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On Fuzzy Implication Classes – Towards Extensions of Fuzzy Rule-Based Systems

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Abstract

There are more than one acceptable fuzzy implication definitions in the current literature dealing with this subject. From a theoretical point of view, this fact demonstrates a lack of consensus regarding logical implication meanings in Boolean and fuzzy contexts. From a practical point of view, this raises questions about the "implication operators" that software engineers must consider to implement a Fuzzy Rule Based System (FRBS). A poor choice of these operators generates less appropriate FRBSs with respect to[1] their application domain. In order to have a better understanding of logical connectives, it is necessary to know the properties that they can satisfy. Therefore, aiming to corroborate with fuzzy implication meaning and contribute to implementing more appropriate FRBSs to their domain, several Boolean laws have been generalized and studied as equations or inequations in fuzzy logics. Those generalizations are called Boolean-like laws and a lot of them do not remain valid in any fuzzy semantics. Within this context, this dissertation presents the investigation of sucient and necessary conditions under which three Boolean-like laws — $y \leq I(x, y)$, $I(x, I(y, x)) = 1$ and $I(x, I(y, z)) = I(I(x, y), I(x, z))$ — hold for six known classes of fuzzy implications and for implications generated by automorphisms. Moreover, an extension to FRBSs is proposed.

[1] w.r.t. for short.

Resumo

Atualmente, há diferentes definições de implicações fuzzy aceitas na literatura. Do ponto de vista teórico, esta falta de consenso demonstra que há discordâncias sobre o real significado de "implicação lógica" nos contextos Booleano e fuzzy. Do ponto de vista prático, isso gera dúvidas a respeito de quais "operadores de implicação" os engenheiros de software devem considerar para implementar um Sistema Baseado em Regras Fuzzy (SBRF). Uma escolha ruim destes operadores pode implicar em SBRF's com menor acurácia e menos apropriados aos seus domínios de aplicação. Uma forma de contornar esta situação é conhecer melhor os conectivos lógicos fuzzy. Para isso se faz necessário saber quais propriedades tais conectivos podem satisfazer. Portanto, a fim de corroborar com o significado de implicação fuzzy e corroborar com a implementação de SBRF's mais apropriados, várias leis Booleanas têm sido generalizadas e estudadas como equações ou inequações nas lógicas fuzzy. Tais generalizações são chamadas de leis *Boolean-like* e elas não são comumente válidas em qualquer semântica fuzzy. Neste cenário, esta dissertação apresenta uma investigação sobre as condições suficientes e necessárias nas quais três leis *Boolean-like* — $y \leq I(x, y)$, $I(x, I(y, x)) = 1$ e $I(x, I(y, z)) = I(I(x, y), I(x, z))$ — se mantém válidas no contexto fuzzy, considerando seis classes de implicações fuzzy e implicações geradas por automorfismos. Além disso, ainda no intuito de implementar SBRF's mais apropriados, propomos uma extensão para os mesmos.

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Chapter 1

Introduction

In Aristotle's book entitled "Prior Analytic" [5], there is an investigation about deductive reasoning (or deductive inference), specifically the "syllogism", which determines ways of validating an argument: a certain form in which two or more propositions (called premises) **infer** another (called conclusion).

Aristotle also discussed about three logical laws (usually called classical laws):

- **Law of Identity:** "The fact that a thing is itself."
- **Law of non-contradiction:** "The fact that one cannot say that something is and that it is not in the same respect and at the same time."
- **Law of Excluded Middle:** "A thing can at the same time be or not be something in fact."

Aristotle's investigations and further studies about deductive inference originated the "Formal Logic". Due to the relevance of Aristotle's logical theory, the first system of formal logic considers his deductive inference approach and also his three classical laws. It was the so-called "Classical Logic".

Floy Andrews [4] states that:

> "There is a very long tradition from the fourth century B.C. to the nineteenth century, in which the logic of Aristotle was studied, commented on, criticised at times, though never dethroned, the logic which dominated western thought until the twentieth century".

Moreover, logical systems that regard the classical laws have been defined and extended over the years, e.g.: propositional, first- and high-order logics and modal logic.

However, Egbertus Brouwer in a paper entitled "The untrustworthiness of the principles of logic", questioned the truth of those laws, more specifically the law of excluded middle [37, p. 46]. This motivated the rise of logics which do not accept (at least one of) the classical laws — such logics are so-called "non-classical logics" —, examples being: intuitionistic, paraconsistent, linear, quantum and fuzzy logics.

This dissertation tends toward the paradigm of non-classical logics. It will focus on fuzzy logics that might not accept the law of non-contradiction or the law of excluded middle.

1.1 Fuzzy mathematics

Fuzzy set theory brings an alternative theory to sets in which an element belongs or not to a set represented by a degree within the range of $[0, 1]$. Such metric is called *degree of possibility*, in which "0" means that there is no possibility for the element belonging to a given set and "1" means that the element belongs to a given set with total possibility. In classical sets (called "crisp sets" in Fuzzy Set Theory), characteristic functions play an important role since they are functional representations of sets. Their fuzzy counterparts are membership functions and they have the form $\mu_A : U \to [0, 1]$, i.e. μ_A, like in classical sets, represents the set A and for each $u \in U$, $\mu_A(u)$ means the *possibility degree* of u being in A.

In Fuzzy Set Theory, relations are subsets of Cartesian products, namely they are sets of the form $R = \{((x, y), \mu_R(x, y)) \mid x \in X \text{ and } y \in Y\}$ [82].

It is well-known that operators and relations involving sets are defined in terms of logical connectives; for example: The intersection of two sets is defined in terms of the conjunction $\wedge : \{0, 1\} \times \{0, 1\} \to \{0, 1\}$ — namely $A \cap B = C \Leftrightarrow \forall x.(x \in C \Leftrightarrow x \in A \wedge x \in B)$. Similarly, the inclusion relation is defined in terms of the operator $\Rightarrow: \{0, 1\} \times \{0, 1\} \to \{0, 1\}$ — namely $A \subseteq B \Leftrightarrow \forall x.(x \in A \Rightarrow x \in B)$.

In the fuzzy context, there are many conjunctions and many implications defined in the literature. In [74], Tsoukalas et al. give some examples of what they call "implication operators" (denoted by ϕ). Some of them can be seen in the following table[1].

Name	$\phi[\mu_A(x), \mu_B(y)] =$
Zadeh Max-Min Operator	$\max(1 - \mu_A(x), \min(\mu_A(x), \mu_B(y)))$
Mamdani Min Operator	$\min(\mu_A(x), \mu_B(y))$
Larsen Product Operator	$\mu_A(x) \cdot \mu_B(y))$
Boolean Operator	$1 - (max(\mu_A(x), \mu_B(y)))$

Table 1.1: Table of "Implication Operators"

It is expected that each implication gives rise to a different type of inclusion relation between fuzzy sets. However, in the previous table, we can see that some of those "implication operators" do not necessarily satisfy the classical implication truth table. For example, the expression "$0 \Rightarrow 1$" when "$\Rightarrow$" is *implemented* as Mamdani Min operator returns "0" instead of "1".

The role of fuzzy relations can be divided in, at least, two categories:

1. In Fuzzy Set Theory, a relation is used as an association between fuzzy sets;

2. In FRBSs (see Figure 1.1) where a relation can be used as:

   a. "production rules"[2]; and

   b. inference rules.

Example 1.1.1 *A frequently used inference rule is the generalized Modus Ponens (GMP) (defined below), where A, A', B and B' are fuzzy sets.*

$$\begin{array}{ll} \textit{Fact:} & x \textit{ is } A' \\ \textit{Production Rule:} & \textit{if } x \textit{ is } A \textit{ then } y \textit{ is } B \\ \hline \textit{Conclusion:} & y \textit{ is } B' \end{array}$$

$B'(y) = A'(x) \circ R(A(x), B(y))$, *where R is a fuzzy relation and "$\circ$" is a composition of relations.*

[1]Note that '$\cdot$' is the algebraic product.

[2]Production rules are divided in two categories: they are called *'if-then rules'* if they are defined by a fuzzy implication operator. When they are defined by a conjunction, they are called *'fuzzy mapping rules'* [80]).

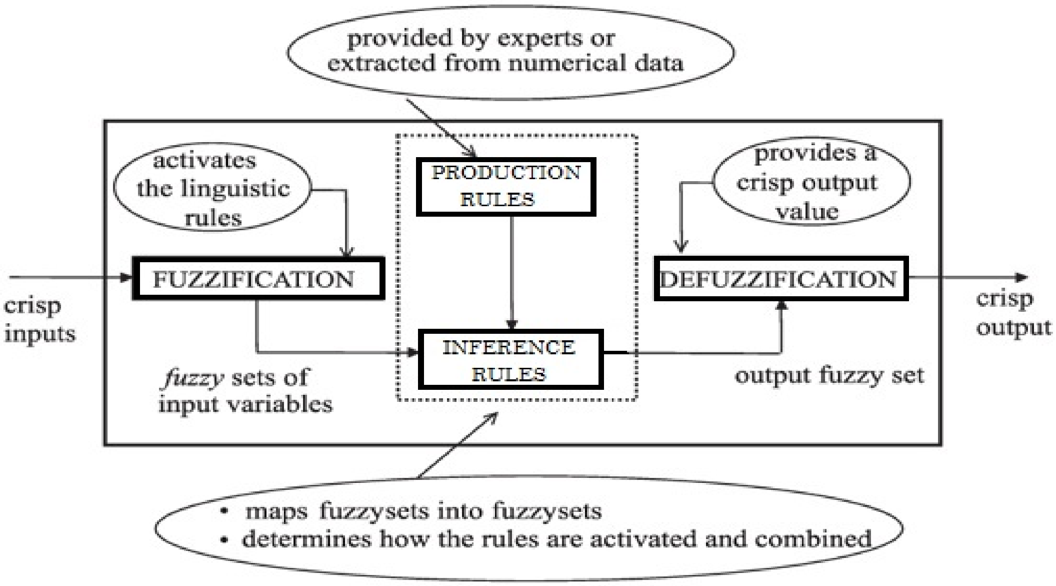

Figure 1.1: Usual FRBSs Architecture adapted from [41]

FRBS belongs to the class of expert systems that deal with approximate reasoning. Figure 1.1 illustrates a usual FRBS with three explicit processes: Fuzzification, Inference Process and Defuzzification. The first process is the Fuzzification where each input ("crisp input") is mapped into a fuzzy degree by a corresponding fuzzy set. So, in the fuzzification, each input becomes a fuzzy degree of a linguistic variable attribute. The inference process starts when those attributes set the relevant production rules which in turn will relate such variables to the variables from output fuzzy sets (as we already saw, this is done by a fuzzy relation). The relevant production rules are combined with some facts by inference rules. The results of inference rules are aggregated[1] to generate the output fuzzy set (also called solution set). Finally, the solution set becomes a crisp output through application of some defuzzification method. The crisp value can be a value from 'x' axis (common in fuzzy control systems) or a categorical value.

