC fuzzy bi-implication.
	$= T(1, I_{T^*}(1,0))$	Because of (LB).
	$= T(1,0)$	By $(I5)$.
	$= 0$	By $(T1)$ and $(T4)$.

Thus B satisfies $(B2)$.

$(B3)$

$B(x,x)$	$= T(I_{T^*}(x,x), I_{T^*}(x,x))$	By def. of BC fuzzy bi-implication.
	$= T(1,1)$	By Theorem 2.4.1 I_{T^*} satisfies (IP).
	$= 1$	Because of $(T4)$.

Hence B satisfies $(B3)$.

$(B4)$

Suppose $w \leq x \leq y \leq z$ then

$B(w,z)$	$= T(I_{T^*}(w,z), I_{T^*}(z,w))$	By def. of BC fuzzy bi-implication.
	$= T(1, I_{T^*}(z,w))$	By (LOP) and $w \leq z$.
	$= I_{T^*}(z,w)$	Because of $(T1)$ and $(T4)$.
	$\leq I_{T^*}(z,x)$	By $(I2)$.
	$\leq I_{T^*}(y,x)$	By $(I1)$.
	$= T(1, I_{T^*}(y,x))$	By $(T4)$ and $(T1)$.
	$= T(I_{T^*}(x,y), I_{T^*}(y,x))$	By (LOP) and $x \leq y$.
	$= B(x,y)$	By def. of BC fuzzy bi-implication.

Therefore B satisfies $(B4)$. □

Thanks to the Proposition 3.1.9 we know that the class of all operators that are BC fuzzy bi-implications is a subclass of the operators that are Fodor-Roubens fuzzy bi-implications.

In the following definition we propose an operator similar to the BC fuzzy bi-implication, but with the difference that it is constructed with only one t-norm and not necessarie a left-continuous one.

Definition 3.1.6. *If T^* is a t-norm and I_{T^*} is an R-implication (Definition 2.4.1), the binary operator B on $[0,1]$ defined as*
$B(x,y) = T^*(I_{T^*}(x,y), I_{T^*}(y,x))$ *is called an* AT fuzzy bi-implication*.*

Theorem 3.1.2. *An operator B is a BC fuzzy bi-implication if and only if it is an AT fuzzy bi-implication.*

Proof. Let T and T^* be two t-norms and I_{T^*} be an R-implication (Definition 2.4.1). Let also an operator B be an AT fuzzy bi-implication based on T^* and I_{T^*}. Suppose that $x \leq y$. Then

$$\begin{array}{lll} B(x,y) & = T^*(I_{T^*}(x,y), I_{T^*}(y,x)) & \text{By def. of AT fuzzy bi-implication.} \\ & = T^*(1, I_{T^*}(y,x)) & \text{By } (LOP) \text{ and hypothesis.} \\ & = I_{T^*}(y,x) & \text{Because of } (T1) \text{ and } (T4). \\ & = min(1, I_{T^*}(y,x)) & \\ & = min(I_{T^*}(x,y), I_{T^*}(y,x)) & \text{By } (LOP) \text{ and hypothesis.} \end{array}$$

In case $y \leq x$ the proof is analogous.

Now let an operator B be a BC fuzzy bi-implication based on T and I_{T^*}.

$$\begin{array}{lll} B(x,y) & = T(I_{T^*}(x,y), I_{T^*}(y,x)) & \text{By def. of BC fuzzy bi-implication.} \\ & = min(I_{T^*}(x,y), I_{T^*}(y,x)) & \text{By Prop. 3.1.7 it satisfies } (B3'). \end{array}$$

By transitivity of equality the operator B constructed as an AT fuzzy bi-implication is the same than when it is constructed as a BC fuzzy bi-implication. □

In the Figure 3.1 the relations between all the fuzzy bi-implication definitions shown until now are depected.

