In this paper one examines analytical solutions for flat and non-flat universes composed by four components namely hot matter (ultra-relativistic), warm matter (WM) (relativistic), cold matter (CM) (non-relativistic) and cosmological constant. The WM is treated as a reduced relativistic gas (RRG) and the other three components are treated in the usual way. The solutions achieved contains one, two or three components of which one component is of WM type. A solution involving all the four components was not found.

Keywords: Reduced relativistic gas; cosmological solutions.

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1. Introduction

At recent years cosmology has lived a golden age. Many observational techniques are being developed and they are producing a lot of data about our universe. We can cite for example techniques involving observation of supernovas\textsuperscript{1-5}, detection of cosmic microwave background anisotropies\textsuperscript{6} and surveys of galaxies.\textsuperscript{7-9} These new data have generated a picture where the universe is constituted by five components: radiation, neutrinos, baryonic matter, dark matter and dark energy. In the simplest models the components evolve independently which means each one has its own equation of state (EoS).

In the standard approach the evolution of the universe is divided in eras, each one dominated by one component. Separate solutions are used for each era and intermediate periods are connected matching the initial conditions. This approach, although approximated, describes the history of the universe in a simple way. However, the precision of new data, make it desirable to have solutions as complete as possible.
Analytic solutions involving cold matter (CM), radiation and cosmological constant were studied in Refs. 10–13. Nevertheless, none of these papers consider relativistic components in their analysis. In principle, if we look for solutions that take into account relativistic particles it would be necessary to deal with equations of state containing modified Bessel functions. This would make it very difficult to obtain analytical solutions. Fortunately, in 2005 a simpler formulation was proposed to describe relativistic particles. This formulation, known as reduced relativistic gas (RRG), is able to represent a gas of relativistic particles with good accuracy. Besides, the RRG model is simple enough which allows us to search for analytic solutions to Friedmann equations.

The first analytic solutions containing the RRG were found in Ref. 14. Here one continues this work extending the analysis for cases involving RRG with other components. We discussed solutions for a universe composed by a RRG component plus non-relativistic matter, radiation and/or cosmological constant. These kind of solutions are important whenever we want to describe a universe which has a component with relativistic behavior. Good examples are models which involves warm dark matter.

The paper is organized as follows: In Sec. 2 a summary about RRG, focusing its connection with standard cosmology is given. The content of this section is a resume of Secs. 2 and 3 of Ref. 14. Section 3 presents analytic solutions for a universe containing one, two and three components where one of these components is modeled by RRG. In general, the flat and non-flat cases were treated separately. The final comments and further perspectives are given in Sec. 4.

2. RRG in the Standard Cosmology

The RRG is a simple model for a relativistic ideal gas of massive particles and was first introduced in Ref. 14. The idea behind this model is to use the kinetic theory of gases attached with relativistic concepts. Elementary considerations show that the average of transferred relativistic moment by particles of mass $m$ to a wall of a vessel produce a pressure $p$ given by

$$p = \frac{1}{V} \frac{mv^2}{\sqrt{1 - v^2/c^2}},$$

where $V$ is the volume of vessel and $v$ is the relativistic velocity.

Supposing that all particles have the same relativistic kinetic energy $\varepsilon$ one can rewrite the equation above as

$$p = \frac{\rho}{3} \left[ 1 - \left( \frac{nm^2c^2}{\rho} \right)^2 \right] = \frac{\rho}{3} \left[ 1 - \left( \frac{\rho_d}{\rho} \right)^2 \right],$$

where $\rho = n\varepsilon$ is the energy density and $n$ is the numerical density of particles. Note that $\rho_d$ is the energy density of non-relativistic particles and thus it is proportional to $V$, i.e. $\rho_d = \rho_1 V$. It is easy to see that if $\rho \simeq \rho_d$ (non-relativistic particles)
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the EoS (1) reduces to $p \simeq 0$, and if $\rho \gg \rho_d$ (ultra-relativistic particles) the EoS becomes $p = \rho/3$.