As already mentioned, production and inference rules are implemented as fuzzy relations as well as the inference rules results are aggregated by a conjunc-

[1]Inference rules are aggregated by a conjunction or disjunction operator which is called aggregation operator in this case.

tion or disjunction operator. Besides that, negations, disjunctions and conjunctions might happen in production rules. Thus the assembly language of a FRBS is an elected collection of fuzzy connectives.[1]

In this scenario, note that conjunctions, disjunctions and negations used in a FRBS might vary. This comes from the fact that in fuzzy logics, unlike classical logics, instead of only one conjunction, disjunction, negation and implication, there is a family of each one of them; the unique requirement is that they behave like the classical one on the values "0" and "1", i.e. they are extensions of classical connectives. Historically, Zadehnian conjunction, disjunction and negation (defined by 'min', 'max' and $1 - \mu_A(x)$, respectively) have been the most used connectives in FRBSs. Due to wide ranging success of Mamdani Min operator in control systems, it is also frequently applied for production rules in FRBSs.

Since the connectives may vary, each n-tuple of chosen connectives gives rise to a different FRBS implementation. We will call fuzzy semantics, a n-tuple composed by the fuzzy connectives, for example: $(\min, \max, \min)$ is the semantics — composed by connectives that interpret conjunction, compositional relation and "implication", respectively — required for the implementation of the Mamdani method; while $(\min, \max, \bullet)$ is the same for the implementation of the Larsen method. Therefore, different fuzzy semantics can be seen as distinct implementations and as distinct definitions for set relation/operator. In order to illustrate this observation, see the following situations:

Let A, B and C be fuzzy sets. Take the production rule

$$\text{If } A \text{ and not } B \text{ then } C \tag{1.1}$$

and two distinct fuzzy semantics $S1 = (\min, \max, N_z, \min)$ and $S2 = (c_L, d_L, n', i_L)$ in which:

- $N_Z(x) = 1 - x$ is the Zadeh negation;
- $c_L(x, y) = \max(x + y - 1, 0)$ is the Łukasiewicz conjunction;
- $d_L(x, y) = \min(x + y, 1)$ is the Łukasiewicz disjunction;

[1] In order to simplify our insight, we conveniently omitted the defuzzification operation, but it also composes the assembly language.

- $n'(x) = 1 - x^2$ is a fuzzy negation; and
- $i_L(x, y) = \min(1, 1 - x + y)$ is the Łukasiewicz implication.

The production rule (1.1) is interpreted by the following relation expression

$$R(c(A(x), n(B(x))), C(y)) \tag{1.2}$$

where $x \in X$, $y \in Y$, X and Y are sets of crisp inputs and outputs, c is a conjunction connective and n is a negation connective. Then, according to semantics $S1$ and $S2$, (1.2) is interpreted, in FRBSs, as follows:

$[\![\textit{if } A \textit{ and not } B \textit{ then } C]\!]_{S1} = \min(\min(A(x), 1 - B(x)), C(y)),$

$[\![\textit{if } A \textit{ and not } B \textit{ then } C]\!]_{S2} = \min(1, 1 - \min(A(x), 1 - (B(x)^2)) + C(y)).$

Moreover, applying GMP on such production rule, we have

$$C'(y) = A'(x) \circ R(c(A(x), n(B(x))), C(y)).$$

So, adopting "max" as the composition operator "$\circ$", $C'(y)$ is interpreted for $S1$ and $S2$ as:

$[\![C'(y)]\!]_{S1} = \max(A'(x), \min(\min(A(x), 1 - B(x)), C(y))),$

$[\![C'(y)]\!]_{S2} = \max(A'(x), \min(1, 1 - \min(A(x), 1 - B(x)^2) + C(y))).$

Using membership function language, the semantics $S1$ determines $\mu_R(x, y) = \min(x, y)$ and $\mu_c(x, y) = \min(x, y)$, and $S2$ determines $\mu_R(x, y) = \min(1, 1x + y)$ and $\mu_c(x, y) = \min(x + y - 1, 0)$. Then the production rule can be rewritten in each semantics as:

$[\![\textit{if } A \textit{ and not } B \textit{ then } C]\!]_{S1} = \min(\min(\mu_A(x), 1 - \mu_B(x)), \mu_C(y));$ and

$[\![\textit{if } A \textit{ and not } B \textit{ then } C]\!]_{S2} = \min(1{,}1- \min(\mu_A(x)+1-\mu_B(x)^2-1{,}0)+\mu_C(y)).$

And $[\![C'(y)]\!]_S 1$ and $[\![D'(y)]\!]_{S2}$ can be interpreted as:

$[\![\mu_{C'}(y)]\!]_{S1} = \max(\mu_{A'}(x), \min(\min(\mu_A(x), 1 - \mu_B(x)), \mu_C(y)));$ and

$[\![\mu_{C'}(y)]\!]_{S2} = \max(\mu_{A'}(x), \min(1, 1- \min(\mu_A(x)+1-\mu_B(x)^2-1, 0)+\mu_C(y))).$

Since each fuzzy semantics is an implementation, then production rules, inference rule(s) and fuzzy sets can be seen as a specification for such implementation (a fuzzy specification).

This work will analyse some classes of fuzzy implications. The investigation of such connectives induces the understanding of both:

1. The implementation of FRBSs in which production rules are interpreted as implications; and

2. The logical implication meaning, specially w.r.t. its behaviour on the truth domain $[0, 1]$.

1.2 Fuzzy implications and Boolean-like laws

It is known that Classical Logic was the first logic to be rigorously studied, so the implication connective associated to the classical inference notion[1] was also the first implication to be defined and disseminated. For this reason, we tend to believe that it is the correct notion (common sense) of what actually is a logical implication. However, other Boolean and non-classical implications, such as the intuitionistic or the quantum implications, also present acceptable inference models and they must be considered to understand the meaning of a logical implication.

Since there is no consensus of what would be a logical implication in Boolean semantics, how should be the implication generalization to the approximate reasoning (i.e. to the truth domain $[0, 1]$)? In other words, which are the suitable axiomatic properties for a fuzzy implication?

Due to the lack of a single answer to such questions, there are non-equivalent acceptable definitions for fuzzy implications (see [26; 28; 48; 61; 67; 76] as examples) and for fuzzy implication classes (see [9; 10; 14; 18; 43; 47; 49; 77]).

The knowledge about a connective is related to which properties it satisfies. Moreover, since fuzzy implications are extensions of classical ones, it is natural to investigate which Boolean-like laws remain valid for the fuzzy implications.

In spite of the classic-like fuzzy semantics[2] [16], a lot of Boolean-like laws are not usually valid in any fuzzy semantics, i.e., in any standard structure (T, S, N, I), where, T is a t-norm, S a t-conorm, N a fuzzy negation and I a

[1] The implication connective of Classical Logic is called "material implication".

[2] Classic-like fuzzy semantics are fuzzy semantics whose set of theorems are equivalent to the set of tautologies.

fuzzy implication.[1] This fact induced a broad investigation of Boolean-like laws. As examples of related works can be seen as follows:

- In [3], Alsina and Trillas studied a class of functional equations [3, Definition 4.1] with Boolean background [3, Definition 4.2] derived from the laws $(A \cup A) \cap (A \cap A)' = \emptyset$, $A \cup B = (A \cap B) \cup [(A \cup B) \cap (A \cap B)']$ and $(A \cup A \cup B) \cup (A \cap B \cap B) = A \cup B$. In this paper they find out the conditions under which each functional equation holds.

- In [70], the authors investigated when the law $(p \wedge q) \rightarrow r \equiv (p \rightarrow r) \vee (q \rightarrow r)$ is valid in fuzzy logics. And Combs and Andrews, in [20], analysed this same law aiming to eliminate combinatorial rule explosion in fuzzy systems.

- The equivalence exposed in the previous item is one of the four well-known equations, listed in [24], which deals with the distributivity of implication operator over t-norms and t-conorms. So, motivated by [70] and [20], the authors of [11] studied the validity of those other three Boolean-like laws.

- In [67], Shi et al. presented characterizations of some classes of fuzzy implications satisfying the Boolean-like law $I(x, y) = I(x, I(x, y))$ where I is defined as an S-, R- or QL-implication. This Boolean-like law was also investigated for implications derived from uninorms[2] in [50].

- In [32], Balasubramaniam studied the law of importation $(x \wedge y) \rightarrow z = (x \rightarrow (y \rightarrow z))$. He investigated under which conditions the general form of the law of importation holds for (S,N)-, R-, f-, g-implications and some specific families of QL-implications.

In this scenario, this work focuses on the three Boolean-like laws: For all $x, y, z \in [0, 1]$,

$$y \leq I(x, y) \tag{1.3}$$

$$I(x, I(y, x)) = 1 \tag{1.4}$$

[1]T-norms and t-conorms are often used as fuzzy operators and they will be described in more details in the next chapter.

[2]Uninorms were introduced by Yager and Rybalov. See [78] for more details.

$$I(x, I(y, z)) = I(I(x, y), I(x, z)) \tag{1.5}$$

From the formal logic point of view, the functional inequation (1.3) generalizes the relation $x \leq (y \Rightarrow x)$ for $x, y \in \{0, 1\}$. Such relation is a well-known Boolean law that entails other important Boolean rules, e.g. $(x \Rightarrow 1) = 1$ which can be translated as *"If the consequent of an implication is true, then the implication is also true"*[1]

The Boolean-like law (1.4) is the fuzzy rewriting of the classical, intuitionistic and Łukasiewicz axiom "$x \Rightarrow (y \Rightarrow x)$" (*).[2] Such Boolean law is called "Weakening" once that it is said to be the weakening of "$x \Rightarrow x$". In a similar sense, we observe that (*) is also a weakening of Aristotle's Law of Excluded Middle — "$\neg x \vee x$" — since, "$\alpha \to (\beta \to \alpha)$" is equivalent to "$\neg x \vee (\neg y \vee x)$" assuming that "$\to$" is the material implication. Moreover, "$x \Rightarrow x$" is equivalent to "$\neg x \vee x$". This describes the Boolean relation among "$\neg x \vee x$", "$x \Rightarrow (y \Rightarrow x)$" and "$x \Rightarrow x$".

The Boolean-like law (1.5) is a distributivity of implications. It generalizes the Boolean law $(x \Rightarrow (y \Rightarrow z)) \equiv (x \Rightarrow y) \Rightarrow (x \Rightarrow z)$. Such law is a stronger form of the classical formal theory axiom $(x \to (y \to z)) \to ((x \to y) \to (x \to z))$.