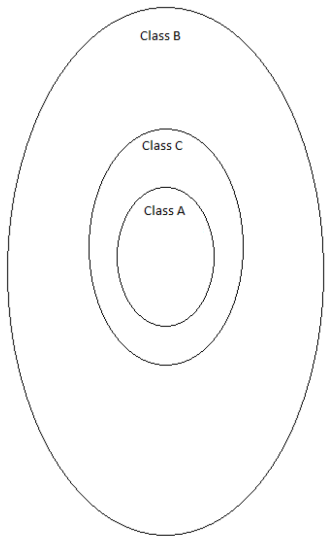

Figure 3.1: Relation between the class of operators that are KMP fuzzy bi-implications (Class A) or LT fuzzy bi-implications (Class A), the class of operators that are Fodor-Roubens fuzzy bi-implications (Class B) or Bedregal-Cruz fuzzy bi-implications (Class B) and the class of operators that are BC fuzzy bi-implications (Class C) or AT fuzzy bi-implications (Class C).

Definition 3.1.7 ([21, 3])**.** *From the most common t-norms the following operators may be defined:*

1. $B_M(x,y) = \begin{cases} 1, & \textit{if } x = y, \\ min(x,y), & \textit{otherwise,} \end{cases}$

2. $B_P(x,y) = \begin{cases} 1, & \textit{if } x = y, \\ \frac{min(x,y)}{max(x,y)}, & \textit{otherwise,} \end{cases}$

3. $B_L(x,y) = 1 - |x-y|,$

4. $B_D(x,y) = \begin{cases} y, & \textit{if } x = 1, \\ x, & \textit{if } y = 1, \\ 1, & \textit{otherwise.} \end{cases}$

Remark 3.1.2. *The operators B_M, B_P and B_L are KMP fuzzy bi-implications and B_D is an AT fuzzy bi-implication and is not a KMP fuzzy bi-implication, because T_D is not a left-continuous t-norm.*

In Figure 3.2 we include the examples B_M, B_P, B_L and B_D.

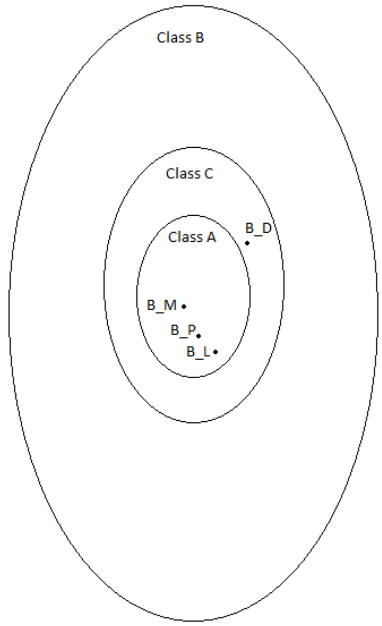

Figure 3.2: Relation between the class of operators that are KMP fuzzy bi-implications (Class A) or LT fuzzy bi-implications (Class A), the class of operators that are Fodor-Roubens fuzzy bi-implications (Class B) or Bedregal-Cruz fuzzy bi-implications (Class B) and the class of operators that are BC fuzzy bi-implications (Class C) or AT fuzzy bi-implications (Class C). Includes examples.

Through the following proposition we show how to get a KMP fuzzy bi-implication by means of a continuous additive generator.

Proposition 3.1.10 (Page 235 of [21]). *Let f be the continuous additive generator of the continuous Arquimedean t-norm T. Then the fuzzy bi-implication B defined as a KMP fuzzy bi-implication can be generated by*

$$B(x,y) = f^{(-1)}(|f(x) - f(y)|). \tag{3.5}$$

3.2 A new defining standard for fuzzy bi-implications

In classic logic it can be proved that the formulae $p \leftrightarrow q$ and $(p \vee q) \rightarrow (p \wedge q)$ are logically equivalent. So, it is reasonable to consider this logical equivalence to introduce a new kind of fuzzy bi-implications which we will prove in the most general case that is not a Fodor-Roubens fuzzy bi-implication. We will also show LT fuzzy bi-implications and KMP fuzzy bi-implications are a particular case of this new kind of fuzzy bi-implications.

Definition 3.2.1. *Let T be a t-norm, T' be a left continuous t-norm, $I_{T'}$ be an R-implication (Definition 2.4.1) and S be a t-conorm. Then the binary operator B on $[0,1]$ defined by*

$$B(x,y) = I_{T'}(S(x,y), T(x,y)) \tag{3.6}$$

is called a TS fuzzy bi-implication.

It may be interesting to ask ourselves if the class of operators that are TS fuzzy bi-implications, are the same than the class of operators that are LT fuzzy bi-implications or Fodor-Roubens fuzzy bi-implications. In the following propositions we show that this new kind of fuzzy bi-implications is not a fuzzy bi-implication in the sense of neither of these definitions.