It is instructive to compare (1) with the correct EoS derived by the statistical mechanics of ensembles. Computing the partition function of classical relativistic ideal gas we can determine $p$ and $\rho$ as functions of $n$ and $kT$:

\[ p = nkT, \]
\[ \rho = nmc^2K_3(mc^2/kT)/K_2(mc^2/kT) - nkT, \]

where $K_\nu$ is a modified Bessel function of index $\nu$. Combining these two equations we obtain

\[ \rho = \rho_d[K_3(\rho_d/p)K_2(\rho_d/p)] - p. \] (2)

At first sight, (1) and (2) are completely different. However, a numerical comparison between these two equations was performed in Ref. 14 and it was shown that they are quite similar. Indeed, the difference between (1) and (2) is at most 2.5% and becomes negligible at ultra-relativistic and non-relativistic regimes. Thus, (1) is a good approximation for EoS of classical relativistic ideal gas with the great benefit of being much simpler than (2).

In order to use (1) as an EoS for a relativistic component of cosmic fluid it is necessary to determine how the energy density depends on scale factor. This is performed by writing the covariant conservation law in terms of volume

\[ \frac{d\rho}{dV} = -\left(\frac{\rho + p}{V}\right), \quad \text{where } V \sim a^3 \] (3)

and replacing (1) in (3). Solving the differential equation we obtain

\[ \rho(V) = \sqrt{\rho_1^2 \left(\frac{V_0}{V}\right)^2 + \rho_2^2 \left(\frac{V_0}{V}\right)^{\frac{4}{3}}} \] (4)

or in terms of scale factor

\[ \rho_{RRG}(a) = \sqrt{\rho_1^2 \left(\frac{a_0}{a}\right)^6 + \rho_2^2 \left(\frac{a_0}{a}\right)^8}. \] (5)

The initial condition used was $\rho(V_0) = \rho_{RRG}(a_0) = \sqrt{\rho_1^2 + \rho_2^2}$.

Analyzing the above equation we can associate the constants $\rho_1$ and $\rho_2$ as the energy densities of dust and radiation, respectively. Indeed, if we take $\rho_2 = 0$, (5) scaling as a dust-like component which means $\rho \sim a^{-3}$. And if we take $\rho_1 = 0$, (5) scales as a radiation-like component which means $\rho \sim a^{-4}$. Although, (5) reproduces these two behaviors it is qualitatively and quantitatively different from a cosmic fluid composed by dust and radiation. In the first case, we have a single
Fig. 1. The energy density of $\rho_{RRG}$ (full line) and $\rho_{m+\gamma}$ (dashed line) in terms of $a/a_0$. In both case it was chosen $\rho_1 = \rho_2$ with the initial condition $\rho(a_0) = 1$. Observe that the transition among the radiation and dust behavior is smoother for $\rho_{RRG}$.

relativistic component represented by (5), and in the second case, we have two distinct components whose energy density is given by

$$\rho_{m+\gamma}(a) = \rho_1 \left(\frac{a_0}{a}\right)^3 + \rho_2 \left(\frac{a_0}{a}\right)^4.$$ (6)

A numerical confrontation between (5) and (6) is shown in Fig. 1.

3. Analytic Solutions

Suppose that the cosmic fluid are composed by four independent components namely radiation ($\gamma$), CM, cosmological constant ($\Lambda$) and WM. Thus, the first Friedmann equation in units of $c = 1$ results in

$$ \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \left[ \rho_\gamma (\frac{a_0}{a})^4 + \rho_{CM0} (\frac{a_0}{a})^3 + \frac{\Lambda}{8\pi G} + \sqrt{\rho_1^2 \left(\frac{a_0}{a}\right)^6 + \rho_2^2 \left(\frac{a_0}{a}\right)^8} \right] - \frac{\kappa}{a^2}, $$ (7)

where $\kappa$ is the spatial curvature and WM is modeled by RRG. Usually, the component of WM represents a warm dark matter (as proposed in Refs. 15 and 16). Nevertheless, another physical possibility to use WM component to describe neutrinos.