In order to enrich the knowledge about what is a logical implication, mainly fuzzy implications, this work will investigate under which conditions those Boolean-like laws are valid for (S,N)- R-, QL-, D-, (N,T)- and h-implications. From a practical point of view, such knowledge is essential to determine the appropriate semantics for a given FRBS, since an implication operator must be defined according to the environment in which the FRBS is inserted[3] [68] and distinct semantics imply distinct implementations.

Moreover, the choice of a fuzzy implication can be given by the requirements of some properties which in turn may imply the requirement of a particular set of operators. For example, an (S,N)-implication satisfies (1.4) iff the pair (S,N) satisfies $S(N(x), x) = 1$[4]; so if a FRBS requires I satisfying (1.4), software en-

[1] $(x \Rightarrow 1) = 1$ can be generalized to fuzzy logic as $I(x, 1) = 1$ (denoted by (I8) in this work) which is called *right boundary condition* (RB) in [7].

[2] "$x \Rightarrow (y \Rightarrow x)$" is also a theorem in basic and product logics.

[3] Truly, all the connectives which compose FRBS semantics must be chosen in accordance to the domain in which the FRBS is inserted.

[4] See Theorem 5.1.1.

gineers have to select a pair (S, N) which satisfies $S(N(x), x) = 1$ in order to maintain semantics consistency[1]. Hence the choice of an implication connective may require a choice of a particular family of other connectives and vice-versa. In other words, in order to define a *consistent semantics* to a FRBS, software engineers must be well-reasoned about fuzzy connectives.

Therefore, we will also propose extensions for FRBSs — Polymorphous FRBSs (PFRBSs for short) — in order to allow more than one interpretation for each fuzzy operation.

In addition to such motivations, the application of automorphisms on fuzzy implications is an alternative way of defining new fuzzy implications. Works like [7] and [19] expose the importance of such *implication builders*. In [7], for example, Baczyński and Balasubramaniam cite the Łukasiewicz implication up to an automorphism as the only intersection between continuous (S,N)- and R-implication, generated from a left-continuous t-norm and a strong negation. Therefore it is relevant to investigate whether Boolean-like laws remain valid for fuzzy implications generated by automorphisms.

1.3 Structure of the thesis

This thesis is organised as follows. Chapter 2 will recall some basic definitions about t-norms, t-conorms, fuzzy negations and properties relating those fuzzy operators. Chapter 3 will focus on fuzzy implications, six of their classes and some properties about these classes. This chapter will also recall automorphism and Φ-conjugate implication definitions.

Chapter 4 will bring solutions of (1.3), and Chapters 5 and 6 will realise solutions of (1.4) and (1.5), respectively. All of these chapters will deal with each cited law for the six fuzzy implication classes and Φ-conjugate fuzzy implications; and they will also show relations between those Boolean-like laws and other fuzzy implication properties.

Chapter 7 will show how this thesis can contribute to the implementation of extensions of Fuzzy Systems. Finally, Chapter 8 will gather the results and present final remarks.

[1]The term *consistency* is used in a literal sense.

Chapter 2

Preliminaries

This chapter will present fundamental concepts related to fuzzy operators: Section 2.1 will show basic definitions about t-norms, t-conorms and fuzzy negations; and Section 2.2 will recall, and also define, properties involving those fuzzy operators.

2.1 Basic definitions

This section presents the state of the art on t-norms, t-conorms and fuzzy negation. Such points will be useful for the following sections and chapters. For more details see [38] and [7].

2.1.1 T-norms

Triangular norms (t-norms, for short) were introduced by [51] and initially used on probabilistic metric spaces. Schweizer and Sklar [63; 64] redefined t-norm's axioms to the form used today. Nowadays t-norms are often applied as a generalization of the classical conjunction.

Definition 2.1.1 *A function $T : [0,1]^2 \to [0,1]$ is a t-norm, for all $x, y, z \in [0,1]$, if it satisfies:*

T1. Commutativity: $T(x,y) = T(y,x)$;

T2. Associativity: $T(x, T(y, z)) = T(T(x, y), z)$*;*

T3. Monotonicity: if $y \leq z$*, then* $T(x, y) \leq T(x, z)$*, i.e.,* $T(x, \cdot)$ *is increasing;*

T4. Boundary condition: $T(x, 1) = x$*.*

Besides these properties, some others can be required, such as:

T5. Continuity: T is continuous in both arguments at the same time;

T6. Left-Continuity: T is left-continuous in each argument;

T7. Idempotency: $T(x, x) = x$, for all $x \in [0, 1]$;

T8. Nilpotency: T is continuous and for all $x \in]0, 1[$ there exists an $n \in \mathbb{N}$ such that $x_T^{[n]} = 0$, where $x_T^{[0]} = 1$ and $x_T^{[i+1]} = T(x, x_T^{[i]})$;

T9. Positiveness: if $T(x, y) = 0$ then either $x = 0$ or $y = 0$.

For the case of positiveness property, an alternative definition based on non-trivial zero divisors is used. An element $x \in]0, 1]$ is a non-trivial zero divisor of a t-norm T, if there exists $y \in]0, 1]$ such that $T(x, y) = 0$. In this case clearly, a t-norm is positive iff it does not have non-trivial zero divisor.

There are other properties for t-norms (see [38]), but they are not directly used in this work.

A very common example of fuzzy conjunction is T_M (defined as (2.1)). It is the only idempotent t-norm [40, Theorem 3.9].

$$T_M(x, y) = \min(x, y), \text{ for all } x, y \in [0, 1]. \tag{2.1}$$

Remark 2.1.1 [1] *Considering the point-wise order on the family of all t-norms induced by the order [0,1],* T_M *is the greatest one. So, by the ordering on all t-norms, we conclude that for any t-norm* T*,* $T(x, 0) = T(0, x) = 0$*.*

[1]This remark is known in the literature and it will be refereed in further proofs.

2.1.2 T-conorms

The dual notion of a t-norm is a t-conorm.

Definition 2.1.2 *A function $S : [0,1]^2 \to [0,1]$ is a t-conorm, for all $x, y, z \in [0,1]$, if it satisfies:*

S1. Commutativity: $S(x,y) = S(y,x)$;

S2. Associativity: $S(x, S(y,z)) = S(S(x,y), z)$;

S3. Monotonicity: if $y \leq z$, then $S(x,y) \leq S(x,z)$, i.e., $S(x,\cdot)$ is increasing;

S4. Boundary condition: $S(x,0) = x$.

Additional properties:

S5. Continuity: S is continuous in both arguments at the same time;

S6. Left-continuity: S is left-continuous in each argument;

S7. Idempotency: $S(x,x) = x$ for all $x \in [0,1]$;

S8. Nilpotency: S is continuous, $x \in]0,1[$ and there exists an $n \in \mathbb{N}$ such that $x_S^{[n]} = 1$, where $x_S^{[0]} = 0$ and $x_S^{[i+1]} = S(x, x_S^{[i]})$;

S9. Positiveness: if $S(x,y) = 1$ then either $x = 1$ or $y = 1$.

Definition 2.1.3 *[16] An element $x \in [0,1[$ is a non-trivial one-divisor of a t-conorm S if there exists $y \in [0,1[$ such that $S(x,y) = 1$.*

Above definition complements the notion of a positive t-conorm analogously to the positive t-norm definition [72]:

> "*A t-norm T is positive or without non-trivial zero divisors if $T(x,y) = 0$ entails that $x = 0$ or $y = 0$*".

Similarly, a t-conorm S is positive iff S does not have non-trivial one-divisors. For example, $S_{LK}(x,y) = \min(x+y, 1)$ has non-trivial one-divisors and it is not positive, and $S_M(x,y) = max(x,y)$ does not have not non-trivial one-divisors and it is positive.

Remark 2.1.2 [1] *Considering the point-wise order on the family of all t-conorms induced by the order [0,1], S_M is the least t-conorm. By definition of S_M, $S_M(x,1) = S_M(1,x) = 1$, so by the ordering on all t-conorms, for any t-conorm S, $S(x,1) = S(1,x) = 1$.*

2.1.3 Fuzzy negations

A fuzzy negation is a generalization of the classical negation '¬' whose truth table is $\neg 0 = 1$ and $\neg 1 = 0$. According to [15], Zadeh's fuzzy negation, defined in his seminal work [81], is the most used one in fuzzy systems. Nevertheless, several approaches have been proposed for fuzzy negations. Many of them appeared at the end of 70's and the beginning of the 80's (e.g. [27; 31; 44; 69]). The axiomatic definition for fuzzy negations, as it is known currently, can be found in [69] and is as follows.

Definition 2.1.4 *A function $N : [0,1] \to [0,1]$ is a fuzzy negation, if $N(0) = 1$, $N(1) = 0$ (N1), and N is decreasing (N2).*

In addition to this,

- *A fuzzy negation is called **strict** (N3) if, in addition, N is continuous and strictly decreasing.*
- *A fuzzy negation is called **strong** (N4) if it is involutive, i.e., $N(N(x)) = x$, for all $x \in [0,1]$.*

Proposition 2.1.1 *[7, Corollary 1.4.6] Every strong negation is strict.*

A fuzzy negation also can be generated from a fuzzy implication. Such one is called natural fuzzy negation.

Definition 2.1.5 *A function $N_I : [0,1] \to [0,1]$ is called a natural negation of I, if there exists a fuzzy implication I such that*

$$N_I(x) = I(x,0). \tag{2.2}$$

[1]This remark will be refereed in further proofs.

Definition 2.1.6 *A function $N_S : [0,1] \to [0,1]$ is called a natural negation of S, if there exists a t-conorm S such that*

$$N_S(x) = inf\{y \in [0,1] \,|\, S(x,y) = 1\}. \tag{2.3}$$

An equilibrium point of a fuzzy negation N is a value $e \in [0,1]$ such that $N(e) = e$.

As examples of fuzzy negations, we cite the classical (standard) Zadeh's fuzzy negation N_Z, the least fuzzy negation $N_\perp$ and the greatest fuzzy negation $N_\top$:

$$N_Z(x) = 1 - x, \text{ for all } x \in [0,1];$$

$$N_\perp(x) = \begin{cases} 1 & \text{, if } x = 0 \\ 0 & \text{, if } x \in]0,1]; \end{cases}$$

$$N_\top(x) = \begin{cases} 0 & \text{, if } x = 1 \\ 1 & \text{, if } x \in [0,1[. \end{cases}$$

2.2 Properties involving t-norms, t-conorms and fuzzy negations

In this subsection we will expose useful relations between fuzzy operators.