Proposition 3.2.1. *TS fuzzy bi-implications satisfy property $(B2')$.*

Proof. Let T be a t-norm, T' be a left continuous t-norm, $I_{T'}$ be an R-implication (Definition 2.4.1) and S be a t-conorm. Let also B be a TS fuzzy bi-implication based on T, $I_{T'}$ and S. We are going to prove that B satisfies $(B2')$.

$$\begin{aligned} B(x,y) &= I_{T'}(S(x,y), T(x,y)) && \text{By Definition of TS fuzzy bi-implication.}\\ &= I_{T'}(S(y,x), T(y,x)) && \text{By } (T1).\\ &= B(y,x) && \text{By Definition of TS fuzzy bi-implication.} \end{aligned}$$

Thus B satisfies $(B2')$. □

Proposition 3.2.2. *There are TS fuzzy bi-implications that fail properties $(B1'), (B3'), (B5')$, e.g. $B_i(x,y) = I_M(S_P(x,y), T_L(y,x))$, or $(B4')$, e.g. $B_j(x,y) = I_M(S_D(x,y), T_D(y,x))$.*

Proof. $(B1')$

Let B be a TS fuzzy bi-implication based on T_L, I_M and S_P. For $x = y = .7$ we have that

$$\begin{aligned} S_P(.7,.7) &= .7 + .7 - (.7 \times .7) && = .91,\\ T_L(.7,.7) &= max(.7 + .7 - 1, 0) && = .4,\\ I_M(.91,.4) &= .4,\\ B(.7,.7) &= I_M(S_P(.7,.7), T_L(.7,.7))\\ &= I_M(.91,.4)\\ &= .4\\ &\neq 1. \end{aligned}$$

Therefore B does not satisfy $(B1')$.

$(B3')$

Let B be a TS fuzzy bi-implication based on T_L, I_M and S_P. For $x = .7$ and $y = .2$ we have that

$$\begin{aligned}
S_P(.7,.2) &= .7+.2-(.7\times .2) && = .76,\\
T_L(.7,.2) &= max(.7+.2-1,0) && = 0,\\
I_M(.76,0) &= 0,\\
I_M(.7,.2) &= .2,\\
I_M(.2,.7) &= 1,\\
min(.2,1) &= .2,\\
B(.7,.2) &= I_M(S_P(.7,.2),T_L(.7,.2))\\
&= I_M(.76,0)\\
&= 0\\
&\neq .2\\
&= min(.2,1)\\
&= min(I_M(.7,.2),I_M(.2,.7)).
\end{aligned}$$

Therefore B does not satisfy $(B3')$.

$(B4')$

Let B be a TS fuzzy bi-implication based on T_D, I_M and S_D. For $x = .3, y = 1$ and $z = .7$ we have that

$$\begin{aligned}
S_D(.3,1) &= 1,\\
T_D(.3,1) &= min(.3,1) && = .3,\\
I_M(1,.3) &= .3,\\
S_D(1,.7) &= 1,\\
T_D(1,.7) &= min(1,.7) && = .7,\\
I_M(1,.7) &= .7,\\
min(.3,.7) &= .3,\\
S_D(.3,.7) &= 1,\\
T_D(.3,.7) &= 0,\\
I_M(1,0) &= 0,
\end{aligned}$$

$$\begin{aligned}
&min(B(.3,1),B(1,.7))\\
&= min(I_M(S_D(.3,1),T_D(.3,1)),I_M(S_D(1,.7),T_D(1,.7)))\\
&= min(I_M(1,.3),I_M(1,.7))\\
&= min(.3,.7)\\
&= .3\\
&\not\leq 0\\
&= I_M(1,0)\\
&= I_M(S_D(.3,.7),T_D(.3,.7))\\
&= B(.3,.7)
\end{aligned}$$

Therefore B does not satisfy $(B4')$.