The main goal of this work is to study the analytical solutions linked with (7). To perform this it is convenient to define the following quantities:

$$\Omega_{\gamma0} = \frac{\rho_{\gamma0}}{\rho_c}, \quad \Omega_{CM0} = \frac{\rho_{CM0}}{\rho_c}, \quad \Omega_{\Lambda} = \frac{\Lambda}{3H_0^2},$$

$$\Omega_1 = \frac{\rho_1}{\rho_c}, \quad b = \frac{\rho_2}{\rho_1} \quad \text{and} \quad \Omega_{\kappa0} = \frac{\kappa}{H_0^2} \quad \text{with} \quad \rho_c = \frac{3H_0^2}{8\pi G}. $$ (8)
Since (7) is a separable differential equation of first-order it can be written as an integral in the scale factor. Thus, using (8) we obtain,

\[ t(a) = \pm \frac{1}{H_0} \int \frac{a \, da}{\sqrt{\Omega_{\gamma 0} + \Omega_{CM0} a + \Omega_\Lambda a^4 + \Omega_1 [a^2 + b^2]^{\frac{1}{2}} - \Omega_\kappa 0 a^2}}, \quad (9) \]

where \( a_0 \equiv 1 \). The sign will be chosen as to get an expanding universe always.

The approach adopted here is to search solutions involving one, two and three components with and without curvature.\(^a\) The integral in (9) can be written as a sum of integrals whose structures are of type

\[ \int F(a, \sqrt{P(a)}) \, , \]

where \( P(a) \) is a polynomial. Associated with this integral we have three possibilities:\(^17\)

(i) If \( P(a) \) is at most second-degree polynomial and \( F \) has a simple structure then the solution of (9) can be expressed in terms of algebraic functions. In this case, an explicit expression sometimes can be found.

(ii) If \( P(a) \) is third- or fourth-degree polynomial and/or \( F \) does not have a simple structure then the solution of (9) can be written in terms of elliptical integrals (see Appendix A). In this case, the solution is only implicit.

(iii) If \( P(a) \) is more than fourth-degree polynomial then it is not possible to obtain a solution for (9) — e.g., the integral with all four components.

Before moving to the specific cases it is noteworthy that the solutions without WM is not treated in this paper. This kind of solution was extensively studied in Refs. 10–13.

3.1. Solutions with one component

The first and most simple case is when only one component (WM) is present. Thus, (9) is reduced to

\[ t(a) = \frac{1}{H_0} \int \frac{a \, da}{\sqrt{\Omega_1 [a^2 + b^2]^{\frac{1}{2}} - \Omega_\kappa 0 a^2}} \, . \]

The solution for flat curvature (\( \kappa = 0 \)) is given by

\[ t_f(a) = \frac{2}{3H_0\sqrt{\Omega_1}} [a^2 + b^2]^{\frac{1}{2}} + \tilde{t} \Rightarrow a_f(t) = \sqrt{\left[ \frac{3}{2} \sqrt{\Omega_1 H_0 (t - \tilde{t})} \right]^{\frac{1}{2}} - b^2}, \quad (10) \]

where \( \tilde{t} \) is the integration constant and \( \Omega_1 (1 + b^2)^{\frac{1}{2}} = 1 \). This result was first derived in Ref. 14 and it is presented here only for completeness.

The structure of solution for non-flat cases (\( \kappa = \pm 1 \)) is as complicated as the solutions involving \( \gamma \) and WM. Therefore, it will be presented in Sec. 3.2.