2.2.1 Distributivity among T-norms and T-conorms

In classical logic, the distributivity of disjunction over conjunction and the distributivity of conjunction over disjunction are well-known properties. Their fuzzy extension involves t-norms and t-conorms. Thus, given a t-conorm S and a t-norm T, S distributes over T if (2.4) is satisfied and T distributes over S if (2.5) is satisfied.

$$S(x, T(y,z)) = T(S(x,y), S(x,z)), \text{ for all } x, y, z \in [0,1]. \tag{2.4}$$

$$T(x, S(y, z)) = S(T(x, y), T(x, z)), \text{ for all } x, y, z \in [0, 1]. \tag{2.5}$$

Proposition 2.2.1 *[38, proposition 2.22] Let T be a t-norm and S a t-conorm.*

i. S is distributive over T iff $T = T_M$;

ii. T is distributive over S iff $S = S_M$;

iii. (T, S) is a distributive pair iff $T = T_M$ and $S = S_M$.

2.2.2 N-duality and De Morgan triples

For any t-conorm S there exists a t-norm T such that, $S(x, y) = 1-T(1-x, 1-y)$. Moreover, let T be a t-norm, S a t-conorm and N a fuzzy negation then S is said the N-dual of T if

$$S(x, y) = N(T(N(x), N(y))), \text{ for all } x, y \in [0, 1]. \tag{2.6}$$

Linked to the N-dual concept is De Morgan's laws:

$$\neg(p \vee q) = \neg p \wedge \neg q;$$
$$\neg(p \wedge q) = \neg p \vee \neg q.$$

In fuzzy context, there are several subtly different notions to express above laws [23]. In this thesis, we will follow [23; 45] and so, we will consider the most faithful to the original generalization of the De Morgan laws:

$$N(S(x, y)) = T(N(x), N(y)), \text{ for all } x, y \in [0, 1]; \tag{2.7}$$

$$N(T(x, y)) = S(N(x), N(y)), \text{ for all } x, y \in [0, 1]. \tag{2.8}$$

Definition 2.2.1 *Let T be a t-norm, S a t-conorm and N a fuzzy negation, (T, S, N) is called a De Morgan triple if it satisfies (2.7) and (2.8).*

A proposition was demonstrated in [38, p.232] and also cited in [7, Theorem 2.3.17]:

Let N be a strict negation, S a t-conorm and T a t-norm.
(T, S, N) is a De Morgan triple iff N is strong and S is the N-dual of T.

The above statement is valid when the De Morgan and the N-Duality properties are defined as in [38], but it is not valid when those properties are defined as we have done in this dissertation[1]. Regarding the De Morgan and the N-Duality definitions, as done in this dissertation, it is trivial that: If N is strong and S is the N-dual of T, then (T, S, N) is a De Morgan triple. However, the converse does not hold.

Proposition 2.2.2 *Let N be a strict negation, T a t-norm and S a t-conorm. (T, S, N) being a De Morgan triple does not imply that N is strong and S is the N-dual of T.*

PROOF: $N^*(x) = 1 - x^2$ is an example of a strict and not strong fuzzy negation and (T_M, S_M, N^*) is a De Morgan triple, but S_M is not N^*-dual of T_M. ■

2.2.3 Distributivity over De Morgan

Distributivity over De Morgan is a property composed by the De Morgan law and the distributivity between t-norms and t-conorms.

Definition 2.2.2 *Let T be a t-norm, S a t-conorm and N a fuzzy negation, (T, S, N) satisfies the distributivity over De Morgan if*

$$S(x, N(S(y, z))) = T(S(x, N(y)), S(x, N(z))) \tag{2.9}$$

or

$$T(x, N(T(y, z))) = S(T(x, N(y)), T(x, N(z))). \tag{2.10}$$

Remark 2.2.1 *Equation (2.9) is a property composed by Equation (2.7) — from the De Morgan law —, and Equation (2.4) — from the distributivity of t-conorms over t-norms. While (2.10) is composed by Equation (2.8) — from the De Morgan law —, and Equation (2.5) — from the distributivity of t-norms over t-conorms. Thus (2.10) is dual of (2.9).*

[1]In this dissertation, we follow the same De Morgan and N-Duality laws fuzzy extension of [7].

Proposition 2.2.3 *Let T be a t-norm, S a t-conorm and N a fuzzy negation. Then* $\min(S(x,N(y)),S(x,N(z))) \geq S(x,N(S(y,z)))$

PROOF: By Remark 2.1.2, $S(x,y) \geq y$, so $N(y) \geq N(S(y,z))$ (by N2) and therefore $S(x,N(y)) \geq S(x,N(S(y,z)))$ (by S3). Analogously, $S(x,N(z)) \geq S(x,N(S(y,z)))$. Hence $\min(S(x,N(z)),S(x,N(y))) \geq S(x,N(S(y,z)))$ for all $x,y,z \in [0,1]$. ■

Proposition 2.2.4 *Let T be a t-norm, S a t-conorm and N a continuous fuzzy negation. (T,S,N) satisfies (2.9) iff (T,S,N) satisfies (2.7) and (2.4).*

PROOF: $\Rightarrow$: For any $x,y \in [0,1]$, by (S4) and (2.9), $T(N(x),N(y))=T(S(0,N(x)),S(0,N(y))) = S(0,N(S(x,y))) = N(S(x,y))$. Thus (T,S,N) satisfies (2.7). Let $y,z \in [0,1]$, by continuity of N and by the intermediate-value theorem, there exist $y',z' \in [0,1]$ such that $N(y') = y$ and $N(z') = z$. Then, by (2.7) and (2.9), $S(x,T(y,z)) = S(x,T(N(y'),N(z'))) = S(x,N(S(y',z'))) = T(S(x,N(y')),S(x,N(z'))) = T(S(x,y),S(x,z))$. Therefore (T,S,N) satisfies (2.4).

$\Leftarrow$: By (2.7) and (2.4), $S(x,N(S(y,z)))=S(x,T(N(y),N(z)))=T(S(x,N(y)),S(x,N(z)))$. Thus, (T,S,N) satisfies (2.9). ■

Corollary 2.2.1 *If (T,S,N) satisfies (2.9) then $T = T_M$.*

PROOF: Straightforward from Propositions 2.2.1(i) and 2.2.4. ■

Remark 2.2.2 *By Proposition 2.2.1, if the triple (T,S,N) is distributive over De Morgan then $T = T_M$. But this does not mean that $S = S_M$, since such conclusion depends on which N is assumed. See the following propositions to illustrate this remark.*

Proposition 2.2.5 *The triple $(T_M,S,N_\top)$ satisfies (2.9) for any t-conorm S.*

PROOF:

On the one hand, by $N_\top$ definition, $S(x,N_\top(S(y,z))) \in \{x,1\}$ for any $x,y,z \in [0,1]$. In more detail, $S(x,N_\top(S(y,z))) = S(x,1) = 1$ if $y < 1$ and $z < 1$ (case 1) and, $S(x,N_\top(S(y,z))) = S(x,0) = x$ when $y < 1$ and $z = 1$ (case 2) or when $y = 1$ and $z < 1$ (case 3) or when $y = z = 1$ (case 4). On the other hand, Case 1: If $y < 1$ and $z < 1$ then $T_M(S(x, N_\top(y)), S(x, N_\top(z))) = \min(1, 1) = 1$;

Case 2: If $y < 1$ and $z = 1$ then $T_M(S(x, N_\top(y)), S(x, N_\top(1))) = \min(1, x) = x$;

Case 3: If $y = 1$ and $z < 1$ then $T_M(S(x, N_\top(1)), S(x, N_\top(z))) = \min(x, 1) = x$;

Case 4: If $y = z = 1$ then $T_M(S(x, N_\top(1)), S(x, N_\top(1))) = \min(x, x) = x$.

■

Proposition 2.2.6 $(T_M, S, N_\perp)$ *satisfies (2.9) for any t-conorm* S.

PROOF: On the one hand, by $N_\perp$ definition, $S(x, N_\perp(S(y, z))) = S(x, 1) = 1$ if $y = z = 0$ (case 1) and, $S(x, N_\perp(S(y, z))) = S(x, 0) = x$ when $y > 0$ and $z > 0$ (case 2) or when $y = 0$ and $z > 0$ (case 3) or when $y > 0$ and $z = 0$ (case 4). On the other hand,

Case 1: If $y = z = 0$ then $T_M(S(x, N_\perp(y)), S(x, N_\perp(z))) = \min(1, 1) = 1$;

Case 2: If $y > 0$ and $z > 0$ then $T_M(S(x, N_\perp(y)), S(x, N_\perp(z))) = \min(x, x) = x$;

Case 3: If $y = 0$ and $z > 0$ then $T_M(S(x, N_\perp(0)), S(x, N_\perp(z))) = \min(1, x) = x$;

Case 4: If $y > 0$ and $z = 0$ then $T_M(S(x, N_\perp(y)), S(x, N_\perp(0))) = \min(x, 1) = x$.

■

Proposition 2.2.7 (T_M, S_M, N) *satisfies (2.9) for any fuzzy negation* N.

PROOF: If $T = T_M$ then any S is distributive over T (Lemma 2.2.1) and for any fuzzy negation N, (T_M, S_M, N) is a De Morgan triple. Hence, by Lemma 2.2.4, (T_M, S_M, N) satisfies (2.9) for any N. ■

The results obtained for (2.9) remain valid for (2.10) regarding dual operators. See the following propositions.

Proposition 2.2.8 *Let* T *be a t-norm,* S *a t-conorm and* N *a fuzzy negation.* $\max(T(x, N(y)), T(x, N(z))) \leq T(x, N(T(y, z)))$.

PROOF: Analogous to the proof of Proposition 2.2.3. ■

Proposition 2.2.9 *Let T be a t-norm, S a t-conorm and N a continuous fuzzy negation. (T, S, N) satisfies (2.10) iff (T, S, N) satisfies (2.8) and (2.5).*

PROOF: Analogous to the proof of Proposition 2.2.4. ■

Remark 2.2.3 *By Proposition 2.2.9 and Lemma 2.2.1(ii), if the triple (T, S, N) satisfies (2.10) then $S = S_M$. But it does not means that $T = T_M$. Moreover if (T, S, N) satisfies (2.9) and (2.10) then $T = T_M$, $S = S_M$ and any fuzzy negation N can be adopted.*

Corollary 2.2.2 *If (T, S, N) satisfies (2.9) then $S = S_M$.*

PROOF: Straightforward from Proposition 2.2.9 and Lemma 2.2.1(ii). ■

Proposition 2.2.10 *The triple $(T_M, S, N_\top)$, $(T_M, S, N_\perp)$ satisfy (2.10) for any t-conorm S. And triple (T_M, S_M, N) satisfies (2.10) for any fuzzy negation N.*

PROOF: Analogous to the proof of Propositions 2.2.5, 2.2.6 and 2.2.7 ■

Definition 2.2.3 *Let T be a t-norm, S a t-conorm and N a fuzzy negation, (T, S, N) satisfies the weak distributivity over De Morgan if*

$$S(x, N(S(y, z))) = T(S(x, N(y)), S(x, N(z))), \text{ when } \min(y, z) \leq x \tag{2.11}$$

or

$$T(x, N(T(y, z))) = S(T(x, N(y)), T(x, N(z))), \text{ when } \max(y, z) \leq x. \tag{2.12}$$

Remark 2.2.4 *Since (2.10) and (2.9) are dual properties, so (2.11) and (2.12) are dual properties too.*

Clearly, (2.9) implies (2.11) but the converse does not hold (see the following example).