$(B5')$

Let B be a TS fuzzy bi-implication based on T_L, I_M and S_P. For $x = .7$ and $y = .2$ we have that

$$\begin{array}{lll} S_P(.7,.2) & = .7+.2-(.7\times .2) & = .76, \\ T_L(.7,.2) & = max(.7+.2-1,0) & = 0, \\ I_M(.76,0) & = 0, & \\ max(.7,.2) & = .7, & \\ min(.7,.2) & = .2, & \\ I_M(.7,.2) & = .2, & \\ B(.7,.2) & = I_M(S_P(.7,.2),T_L(.7,.2)) & \\ & = I_M(.76,0) & \\ & = 0 & \\ & \neq .2 & \\ & = I_M(.7,.2) & \\ & = I_M(max(.7,.2),min(.7,.2)). & \end{array}$$

Therefore B does not satisfy $(B5')$. □

Proposition 3.2.3. *TS fuzzy bi-implications satisfy properties* $(B1)$ *and* $(B2)$.

Proof. Let T be a t-norm, T' be a left continuous t-norm, $I_{T'}$ be an R-implication (Definition 2.4.1) and S be a t-conorm. Let also B be a TS fuzzy bi-implication based on T, $I_{T'}$ and S. We are going to prove that B satisfies $(B1)$.

$$\begin{array}{lll} B(x,y) & = I_{T'}(S(x,y),T(x,y)) & \text{By definition of TS fuzzy bi-implication.} \\ & = I_{T'}(S(y,x),T(y,x)) & \text{By } (T1). \\ & = B(y,x) & \text{By definition of TS fuzzy bi-implication.} \end{array}$$

Thus B satisfies $(B1)$.

Through the following equations we are going to prove that B satisfies $(B2)$.

$$\begin{array}{lll} B(0,1) & = I_{T'}(S(0,1),T(0,1)) & \text{By definition of TS fuzzy bi-implication.} \\ & = I_{T'}(S(0,1),0) & \text{By } (T4). \\ & = I_{T'}(1,0) & \text{By } (T1) \text{ and } (S4). \\ & = 0 & \text{By } (I5). \end{array}$$

Thus B satisfies $(B2)$. □

Proposition 3.2.4. *There are TS fuzzy bi-implications that fail properties* $(B3)$*, e.g.* $B_A(x,y) = I_M(S_P(x,y),T_L(x,y))$ *or* $(B4)$*, e.g.* $B_B(x,y) = I_M(S_L(x,y),T_P(x,y))$.

Proof. $(B3)$

Let B be a TS fuzzy bi-implication based on T_L, I_M and S_P. For $x = .7$ we have that

$$\begin{array}{lll} S_P(.7,.7) & = .7+.7-(.7\times .7) & = .91, \\ T_L(.7,.7) & = max(.7+.7-1,0) & = .4, \\ I_M(.91,.4) & = .4, & \\ B(.7,.7) & = I_M(S_P(.7,.7),T_L(.7,.7)) & \\ & = I_M(.91,.4) & \\ & = .4 & \\ & \neq 1. & \end{array}$$

Therefore B does not satisfy $(B3)$.

$(B4)$

Let B be a TS fuzzy bi-implication based on T_P, I_M and S_L. For $w = .2, x = .3, y = .5$ and $z = .8$ we have that

$$
\begin{array}{lll}
S_L(.2,.8) & = min(.2+.8,1) & = 1, \\
T_P(.2,.8) & = .2 \times .8 & = .16, \\
I_M(1,.16) & = .16, & \\
S_L(.3,.5) & = min(.3+.5,1) & = .8, \\
T_P(.3,.5) & = .3 \times .5 & = .15, \\
I_M(.8,.15) & = .15, & \\
B(.2,.8) & = I_M(S_L(.2,.8), T_P(.2,.8)) & \\
 & = I_M(1,.16) & \\
 & = .16 & \\
 & \not\leq .15 & \\
 & = I_M(.8,.15) & \\
 & = I_M(S_L(.3,.5), T_P(.3,.5)) & \\
 & = B(.3,.5). &
\end{array}
$$

Therefore B does not satisfy $(B4)$. □

Trivially we can verify that $(B2')$ is identical to $(B1)$, so by the propositions 3.2.1 and 3.2.3 we know that TS fuzzy bi-implications satisfy $(B1)$ and $(B2)$, but by Proposition 3.2.4 not necessarily satisfies $(B3)$ or $(B4)$. Notice also that since each KMP fuzzy bi-implication or LT fuzzy bi-implication satisfies $(B5')$ (see Proposition 3.1.2) and max and min are a t-conorm and a t-norm, respectively, then trivially each KMP fuzzy bi-implication or LT fuzzy bi-implication is a TS fuzzy bi-implication. Therefore, the class of operators that are LT fuzzy bi-implications and KMP fuzzy bi-implications is a proper subclass of the fuzzy operators that are TS fuzzy bi-implications.