\(^a\)An analytic solution involving all the four components was not achieved.
3.2. Solutions with two components

Solutions with two components are of type (WM, $\gamma$), (WM, CM) and (WM, $\Lambda$) with and without curvature. Because of their complicated structure, the cases (WM, $\Lambda$) with and without curvature will be treated in Sec. 3.3. Let us perform the analysis of the two other cases.

3.2.1. Universe with WM and $\gamma$

For an universe constituted by WM and $\gamma$, Eq. (9) is reduced to

$$t(a) = \pm \frac{1}{H_0} \int \frac{a \, da}{\sqrt{\Omega_\gamma + \Omega_1 [a^2 + b^2]^{\frac{1}{2}} - \Omega_{\kappa 0} a^2}}. \tag{11}$$

The solution for flat curvature is given by

$$t_f(a) = \frac{1}{H_0} \left[ \frac{(2\sqrt{a^2 + b^2} - 4\Omega_{\gamma 0}) \sqrt{\Omega_{\gamma 0} + \Omega_1 \sqrt{a^2 + b^2}}}{3\Omega_1} \right] + \bar{t}, \tag{12}$$

where $\bar{t}$ is the integration constant and $\Omega_{\gamma 0} + \Omega_1 (1 + b^2)^{\frac{1}{2}} = 1$. Note that if we take $\Omega_{\gamma 0} = 0$ the result (10) is recovered. This result was first derived in Ref. 14 and again it is presented here only for completeness. Unfortunately, Eq (11) cannot be inverted and thus it is not possible to derive an explicit solution.

For positive curvature the solution is

$$t_p(a) = \frac{-1}{|\Omega_{\kappa 0}|^{\frac{3}{2}} H_0} \left[ y_p + \frac{\Omega_1}{2} \arctan \left( \frac{\Omega_1 - 2|\Omega_{\kappa 0}| \sqrt{a^2 + b^2}}{2y_p} \right) \right] + \bar{t}, \tag{12}$$

where

$$y_p \equiv \sqrt{-\Omega_{\kappa 0}^2 a^2 + \Omega_1 |\Omega_{\kappa 0}| \sqrt{a^2 + b^2} + \Omega_{\gamma 0} |\Omega_{\kappa 0}|}$$

and for negative curvature the solution is

$$t_n(a) = \frac{1}{|\Omega_{\kappa 0}|^{\frac{3}{2}} H_0} \left[ y_n - \frac{\Omega_1}{2} \ln \left( \Omega_1 + 2|\Omega_{\kappa 0}| \sqrt{a^2 + b^2} + 2y_n \right) \right] + \bar{t}, \tag{13}$$

where

$$y_n \equiv \sqrt{\Omega_{\kappa 0}^2 a^2 + \Omega_1 |\Omega_{\kappa 0}| \sqrt{a^2 + b^2} + \Omega_{\gamma 0} |\Omega_{\kappa 0}|}.$$ 

In both cases $\bar{t}$ is an integration constant and $\Omega_{\gamma 0} + \Omega_1 (1 + b^2)^{\frac{1}{2}} - \Omega_{\kappa 0} = 1$. As it should be, Eqs. (12) and (13) are reduced to (11) in the limit $|\Omega_{\kappa 0}| \to 0$. This statement can be verified expanding the functions $\arctan[\cdots]$ and $\ln[\cdots]$ in powers of $\sqrt{|\Omega_{\kappa 0}|}$ until third-order.
3.2.2. *Universe with WM and CM*

For a flat universe composed by WM and CM, Eq. (9) becomes

\[ t_f(a) = \pm \frac{1}{H_0} \int \frac{a \ da}{\sqrt{\Omega_{CM0}a + \Omega_1[a^2 + b^2]^\frac{1}{2}}}. \tag{14} \]