Example 2.2.1 *The triple $(T_P, S_M, N_\top)$, where $T_P(x, y) = x \cdot y$ is the product t-norm, satisfies (2.11), but it does not satisfy (2.9). Notice that only when $y, z{=}1$ and $x{<}1$, $S(x, N(S(y, z))) \neq T(S(x, N(y)), S(x, N(z)))$: $\max(x, N_\top(\max(y, z))) = x$ and $T_P(\max(x, N_\top(y)), \max(x, N_\top(z))) = x \cdot x$.*

The same is seen between (2.10) and (2.12). I.e. (2.10) implies (2.12) but the converse does not hold (see the following example).

Example 2.2.2 *The triple* $(T_M, S_P, N_\perp)$*, where* $S_P(x,y) = x+y-x\cdot y$ *is the product t-conorm, satisfies (2.12), but it does not satisfy (2.10). Note that only when* $y, z{=}0$ *and* $x{>}0$*,* $T(x, N(T(y,z))) \neq S(T(x,N(y)), T(x,N(z)))$*:* $\min(x, N_\perp(\min(y,z))) = x$ *and* $S_P(\min(x, N_\perp(y)), \min(x, N_\perp(z))) = 2x - x^2$*.*

2.2.4 Law of excluded middle

The Law of Excluded Middle (LEM) is one of the well-known fundamental Boolean laws of classical theory. As the LEM, in classical logic, means that $\neg p \vee p$ is always true, so we have the following definition.

Definition 2.2.4 *Let* S *be a t-conorm and* N *a fuzzy negation, the pair* (S, N) *satisfies the LEM if*

$$S(N(x), x) = 1, \text{ for all } x \in [0,1]. \tag{LEM}$$

Note that any t-conorm S with the $N_\top$ satisfies (LEM). Also, no t-conorm with the $N_\perp$ satisfies (LEM) [7, Remark 2.3.10(i) and (iii)]. Another result is given in the following lemma.

Lemma 2.2.1 *[7, Lemma 2.3.9] If a t-conorm* S *and fuzzy negation* N *satisfy (LEM), then*

i. $N \geq N_S$*; and*

ii. $N_S \circ N(x) \leq x$*, for all* $x \in [0,1]$*.*

Proposition 2.2.11 *Let* S *be a positive t-conorm and* N *a fuzzy negation.* (S, N) *satisfies (LEM) iff* $N = N_\top$*.*

Proof: $\Rightarrow$: By Lemma 2.2.1(ii) and let S be a positive t-conorm, since (S, N) satisfies (LEM), then: $N_S(x) = 0$, if $x = 1$, and $N_S(x) = 1$, if $x < 1$. I.e.:

$$N_S(x) = \begin{cases} 0 & \text{, if } x = 1 \\ 1 & \text{, if } x \in [0,1[\end{cases} = N_\top(x)$$

Again by Lemma 2.2.1(i), if (S, N) satisfies (LEM) then $N \geq N_S$. Therefore, since $N_\top$ is the greatest negation, $N = N_\top$.

$\Leftarrow$: It is known that, for any S, if $N = N_\top$ then (LEM) [7, pp. 53]. ■

2.2.5 Law of non-contradiction

Aristotle's law of non-contradiction is defined in classical logic as "$\neg x \wedge x$ is always false". So its fuzzy generalization is defined as follows.

Definition 2.2.5 *Let T be a t-norm and N a fuzzy negation, the pair (T, N) satisfies the law of non-contradiction if*

$$T(N(x), x) = 0. \tag{2.13}$$

2.3 Final considerations

This chapter presented fundamental concepts about fuzzy operators and their properties. All of them were already known in the literature, with the exception of *Distributivity over De Morgan*. This property relates t-norms, t-conorms and fuzzy negations and it is originated by the combination of the Distributivity between t-norms and t-conorms and the De Morgan Law. Many results about this property were demonstrated in this chapter, such as Propositions 2.2.3 – 2.2.9 as well as Corollaries 2.2.1 and 2.2.2.

Chapter 3

Fuzzy Implications

In search of a classical implication generalization, Baldwin and Pilsworth in [12] and Blander and Kohout in [13], proposed a few basic properties that should be required by a fuzzy implication. Furthermore, Trillas and Valverde in [73] give the first axiomatic for fuzzy implications. Other definitions have been proposed and nowadays there are different acceptable definitions (see [26; 28; 61; 67] as examples). In this scenario some classes of fuzzy implications have been proposed, such as [9; 10; 14; 18; 43; 47; 49; 77].

In this dissertation, we will regard only the boundary conditions to define a fuzzy implication as done in [19; 57; 60].

Definition 3.0.1 *A function $I : [0,1]^2 \to [0,1]$ is called a fuzzy implication if it satisfies the following boundary conditions.*

I1. : $I(0,0) = 1$;

I2. : $I(0,1) = 1$;

I3. : $I(1,0) = 0$;

I4. : $I(1,1) = 1$.

Some other potential properties for fuzzy implications are:

I5. Left antitonicity: if $x_1 \leq x_2$ then $I(x_1, y) \geq I(x_2, y)$, for all $x_1, x_2, y \in [0,1]$;

I6. Right isotonicity: if $y_1 \leq y_2$ then $I(x, y_1) \leq I(x, y_2)$, for all $x, y_1, y_2 \in [0,1]$;

I7. Left boundary condition: $I(0, y) = 1$, for all $y \in [0,1]$;

I8. Right boundary condition: $I(x, 1) = 1$, for all $x \in [0,1]$;

I9. Left neutrality: $I(1, y) = y$, for all $y \in [0,1]$;

I10. Identity property: $I(x, x) = 1$, for all $x \in [0,1]$;

I11. Exchange principle: $I(x, I(y, z)) = I(y, I(x, z))$, for all $x, y, z \in [0,1]$;

I12. Ordering property[1]: $x \leq y$ iff $I(x, y) = 1$, for all $x, y, \in [0,1]$;

  I12a. Left ordering property: if $x \leq y$ then $I(x, y) = 1$, for all $x, y, \in [0,1]$;

  I12b. Right ordering property: if $I(x, y) = 1$ then $x \leq y$, for all $x, y, \in [0,1]$;

I13. Continuity.

I14. Iterative Boolean-like Law: $I(x, I(x, y)) = I(x, y)$, for all $x, y, z \in [0,1]$;[2]

There is also a property that relates fuzzy implications and negations:

I15. Contraposition: $I(x, y) = I(N(y), N(x))$, for all $x, y, \in [0,1]$;

  I15a. Left contraposition: $I(I(x, y), I(N(y), N(x))) = 1$, for all $x, y, \in [0,1]$;

  I15b. Right contraposition: $I(I(N(y), N(x)), I(x, y)) = 1$, for all $x, y, \in [0,1]$;

A solution that has been adopted is to give a general definition for fuzzy implication and so to specify classes for them. Each class may generalize an implication concept given by a specific Boolean logic (such as (S,N)- R and QL-implications [9; 10]) or it is defined in terms of these concepts plus some well-acceptable property (such as D-, (N,T)-, Xor and E-implications [14; 18; 71]) or one may try to give another acceptable fuzzy implication concept (such as f-, g-, h- and (g,min)-implications [43; 49; 77]) or even use other kind of fuzzy operators — uninorms instead of t-norms (such as (U,N)- and RU-implications [7]).

[1] Also called confinement property.

[2] Such iterative Boolean-like law was investigated in [67].

Remark 3.0.1 *Although, the Definition 3.0.1 consider functions that would not be usually considered as an implication (Implication 3.1, for example). A more rigorous definition could discard some functions that are well-acceptable as implications. For example, the fuzzy implication definition regarded in [28] does not consider QL-implications as a fuzzy implication. Also, in the following chapters, the investigations will be done on each fuzzy implication class, so the fuzzy implication definition will not have much impact on the thesis results. Therefore, without generating any loss to the thesis, we have chosen a broader definition of fuzzy implications.*

$$I(x,y) = \begin{cases} 1 & , \textit{ if } x = 0 \textit{ or } y = 1 \\ 0 & , \textit{ if } x = 1 \textit{ or } y = 0 \\ \frac{\sqrt{x}y^2}{2} & , \textit{ otherwise.} \end{cases} \tag{3.1}$$

3.1 Fuzzy implication classes

There are three best-known classes of fuzzy implications, namely: (S,N)-, R- and QL-implications. All of them are defined by other fuzzy operators and each one has a specific motivation. Other classes are defined by application of some property on one of those classes, by some generator function or from uninorms (see [78]). In this thesis we will study the three best-known together with D-, (N,T)- and h-implications. D- and (N,T)-implications rises from QL- and (S,N)-implications (respectively) and the last one is generated from the function h.

Lemmas and theorems contained in this section will be useful in the following chapters when we will demonstrate the necessary and sufficient conditions to (1.3), (1.4) and (1.5) be held for those classes.

3.1.1 (S,N)-implications

In classical logic, the operator '$\to$' (material implication) is generated by Boolean negation '$\neg$' and disjunction '$\vee$': $p \to q \equiv \neg p \vee q$. The (S,N)-implication is a generalization of this material implication to the fuzzy logic.

Definition 3.1.1 *A function $I : [0,1]^2 \to [0,1]$ is called an (S,N)-implication if there exist a t-conorm S and a fuzzy negation N, such that*

$$I(x,y) = S(N(x),y), \text{ for all } x,y \in [0,1]. \tag{3.2}$$

If N is a strong fuzzy negation, then I is called strong implication (S-implication, for short). If N is not a strong negation, I is called non-strong (S,N)-implication (nS-implication, for short).

The (S,N)-implication generated by a t-conorm S and a fuzzy negation N is denoted by $I_{S,N}$. The S- and the nS-implication are denoted, respectively, by I_S and I_{nS}.

A relation between fuzzy negations and (S,N)-implication is given in the next proposition.