Now we are going to check whether the class of operators that are BC fuzzy bi-implications and the class of operators that are TS fuzzy bi-implications are incomparable. BC fuzzy bi-implications satisfy $(B5')$ (see Proposition 3.1.7) and max and min are a particular case of a t-conorm S and a t-norm T, respectively. If we restrict the class of operators that are BC fuzzy bi-implications to only those that have the R-implication based on a left-continuous t-norm, then in this case we get a subclass of the operators that are TS fuzzy bi-implications. On the other hand, if we restrict the class of operators that are TS fuzzy bi-implications to those that have the t-conorm S and the t-norm T defined as max and min, respectively, then since the R-implication of BC fuzzy bi-implications can be any t-norm (not necessary left-continuous t-norms) we get a subclass of the operators that are BC fuzzy bi-implications.

In the Figure 3.3 we can see the relations between the classes of fuzzy bi-implications studied until now.

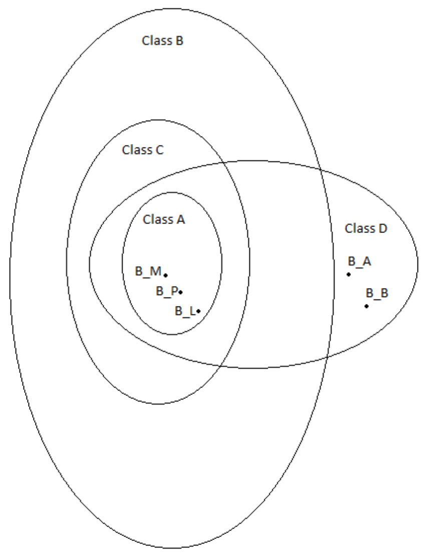

Figure 3.3: Relation between the class of operators that are KMP fuzzy bi-implications (Class A) or LT fuzzy bi-implications (Class A), the class of operators that are Fodor-Roubens fuzzy bi-implications (Class B) or Bedregal-Cruz fuzzy bi-implications (Class B), the class of operators that are BC fuzzy bi-implications (Class C) or AT fuzzy bi-implications (Class C) and the class of operators that are TS fuzzy bi-implications (Class D). Includes examples.

Chapter 4

Conclusions and future works

4.1 Conclusions

We presented four classes of fuzzy bi-implications, say, the class A that corresponds to KMP fuzzy bi-implications which is coextensive with LT fuzzy bi-implications, the class C that corresponds to BC fuzzy bi-implications which is coextensive with AT fuzzy bi-implications, the class B that corresponds to the Fodor-Roubens fuzzy bi-implications which is coextensive with Bedregal-Cruz fuzzy bi-implications and the class D corresponding to TS fuzzy bi-implications.

We have seen that the class A is a proper subclass of class C, that class C is a subclass of class B and that class D intersects with class B by at least class A.

4.2 Future works

The following is a collection of ideas for possible research that may be conducted by us to deepen the study contained in this dissertation.

1. By means of TS fuzzy bi-implications and some other fuzzy operators, study the defining standard of other fuzzy operators, as t-norms, t-conorms, implications and negations, e.g., try to obtain a t-norm by the use of a TS fuzzy bi-implication and an R-implication. It may be defined a restricted TS fuzzy bi-implication in the case that some of those fuzzy operators can not be obtained.

2. If all the fuzzy operators mentioned on the point (1) could be obtained, then possibly it can be build a new sound and complete fuzzy logic with

respect to those obtained operators.

3. Study several properties in order to verify if TS fuzzy bi-implications satisfy them.

4. After point (3) is deeply studied one may check the relation between the properties that are satisfied by the TS fuzzy bi-implications, i.e., it can be verified which properties are special cases or consequence of others.

5. Axiomatize the TS fuzzy bi-implications in order to obtain the definition of a primitive connective. The more exhaustively the points (3) and (4) are studied, the more it can facilitate the quest for this item.