Using the definitions

\[ y^2 = \frac{rb}{a + \sqrt{a^2 + b^2}}, \quad \phi = \arcsin(y) \quad \text{and} \quad r = \sqrt{\frac{\Omega_{CM0} - \Omega_1}{\Omega_1 + \Omega_{CM0}}}, \]

the solution for (14) is written as

\[ t_f(a) = \frac{\sqrt{2b^3}}{6H_0} \left[ \frac{(\Omega_1 + \Omega_{CM0})^3}{(\Omega_{CM0} - \Omega_1)^2} \right]^\frac{1}{2} \left[ (1 - r^4)F(\phi, -1) + \frac{(r^4 - y^4)\sqrt{1 - y^4}}{y^4} \right] + \bar{t}, \tag{15} \]

where \( \bar{t} \) is the integration constant and \( F(\phi, m) \) is the elliptical integral of the first kind (see Appendix A). Besides, we have the following constraint \( \Omega_{CM0} + \Omega_1(1 + b^2)^\frac{1}{2} = 1 \). This solution is valid for all physical values of \( \Omega_{CM0} \) and \( \Omega_1 \) avoiding \( \Omega_{CM0} = \Omega_1 \). If we take \( \Omega_{CM0} = 0 \) then \( r = i \). Using this result and performing some simple algebra we recover the solution (10) for a universe composed only by RRG.

The structure of solution for non-flat cases is as complicated as the solutions involving \( \gamma \), WM and CM, and thus, they will be discussed in Sec. 3.3.

3.3. *Solutions with three components*

Solutions with three components are of type (WM, CM, \( \gamma \)), (WM, \( \gamma \), \( \Lambda \)) and (WM, CM, \( \Lambda \)). It is not possible to achieve an analytic solution for the case (WM, CM, \( \Lambda \)) because the term inside the square root in (9) is a polynomial of degree greater than four. Let us perform the analysis of the two other cases.

3.3.1. *Universe with WM, CM and \( \gamma \)*

Suppose an universe composed by WM, CM and \( \gamma \). In this case, Eq. (9) becomes

\[ t(a) = \pm \frac{1}{H_0} \int \frac{a \ da}{\sqrt{\Omega_{\gamma 0} + \Omega_{CM0}a + \Omega_1[a^2 + b^2]^\frac{1}{2} - \Omega_{\kappa 0}a^2}}. \tag{16} \]

This integration can be solved for flat and non-flat cases, but for \( \kappa = \pm 1 \) the expressions are rather complicated. For negative and positive curvatures the solutions involve four and seven elliptic integrals, respectively. Besides, each solution has constraint related to the cosmological parameters. Thus, they will not be presented in this paper.
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On the other hand, the flat case is relatively simpler than non-flat cases. Indeed, if we make the following definitions

\[ r \equiv -\Omega_0 + \sqrt{\Omega_0^2 + b^2(\Omega_{CM0} - \Omega_1^2)} \]

\[ s \equiv -\Omega_0 - \sqrt{\Omega_0^2 + b^2(\Omega_{CM0} - \Omega_1^2)} \]

\[ y^2 \equiv \frac{rb}{a + \sqrt{a^2 + b^2}} \]

\[ \phi \equiv \arcsin(y) \quad \text{and} \quad m \equiv \frac{s}{r} \]

the \( \kappa = 0 \) solution of (16) is given by

\[ t_f(a) = \pm \frac{1}{6H_0} \int \frac{a da}{\sqrt{\Omega_0 + \Omega_1 a^4 + \Omega_1 [a^2 + b^2]^2 - \Omega_0 a^2}} \]

(17)

where \( \bar{t} \) is the integration constant and \( F(\phi, m) \) and \( E(\phi, m) \) are the elliptical integral of the first and second kind, respectively (see Appendix A). This solution is valid for almost all physical values of \( \Omega_0, \Omega_{CM0}, \Omega_1 \) and \( b \) satisfying the constraint \( \Omega_0 + \Omega_{CM0} + \Omega_1 (1 + b^2) = 1 \). Nevertheless, the choice of sign depends on relation between \( \Omega_{CM0} \) and \( \Omega_1 \). If \( \Omega_{CM0} > \Omega_1 \) (\( \Omega_{CM0} < \Omega_1 \)) the sign plus (minus) must be used.