Proposition 3.1.1 *[7, Prop. 2.4.3] Let $I_{S,N}$ be an (S,N)-implication, then $N_{I_{S,N}} = N$.*

The law of distributivity of implication over t-conorm — when $I = I_{S,N}$. This law is the fuzzy generalization of $(p \vee q) \to r = (p \to r) \wedge (q \to r)$ and it was investigated in [11]. In such work, some results were obtained to S- and R-implication[1]. Note that (2.9) is equivalent to (3.3) so we consider the following lemmas.

$$I(S(x,y),z) = T(I(x,z),I(y,z)), \text{ for all } x,y,z \in [0,1]. \tag{3.3}$$

Lemma 3.1.1 *[7, Theorem 7.2.5] Let $I_{S,N}$ be an (S,N)-implication generated by a t-conorm S and a strict fuzzy negation N, then the triple (I,T,S) satisfies (3.3) iff $S = S_M$ and $T = T_M$.*

Lemma 3.1.2 *Let T, be a t-norm, S a t-conorm and N a strict fuzzy negation. (T,S,N) satisfies (2.9) iff $S = S_M$ and $T = T_M$.*

PROOF: Straightforward from Lemma 3.1.1 and by the equivalence between $I_{S,N}(S(x,y),z) = T(I_{S,N}(x,z), I_{S,N}(y,z))$ and (2.9). ■

[1]Similar results can be found in [7, p.215-216].

Note that, let T, be a t-norm, S a t-conorm and N a strict fuzzy negation, if (T, S, N) satisfies (2.9) then $T = T_M$ and $S = S_M$. But we cannot conclude that if (T, S, N) satisfies (2.9) and (LEM) then $N = N_{\top}$, since $N_{\top}$ is not strict.

3.1.2 R-implications

The residual implication (R-implication, for short) is motivated by the residuation concept employed in intuitionistic logic which is founded by the relative pseudo-complement definition [54, pp.54].[1]

Definition 3.1.2 *A function $I : [0, 1]^2 \to [0, 1]$ is called an R-implication if there exists a t-norm T such that*

$$I(x, y) = sup\{t \in [0, 1] \,|\, T(x, t) \leq y\}. \tag{3.4}$$

The R-implication generated by a t-norm T is denoted by I_T.

Lemma 3.1.3 *[7, Theorem 2.5.4] and [16, pp.359] Every R-implication satisfies (I1)-(I10) and (I12a).*

If T is a left-continuous t-norm, I_T presents interesting features (see [30, Theorem 5.4.2] and [7, Theorem 2.5.7]). Let T be a left-continuous t-norm, (I_T, T) is called adjoint pair (it has the adjunction property: $T(x, z) \leq y$ iff $I(x, y) \geq z$), (3.4) is equivalent to

$$I_T(x, y) = max\{t \in [0, 1] \,|\, T(x, t) \leq y\}; \tag{3.5}$$

and (3.5) satisfies all fuzzy implication properties announced in pages 23-24 (except (I13) and (I14)).

Considering the dual situation of I_T, we have the definition of T_I:

Definition 3.1.3 *[6, Def. 8] Let I be a fuzzy implication which satisfies (I1)-(I6), then $T_I : [0, 1]^2 \to [0, 1]$ is defined as*

$$T_I(x, y) = inf\{t \in [0, 1] \,|\, I(x, t) \geq y\}. \tag{3.6}$$

[1] Let a, b be elements of a lattice A, an element $c \in A$ is a pseudo-complement of a relative to b if c is the greatest element such that $a \wedge c \leq b$.

Analogously to (3.5), if I_T is right continuous with respect to (w.r.t. for short) the second variable, the infimum in (3.6) is the minimum, i.e.,

$$T_I(x,y) = \min\{t \in [0,1] \mid I(x,t) \geq y\}. \tag{3.7}$$

Lemma 3.1.4 *([28, Theorem 1.14] or [30, Theorem 5.4.1]) A function I:$[0,1]^2 \to [0,1]$ is an R-implication based on some left-continuous t-norm iff I satisfies (I6), (I11), (I12) and it is right-continuous in the second argument.*

In the proof of the above lemma it is demonstrated that for a function I that satisfies above conditions and for a fixed left-continuous t-norm T, we have that $I = I_{T_I}$ and $T = T_{I_T}$ where I_T is defined by (3.4) or (3.5) and T_I is defined by (3.6) or (3.7). Based on these, we have the following theorem.

Theorem 3.1.1 *[6, Corollary 10] A function $T : [0,1]^2 \to [0,1]$ is a left-continuous t-norm iff T can be represented by (3.7) for some function $I : [0,1]^2 \to [0,1]$ which satisfies (I6), (I11), (I12) and it is right-continuous in the second argument.*

3.1.3 QL-implications

The QL-implication is the generalization of the quantum implication: $p \to q \equiv \neg p \vee (p \wedge q)$ [8].

Definition 3.1.4 [1] *A function $I : [0,1]^2 \to [0,1]$ is called a QL-implication if there exist a t-norm T, a t-conorm S and a fuzzy negation N such that*

$$I(x,y) = S(N(x), T(x,y)), \text{ for all } x, y \in [0,1]. \tag{3.8}$$

The QL-implication generated by a t-norm T, a t-conorm S and a fuzzy negation N is denoted by $I_{S,N,T}$.

Lemma 3.1.5 *[7, Theorem 2.6.2] Every QL-implication satisfies (I1)-(I4), (I6), (I7) and (I9).*

[1] Some authors define Ql-implications requiring that their underlying negation must be strong (see [19; 48; 57]), but others do not (see [7]).

Lemma 3.1.6 *If (S, N) satisfies (LEM) and $T = T_M$, then a QL-implication $I_{S,N,T}$ satisfies (I10).*

PROOF: Let $I_{S,N,T}$ be a QL-implication and by Proposition 2.2.1(i) $T = T_M$ iff S is distributive over T, so:

$$\begin{aligned} I_{S,N,T}(x,x) &= S(N(x), T(x,x)) && \textit{by } (3.8) \\ &= T(S(N(x),x), S(N(x),x))) && \textit{by } (2.4) \\ &= T(1,1) && \textit{by } (LEM) \\ &= 1 && \textit{by } (T4). \end{aligned}$$

■

3.1.4 D-implications

Diskant-implication (D-implication, for short) is the contraposition (I15) of the QL-implication [48]. Another way to obtain a D-implication from a QL-implication — or a QL- from a D-implication — is by the application of (LEM) and (2.4) which will be seen in Lemma 3.1.2.

Definition 3.1.5 [1] *A function $I : [0,1]^2 \to [0,1]$ is called a D-implication if there exist a t-norm T, a t-conorm S and a fuzzy negation N such that*

$$I(x,y) = S(T(N(x), N(y)), y). \tag{3.9}$$

The D-implication generated by a t-conorm S, a t-norm T and a fuzzy negation N is denoted by $I_{S,T,N}$.

Lemma 3.1.7 [2] *Every D-implication satisfies (I1)-(I4), (I5) and (I9).*

PROOF: Let $I_{S,T,N}$ be a D-implication, so $I_{S,T,N}$ satisfies (I1)-(I4) since:

[1]Generally, D-implications are defined from strong negations [47; 56; 57; 71], but in order to maintain the similarity to QL-implication definition, we will not require that D-implication underlying negation must be strong. Observe that D-implications defined from a non-strong negation are not contraposition of QL-implications.

[2]Such Lemma is known for D-implications, but its demonstration is required in terms of the alternative D-implication definition proposed in this thesis.

$$\begin{aligned}
I_{S,T,N}(0,0) &= S(T(N(0),N(0)),0) &&= S(T(1,1),0) = S(1,0) = 1.\\
I_{S,T,N}(0,1) &= S(T(N(0),N(1)),1) &&= 1.\\
I_{S,T,N}(1,0) &= S(T(N(1),N(0)),0) &&= S(T(0,1),0) = S(0,0) = 0.\\
I_{S,T,N}(1,1) &= S(T(N(1),N(1)),1) &&= 1.
\end{aligned}$$

Now, assume that $x_1, x_2, y \in [0,1]$ and $x_1 \leq x_2$. Then, by (N2), $N(x_1) \geq N(x_2)$. By (T3), $T(N(x_1), N(y)) \geq T(N(x_2), N(y))$, and by (S3) we have $S(T(N(x_1), N(y)), y) \geq S(T(N(x_2), N(y)), y)$. Hence $I_{S,T,N}(x_1, y) \geq I_{S,T,N}(x_2, y)$. Hence $I_{S,T,N}$ satisfies (I5).

For any $y \in [0,1]$, $I_{S,T,N}(1,y) = S(T(N(1), N(y)), y) = S(T(0, N(y)), y)$ and $T(0, N(y)) = 0$ (by the ordering on all t-norms). Since $S(0,y) = y$, so $I_{S,T,N}(1,y) = y$. Hence $I_{S,T,N}$ satisfies (I9). ∎

Example 3.1.1 *As already mentioned in [47, Remark 1], Property (I5) is not satisfied by every QL-implication and (I6) is not satisfied by every D-implication. The triple (T_M, S_M, N_Z) is an example for both implications, i.e., I_{S_M,N_Z,T_M} is an example of a QL-implication that does not satisfy (I5) and I_{S_M,T_M,N_Z} is a D-implication that does not satisfy (I6).*

3.1.5 (N,T)-implications

The (N,T)-implication is the N-dual implication of an (S,N)-implication. It was firstly defined in [14] as an N-dual implication of a t-norm T, since $T(x,y) = N(I_{N,T}(x, N(y)))$ and $I_{N,T}(x,y) = N(T(x, N(y)))$ — for more details see the [14, Proposition 2.7].

Definition 3.1.6 *A function $I : [0,1]^2 \to [0,1]$ is called a N-dual fuzzy implication of T ((N,T)-implication, for short) if there exist a t-norm T, and a negation N such that*

$$I(x,y) = N(T(x, N(y))). \tag{3.10}$$

The (N,T)-implication generated by a t-norm T and a fuzzy negation N is denoted by $I_{N,T}$.

Lemma 3.1.8 *[14, Prop. 2.6] Every (N,T)-implication satisfies (I1)-(I6).*

Lemma 3.1.9 *An (N,T)-implication $I_{N,T}$ satisfies (I9), if N is a strong negation.*

PROOF: Since N is a strong negation, $I_{N,T}(1,y) = N(T(1,N(y))) = N(N(y)) = y$. ∎

3.1.6 h-implications

The h-implications were defined by Massanet et al. in [49] in a similar way realised by Yager in [77].

Definition 3.1.7 *A function $I : [0,1]^2 \to [0,1]$ is called an h-implication if it exists an $e \in]0,1[$ and a strictly increasing and continuous function $h : [0,1] \to [-\infty,+\infty]$ for which $h(0) = -\infty$, $h(e) = 0$ and $h(1) = +\infty$, such that*

$$I(x,y) = \begin{cases} 1 & , \text{ if } x = 0 \\ h^{-1}(x \cdot h(y)) & , \text{ if } x > 0 \text{ and } y \leq e \\ h^{-1}(\frac{1}{x} \cdot h(y)) & , \text{ if } x > 0 \text{ and } y > e. \end{cases} \tag{3.11}$$

The function h is called an h-generator (with respect to e) of the implication function I. The h-implication generated by a continuous and strictly increasing function h is denoted by I^h.