6. If point (5) is concluded, then a research may be made by us with the goal of verifying if with the axioms of the TS fuzzy bi-implications the axioms of the other fuzzy connectives, as t-norms, t-conorms, implications and negations may be derivated. In the case that this can not be successfully done, it should be determined what additional axioms are needed for deriving the axioms of the other fuzzy operators.

7. In [25] it was studied the concept of operations fitting the KMP fuzzy bi-implications. It may be made a research in order to study that concept with respect the TS fuzzy bi-implications.

8. In Proposition 3.1.10 a way to obtain KMP fuzzy bi-implications by means of an additive generator was shown. It can be studied if there is a manner to obtain TS fuzzy bi-implications through the use of additive generators. Since with additive generators we can obtain t-norms, fuzzy implications and t-conorms, we have the intuition that by composition of those additive generators we may build TS fuzzy bi-implications.

9. Analyze if the TS fuzzy bi-implications are continuous for any t-norm T, any left-continuous t-norm T' and any t-conorm S. In case this is not satisfied, it could be studied for which T, T' and S the TS fuzzy bi-implications are continuous.

10. Study if there exists an isomorphism between all the operators in the class of operators that are TS fuzzy bi-implications, or if there is an isomorphism in a sub-class of that class.

11. Instead of having 1 as the only designated value, have a subset of $[0, 1]$ as the designated values. This extension can be studied in all 7 definitions of fuzzy bi-implications presented in this dissertation. We have the intuition that in this case the TS fuzzy bi-implications may satisfy a more weaker notion of identity principle.

12. Check if the fuzzy bi-implications obtained from the most common t-norms, $B_M(x, y), B_P(x, y), B_L(x, y)$ and $B_D(x, y)$, are equivalent to $I_M(S_M((x, y), T_M(x, y)), I_P(S_P((x, y), T_P(x, y)), I_L(S_L((x, y), T_L(x, y))$ and $I_D(S_D((x, y), T_D(x, y))$, respectively.

13. Study subclasses of the TS fuzzy bi-implications, e.g., by restricting T and T' to be the same left-continuous t-norm and/or letting the t-conorm S to be the dual of T'. If the particular case where T and T' are the same t-norm and S is the dual t-conorm of T is studied, we may gain a deeper knowledge of the common properties of Luckasiewicz, Godel and Product constuctions of the TS fuzzy bi-implication.

14. Since the class of operators that are Fodor-Roubens fuzzy bi-implications and the class of operators that are TS fuzzy bi-implications, intersect there self, but neither of both classes contains the other, a generalization that involves both classes may be defined. For instance we refer to that future generalization as *B1-B2 fuzzy bi-implication*, because ($B1$) and ($B2$) are the axioms of the Fodor-Roubens fuzzy bi-implications that the TS fuzzy bi-implications satisfy. The intuition of this idea may be seen in the Figure 4.1.

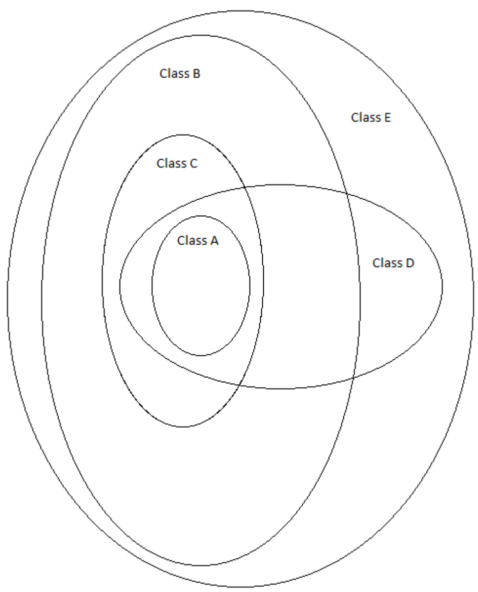

Figure 4.1: Relation between the class of operators that are KMP fuzzy bi-implications (Class A) or LT fuzzy bi-implications (Class A), the class of operators that are Fodor-Roubens fuzzy bi-implications (Class B) or Bedregal-Cruz fuzzy bi-implications (Class B), the class of operators that are BC fuzzy bi- implications (Class C) or AT fuzzy bi-implications (Class C), the class of operators that are TS fuzzy bi-implications (Class D) and the class of operators that are B1-B2 fuzzy bi-implications (Class E).

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