As it should be, the solution (17) contains the previous case involving only WM and CM. Taking \( \Omega_0 = 0 \) we get \( m = -1 \) and after some straightforward algebra we recover the solution (15).

3.3.2. Universe with WM, \( \gamma \) and \( \Lambda \)

For an universe constituted by WM, \( \gamma \) and \( \Lambda \), Eq. (9) is given by

\[ t(a) = \pm \frac{1}{6H_0} \int \frac{a da}{\sqrt{\Omega_0 + \Omega_1 a^4 + \Omega_1 [a^2 + b^2]^2 - \Omega_0 a^2}} \]

(18)

It is convenient to change the variable of integration \( a \) using the relation \( a^2 = x^2 - b^2 \). Thus,

\[ t(a) = \pm \frac{1}{H_0 \sqrt{\Omega}} \int \frac{x dx}{\sqrt{L x^2 + M x + P}} \]

where

\[ L = 2b^2 + \frac{\Omega_0}{\Omega_{CM0}} \]

\[ M = \frac{\Omega_1}{\Omega_{CM0}} \] and \( P = \frac{\Omega_0}{\Omega_{CM0}} + b^4 + \frac{\Omega_{CM0} b^2}{\Omega_{CM0}} \).

\(^b\)The values \( b = 0 \) or \( \Omega_{CM0} = \Omega_1 \) are not allowed.
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The new integral is not too simple but it can be solved through the following steps:

1. Rewrite the fourth-degree polynomial in terms of the roots \( r_i \):
   \[
   x^4 - Lx^2 + Mx + P = (x - r_1)(x - r_2)(x - r_3)(x - r_4).
   \]

2. Introduce some constants, in terms of which the roots will be expressed later on:
   \[
   S = \sqrt[3]{2}(L^2 + 12P),
   \]
   \[
   W = -2L^3 + 27M^2 + 72LP,
   \]
   \[
   U = -4(L^2 + 12P)^3 + (-2L^3 + 27M^2 + 72LP)^2,
   \]
   \[
   V = \left[ \frac{S}{3(W + \sqrt{U})^{\frac{2}{3}}} + \frac{(W + \sqrt{U})^{\frac{2}{3}}}{3\sqrt[3]{2}} \right] + \frac{2L}{3}.
   \]

3. The roots will then be:
   \[
   r_1 = -\frac{1}{2} \left[ \sqrt{V} + \sqrt{\left(2L - V + \frac{2M}{\sqrt{V}}\right)} \right];
   \]
   \[
   r_2 = -\frac{1}{2} \left[ \sqrt{V} - \sqrt{\left(2L - V + \frac{2M}{\sqrt{V}}\right)} \right],
   \]
   \[
   r_3 = \frac{1}{2} \left[ \sqrt{V} + \sqrt{\left(2L - V - \frac{2M}{\sqrt{V}}\right)} \right];
   \]
   \[
   r_4 = \frac{1}{2} \left[ \sqrt{V} - \sqrt{\left(2L - V - \frac{2M}{\sqrt{V}}\right)} \right].
   \]

4. Sets the parameters \( m \), \( n \) and the amplitude \( \phi \) as:
   \[
   n = \frac{r_2 - r_4}{r_1 - r_4}, \quad m = \frac{r_1 - r_3}{r_2 - r_3},
   \]
   \[
   \phi = \arcsin \left[ \sqrt{\left(\frac{x - r_2}{n(x - r_1)}\right)} \right].
   \]

5. The implicit solution will then be:
   \[
   t(a) = \frac{2(x - a)^2}{H_0\sqrt{\Omega_\Lambda(x - r_1)^2(r_4 - r_1)}} \times \left[ \frac{r_1\sqrt{(x - r_2)}}{\sqrt{(r_3 - r_2)^2(x - r_1)}} F[\phi, m] \right.
   \]
   \[
   \left. + \frac{r_4 - r_2}{\sqrt{(r_4 - r_2)^2}} \sqrt{\frac{(r_1 - r_2)^2}{(r_3 - r_2)^2(x - r_1)^2}} \Pi[n, \phi, m] \right] + \tilde{t},
   \]