Lemma 3.1.10 *[49, Prop. 1 and Theo. 5(i) and (v)] Let h be an h-generator w.r.t. a fixed $e \in]0,1[$, then I^h satisfies (I1)-(I6) and (I9).*[1] *Moreover, $I^h(x,x) = 1$ iff $x = 0$ or $x = 1$ i.e., I^h does not satisfy (I8).*

Lemma 3.1.11 *Let h be an h-generator w.r.t. a fixed $e \in]0,1[$, then:*

i. $I^h(0,y) = 1$;

ii. $I^h(x,0) = 0$;

iii. $I^h(x,1) = 1$;

[1] I^h satisfies (I2) is not demonstrated in [49, Prop. 1 and Theo. 5(i)], it is trivially deduced from (I4) and (I5). Besides that, we also can deduce straightforward (I7) from (I1) and (I6), and (I8) from (I4) and (I5).

iv. $I^h(1, y) = y$*;*

v. $I^h(x, e) = e$*;*

vi. Let $x > 0$. $y \leq e$ *iff* $I^h(x, y) \leq e$*;*

vii. Let $x > 0$. $y > e$ *iff* $I^h(x, y) > e$*;*

PROOF: Straightforward from the Definition 3.11. ■

3.1.7 Equivalences between fuzzy implication classes

Lemma 3.1.12 *[7, Proposition 4.2.2] Given an (S,N)-implication* $I_{S,N}$ *and a QL-implication* $I_{S,N,T}$*. If* $T = T_M$ *and* (S, N) *satisfies (LEM), then* $I_{S,N,T} = I_{S,N}$*.*

Lemma 3.1.13 *Given a D-implication* $I_{S,T,N}$ *and an (S,N)-implication* $I_{S,N}$*. If* $T = T_M$ *and* (S, N) *satisfies (LEM), then* $I_{S,T,N} = I_{S,N}$*.*

PROOF: By Proposition 2.2.1(i), $T = T_M$ iff (S, T) satisfies (2.4). So, for all $x, y \in [0, 1]$:

$$
\begin{aligned}
I_{S,T,N}(x, y) &= S(T(N(x), N(y)), y) && \textit{by } (3.9) \\
&= S(y, T(N(x), N(y))) && \textit{by } (S1) \\
&= T(S(y, N(x)), S(y, N(y))) && \textit{by } (2.4) \\
&= T(S(N(x), y), S(N(y), y))) && \textit{by } (S1) \\
&= T(S(N(x), y), 1) && \textit{by } (LEM) \\
&= S(N(x), y) && \textit{by } (T4) \\
&= I_{S,N}(x, y) && \textit{by } (3.2).
\end{aligned}
$$

■

Theorem 3.1.2 *Given a QL-implication* $I_{S,N,T}$*, a D-implication* $I_{S,T,N}$ *and an* (S, N)*-implication* $I_{S,N}$*. If* $T = T_M$ *and* (S, N) *satisfies (LEM), then* $I_{S,T,N} = I_{S,N} = I_{S,N,T}$*.*

PROOF: Straightforward from Lemmas 3.1.12 and 3.1.13. ■

If we assume that N is a strong negation, then we get another result relating (S,N)-, QL- and D-implications.

Proposition 3.1.2 *[47, Prop. 6] Let N be a strong negation and given a QL-implication $I_{S,N,T}$, a D-implication $I_{S,T,N}$ and an (S,N)-implication $I_{S,N}$. If $I_{S,T,N}$ and $I_{S,N}$ satisfy (I1)-(I6), then the corresponding QL- and D-implication coincide and are given by:*

$$I_{S,N,T}(x,y) = I_{S,T,N}(x,y) = \begin{cases} 1 & , \text{ if } x \leq y \\ I_{S,N}(x,y) & , \text{ otherwise.} \end{cases}$$

Lemma 3.1.14 *[pp.139][14] Let N be a strong negation. $I_{S,N} = I_{N,T}$ iff S is N-dual of T.*

Theorem 3.1.3 *Given a D-implication $I_{S,T,N}$, an (S,N)-implication $I_{S,N}$ and a QL-implication $I_{S,N,T}$. If $T = T_M$, N is strong, S is N-dual of T and (S,N) satisfies (LEM). Then $I_{S,T,N} = I_{S,N} = I_{S,N,T} = I_{N,T}$.*

PROOF: Straightforward from Lemma 3.1.14 and Theorem 3.1.2. ∎

Proposition 3.1.3 *[47; 48] Let N be a strong negation and, given a D-implication $I_{S,T,N}$ and a QL-implication $I_{S,N,T}$. If $I_{S,T,N}$ or $I_{S,N,T}$ satisfies the contraposition (I15), then $I_{S,T,N} = I_{S,N,T}$.*

3.2 Φ-conjugate fuzzy implications

An automorphism is a continuous and strictly increasing function $\varphi : [0,1] \to [0,1]$ such that $\varphi(0) = 0$ and $\varphi(1) = 1$ [19].[1] Automorphisms are closed under composition, i.e., if φ and φ' are automorphisms then $\varphi \circ \varphi'(x) = \varphi(\varphi'(x))$. Also, since an automorphism is an increasing bijection, then the inverse of an automorphism (φ^{-1}) is also an automorphism. In addition, the set of automorphisms, denoted by Φ, with the composition is a group [29].

Definition 3.2.1 *We say that functions $f, g : [0,1]^n \to [0,1]$ are Φ-conjugate if there exists an automorphism $\varphi \in \Phi$ such that $g = f_\varphi$, where*

$$f_\varphi(x_1, ..., x_n) = \varphi^{-1}(f(\varphi(x_1), ..., \varphi(x_n))), \quad x_1, ..., x_n \in [0,1].$$

[1] An alternative definition is that automorphisms are increasing bijections functions from [0,1] to [0,1] where [39].

By this definition, two fuzzy implications I and J are Φ-conjugate, if there exists a $\varphi \in \Phi$ such that $J = I_\varphi$, where

$$I_\varphi(x, y) = \varphi^{-1}(I(\varphi(x), \varphi(y))), \text{ for all } x, y \in [0, 1]. \tag{3.12}$$

3.3 Final considerations

In this chapter, we presented an overview of the history of fuzzy implication and then we chose one of the well-accepted definitions of fuzzy implication. We also showed six known classes of fuzzy implications. D-implications, in particular, had its definition relaxed. We will see that this definition does not interfere with the main results of this thesis w.r.t. D-implications. Moreover, this relaxation is interesting to increase the set of D-implications that can satisfy boolean-like laws.

Some results were demonstrated w.r.t. equivalences between fuzzy implications in subsection 3.1.7.

Chapter 4

On the Boolean-like Law $\mathbf{y \leq I(x,y)}$

This chapter will present some relations between (1.3) and other fuzzy implication properties and the characterization of (S,N)-, R-, QL-, D-, (N,T)-, h-implications satisfying (1.3).

There are some relations between (1.3) and the properties announced in pages 23-24, as exposed in [7; 19; 65; 66]. We highlight the previous studies about (1.3) performed by Bustince, Shi et al. in [19; 66]. In [19] and [66], the authors investigated the interrelation between some fuzzy implications properties. Within this context, we will also show some relations between (1.3) and other fuzzy implication properties.

Proposition 4.0.1 *[19] If a fuzzy implication I satisfies (1.3) then I satisfies (I8).*

PROOF: Trivial. ∎

Lemma 4.0.1 *[19, Lemma 1 viii] If a fuzzy implication I satisfies (I9) and (I5), then I satisfies (1.3).*

Adapting the results of [66, Remark 7.5] we have the next proposition.

Proposition 4.0.2 *Let I be a fuzzy implication:*
If I satisfies (I11) and (I12) then I satisfies (1.3);
If I satisfies (I5), (I11) and $I(x,0) = N_I(x)$ then I satisfies (1.3);
If I satisfies (I5), (I11) and (I15) then I satisfies (1.3).

Proposition 4.0.3 *If a fuzzy implication I satisfies (1.3) and (I12a) then I satisfies (1.4).*

PROOF: Let I be a fuzzy implication. If $x \leq I(y,x)$, then (by (I12a)) $I(x, I(y,x)) = 1$. ■

Proposition 4.0.4 *If a fuzzy implication I satisfies (1.4) and (I12b) then I satisfies (1.3).*

PROOF: Let I be a fuzzy implication. If $I(x, I(y,x)) = 1$, then (by (I12b)) $x \leq I(y,x)$. ■

Corollary 4.0.1 *If a fuzzy implication I satisfies (I12), then I satisfies (1.3) iff I satisfies (1.4).*

PROOF: Straightforward from Propositions 4.0.3 and 4.0.4. ■

The following section will bring results about when the Boolean-like law $y \leq I(x,y)$ remain valid in fuzzy logics (Section 4.1). We will analyze those results in Section 4.2 regarding aspects announced in the introduction.

4.1 Solutions of y $\leq$ I(x,y) for (S,N)-, R-, QL-, D-, (N,T)- and h-implications

Theorem 4.1.1 *Every (S,N)-implication satisfies (1.3).*

PROOF: Trivially, by (3.2), $I_{S,N}(x,y) = S(N(x),y)$. $S(N(x),y) \geq max(N(x),y)$ (by Remark 2.1.2) and $max(N(x),y) \geq y$. Hence $y \leq I_{S,N}(x,y)$. ■

Theorem 4.1.2 *Every R-implication satisfies (1.3).*

PROOF: Straightforward from Lemmas 4.0.1 and 3.1.3. ■

We demonstrated that (S,N)-implications satisfy (1.3). Since there is an intersection between the (S,N)- and QL-implications classes, we verify the sufficient and necessary conditions for which the elements of such an intersection satisfy (1.3).

Theorem 4.1.3 *If (S,N) satisfies (LEM) and $T = T_M$ then a QL-implication $I_{S,N,T}$ satisfies (1.3).*

PROOF: Straightforward from Theorem 4.1.1 and Lemma 3.1.12. ■

It can be noted that Lemma 3.1.6 and Theorem 4.1.3 provide the sufficient conditions for $I_{S,N,T}$ to satisfy (1.3). In the sequel we present results that provide the necessary conditions.