These steps were first developed in Ref. 10.
where \( x = \sqrt{a^2 + b^2} \), \( t \) is the integration constant and \( F(\phi, m) \) and \( \Pi(n, \phi, m) \) are the elliptical integral of the first and third kind, respectively (see Appendix A). Besides, we have the following constraint \( \Omega_{\gamma 0} + \Omega_{\Lambda} + \Omega_1 (1 + b^2)^{\frac{1}{2}} - \Omega_{\kappa 0} = 1 \).

At this point, some features about this solution must be clarified. At first sight it seems that (20) could be simplified. However, as \((r_i - r_j)\) and \((x - r_i)\) could be complex numbers, any extra desirable simplification must be done with caution and only when the values of \( \Omega_{\gamma 0}, \Omega_{\Lambda}, \Omega_1 \) and \( b \) are specified. Other important point is that (20) is not valid for all physical values of \( \Omega_{\gamma 0}, \Omega_{\Lambda}, \Omega_1 \) and \( b \). It happens because there is an arbitrariness in choice of which root will be \( r_1, r_2, r_3 \) or \( r_4 \). Nevertheless, the choice it was made in (19) include wide ranges for the parameters comprising inclusively the ΛCDM case. For practical purposes, a set of conditions that ensure a physical solution are

\[
0 < \Omega_{\Lambda} \leq 2, \quad 0 \leq \Omega_1 \leq 2, \quad 0 \leq b \leq 2, \quad \Omega_{\gamma 0} \ll \Omega_{\Lambda} \quad \text{and} \quad \Omega_{\gamma 0} \ll \Omega_1 .
\]

4. Final Comments

In this paper, we derived analytical solutions for a universe composed by one, two and three components where one of them represents WM. The first solution obtained is one that involving only WM. It is very simple but it serves such a guide for the complex ones. The next step was to derive solutions containing WM plus radiation or CM. As expected, these kind of solutions are more complicated than the previous one and only implicit solutions were found. The most complicated solution which were achieved are ones involving WM, radiation and cosmological constant or WM, radiation and CM. These type of solutions, with three components, always involve elliptic integrals. Unfortunately, an analytic solution containing all the four components was not obtained.

WM could mimic dark matter, neutrinos and even baryonic matter. Thus, these solutions can be applied in different context. For example, we can use them to analyze the effects of warm dark matter in structure formation.\(^{15}\) Other possibility is using them to study massive neutrinos in cosmology. It is noteworthy that although the results obtained concerns only the background, they are also important in perturbative cosmology. Indeed, the perturbative analysis becomes simpler when the analytical solution for the background is known.

Finally, it is important to emphasize that WM is represented by the RRG model which is an approximation for a classical relativistic gas. In the context of thermodynamics, this approximation differs from the real situation at most 2.5\%.\(^{14}\) Nevertheless, no comparison was done at cosmological context. We expect to explore this issue in the near future.
Appendix A

Definition of elliptic integrals\textsuperscript{17}:

- First kind:
  \[ F(\phi, m) = \int_{0}^{\sin \phi} \frac{dx}{\sqrt{(1 - x^2)(1 - mx^2)}}. \]

- Second kind:
  \[ E(\phi, m) = \int_{0}^{\sin \phi} \frac{\sqrt{1 - mx^2}}{\sqrt{1 - x^2}} dx. \]

- Third kind:
  \[ \Pi(n, \phi, m) = \int_{0}^{\sin \phi} \frac{dx}{(1 - nx^2)\sqrt{(1 - x^2)(1 - mx^2)}}. \]

Acknowledgments

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