Lemma 4.1.1 *If a QL-implication $I_{S,N,T}$ satisfies (1.3) then (S, N) satisfies (LEM).*

PROOF: By (1.3), $1 \leq I_{S,N,T}(y, 1)$, then $I_{S,N,T}(y, 1) = 1$ (i.e. $I_{S,N,T}$ satisfies (I8)). So $S(N(y), T(y, 1)) = 1$, and by (T4) $S(N(y), y) = 1$. Hence (S, N) satisfies (LEM). ■

The converse of Theorem 4.1.3 is not true. As illustrated in the next example, (1.3) and (LEM) are not sufficient conditions to imply $T = T_M$.

Example 4.1.1 *$I_{T_L,S_L,N_Z}(y, x) = max(1 - y, x)$ is a QL-implication[1] generated from the Lukasiewicz t-norm $T_L(x, y) = max(x + y - 1, 0)$, the Lukasiewicz t-conorm $S_L(x, y) = min(x + y, 1)$ and the Zadeh's fuzzy negation. I_{T_L,S_L,N_Z} satisfies (1.3) once that $I_{T_L,S_L,N_Z}(y, x) = max(1 - y, x) \geq x$. (S_L, N_Z) satisfies (LEM): $S_L(x, N_Z(x)) = min(x+(1-x), 1) = min(1, 1) = 1$. However, $T_L \neq T_M$ (by Proposition 2.2.1(i)).*

Theorem 4.1.4 *Let S be a strictly increasing in [0,1[t-conorm. If a QL-implication $I_{S,N,T}$ satisfies (1.3) and (I10), then (S, N) satisfies (LEM) and S is distributive over T.*

PROOF: By Lemma 4.1.1, if $I_{S,N,T}$ satisfies (1.3), then (S, N) satisfies (LEM). Now, by (I10), $S(N(x), T(x, x)) = 1$, and since (S, N) satisfies (LEM), then for any $x \in [0, 1]$, $S(N(x), T(x, x)) = 1 = S(N(x), x)$. Case $x = 1$, so, trivially $T(x, x) = x$. Case $x < 1$, since S is strictly increasing in [0,1[, then $S(N(x), T(x, x)) = S(N(x), x)$ implies $T(x, x) = x$. Therefore $T = T_M$. ■

Corollary 4.1.1 *Let S be a strictly increasing in [0,1[t-conorm. Then the following statements are equivalent.*

1. *A QL-implication $I_{S,N,T}$ satisfies (1.3) and (I10);*
2. *(S, N) satisfies (LEM) and $T = T_M$.*

[1] The fuzzy implication $I(y, x) = max(1 - y, x)$ [55] is the known Reichenbach implication. Clearly, such implication is also an (S,N)-implication.

PROOF: Straightforward from Lemma 3.1.6 and Theorems 4.1.3 and 4.1.4. ■

Note that, if $x \leq y$ iff $I(x,y) = 1$, then $I(x,x) = x$. In other words, if I satisfies (I12), then I satisfies (I10). Therefore, by Theorem 4.1.4, we deduce the following Corollary.

Corollary 4.1.2 *Let S be a strictly increasing in [0,1[t-conorm. If a QL-implication $I_{S,N,T}$ satisfies (1.3) and (I12), then (S,N) satisfies (LEM) and S is distributive over T.*

The converse of Corollary 4.1.2 is not true. The counter-example is $I_{S',T_M,N_\top}$[1] (given below), since $(S', N_\top)$ satisfies (LEM) and $I_{S',T_M,N_\top}$ satisfies (1.3), but it does not satisfy (I12).

$$I_{S',T_M,N_\top}(x,y) = \begin{cases} 1 & \text{, if } x < 1 \\ y & \text{, if } x = 1. \end{cases}$$

Theorem 4.1.5 *Every D-implication satisfies (1.3).*

PROOF: Let $I_{S,T,N}$ be a D-implication, by (3.9), $I_{S,T,N}(y,x) = S(T(N(y),N(x)),x)$ and, regardless of the value of $T(N(y),N(x))$, by Remark 2.1.2, $S(T(N(y),N(x)),x)$ $\geq x$. Hence $I_{S,T,N}$ satisfies (1.3). ■

Theorem 4.1.6 *If N is a strong negation then $I_{N,T}$ satisfies (1.3).*

PROOF: Straightforward by Lemmas 3.1.8 and 3.1.9, or by Lemma 3.1.14 and Theorem 4.1.1. ■

Theorem 4.1.7 *Let h be an h-generator w.r.t. a fixed $e \in]0,1[$, then I^h satisfies (1.3).*

PROOF: Straightforward from Lemmas 4.0.1 and 3.1.10. ■

W.r.t. the automorphisms on I we present the following theorem.

Theorem 4.1.8 *Let $\varphi \in \Phi$. I satisfies (1.3) iff I_φ also satisfies (1.3).*

PROOF: $y \leq I_\varphi(x,y)$ iff
$y \leq \varphi^{-1}(I(\varphi(x),\varphi(y)))$ iff
$\varphi(y) \leq \varphi \circ \varphi^{-1}(I(\varphi(x),\varphi(y)))$ iff
$\varphi(y) \leq I(\varphi(x),\varphi(y))$.

■

[1]In which S' is any strictly increasing in [0,1[t-conorm.

4.2 Final considerations

The main results of this chapter are stated by Theorems 4.1.1, 4.1.2, 4.1.5, 4.1.6, 4.1.7 and 4.1.8 and Corollary 4.1.1. From those results, we can conclude that any (S,N)-, R-, D- and h-implication satisfy (1.3). (N,T)-implications generated by strong negations also satisfy (1.3), but only a particular class of QL-implications satisfies (1.3). We had already seen that a QL-implication is equivalent to an (S,N)-implication and it satisfies (I10) whenever (S,N) satisfies (LEM) and $T = T_M$. On the other hand, considering that S is strictly increasing in [0,1[, (S,N) satisfies (LEM) and $T = T_M$ then $I_{T,S,N}$ satisfies (1.3) and (I10).

Note that there is a close relationship between (I10) and (1.3): Every R-implication satisfies both; every (S,N)-implication where (S,N) satisfies (LEM), also satisfies both; and only QL-implications which satisfy (I10) guarantee the converse of Theorem 4.1.3. We also demonstrated that: if $I_{T,S,N}$ satisfies (1.3) and (I12), then (S,N) satisfies (LEM) and $T = T_M$ (Corollary 4.1.2). But the converse of this Corollary is not true.

Chapter 5

On the Boolean-like Law I(x,I(y,x)) = 1

This chapter will present the relationship between (1.4) and other properties of fuzzy implications and it will present the conditions under which (1.4) holds for the six mentioned classes of fuzzy implications.

The subsection 1.2 showed the relationship between the Boolean laws "$\alpha \Rightarrow (\beta \Rightarrow \alpha)$", LEM and "$\alpha \Rightarrow \alpha$". In a fuzzy context, this is a relationship among (1.4), (LEM) and (I8) which can be analyzed in each fuzzy implication class:

- $I_{S,N}$ satisfies (I8) iff $I_{S,N}$ satisfies (LEM) iff $I_{S,N}$ satisfies (1.4).
- Every R-implication satisfies (I8) and (1.4).
- $I_{S,N,T}$ satisfies (I8), if $N = N_T$ or $T = T_M$ and (S, N) satisfies LEM or $S = S_D$ and $N = \mathtt{N}$ where $\mathtt{N}(x) = 0$ iff $x = 1$ [7, Prop.2.6.21]. Moreover, $I_{S,N_\top,T}$ satisfies (1.4) and if (S,N) satisfies (LEM) then $I_{\mathrm{S,N},T_M}$ also satisfies (1.4).[1]

Other relations between (1.4) and other fuzzy implication properties were already presented in Propositions 4.0.3 and 4.0.4, and Corollary 4.0.1.

In the following section we will demonstrate results about when the Boolean-like law $I(x, I(y, x)) = 1$ remain valid in fuzzy logics (Section 5.1). In Section 5.2 those results will be analyzed regarding aspects presented in the introduction of this dissertation.

[1]Note that $(S_D, \mathtt{N})$ satisfies (LEM) and $I_{S_D,\mathtt{N},T}$ does not satisfy (1.4) for any T.

5.1 Solutions of $I(x, I(y, x)) = 1$ for (S,N)-, R-, QL-, D-, (N,T)- and h-implications

The results contained in this section will be shown for each fuzzy implication class separately.

5.1.1 (S,N)-implications

Theorem 5.1.1 *An (S,N)-implication $I_{S,N}$ satisfies (1.4) iff (S, N) satisfies (LEM).*

Proof: $\Leftarrow$:

$$\begin{aligned} I_{S,N}(x, I_{S,N}(y, x)) &= S(N(x), S(N(y), x)) && \text{by Def. 3.1.1} \\ &= S(N(x), S(x, N(y))) && \text{by } S1 \\ &= S(S(N(x), x), N(y)) && \text{by } S2 \\ &= S(1, N(y)) && \text{by (LEM)} \\ &= 1 && \text{by Remark 2.1.2.} \end{aligned}$$

$\Rightarrow$: For all $x \in [0, 1]$, $I(x, I(1, x)) = 1$ and $I(1, x) = S(N(1), x) = S(0, x) = x$, so $S(N(x), x) = I(x, I(1, x)) = 1$. Hence (S, N) satisfies (LEM). ■

5.1.2 R-implications

It is already known from the theory of residuated lattices that when I is an R-implication defined from a left-continuous t-norm T, case (T, I) is an adjoint pair [7] then I satisfies (1.4). In the following theorem, we will prove that the left-continuity of underlying t-norm is not required for an R-implication satisfy (1.4).

Theorem 5.1.2 *Every R-implication satisfies (1.4).*

Proof: Let T be any t-norm and I_T an R-implication which is generated from it. Fixing arbitrarily $x, y \in [0, 1]$, it is well known that $T(y, x) \leq \min(y, x) \leq x$, so $I_T(y, x) = sup\{t \in [0, 1] \,|\, T(y, t) \leq x\} \geq x$. It is already known (subsection 3.1.2) that R-implications satisfy I3 and I8 [7, Theorem 2.5.4]. Therefore

$$I_T(x, I_T(y, x)) \geq I_T(x, x) = 1.$$

Hence $I_T(x, I_T(y, x)) = 1$. ■

5.1.3 QL-implications

Lemma 5.1.1 *A QL-implication $I_{S,N,T}$ satisfies (1.4), whenever (S,N) satisfies (LEM) and $T = T_M$.*

PROOF: Since $T = T_M$ iff S is distributive over T (Proposition 2.2.1(i)), then

$I(x, I(y,x)) =$	
$= S(N(x), T(x, S(N(y), T(y,x))))$	*by* Def. 3.1.4
$= S(N(x), T(x, T(S(N(y),y), S(N(y),x))))$	*by* (2.5)
$= S(N(x), T(x, T(1, S(N(y),x))))$	*by* (LEM)
$= S(N(x), T(x, S(N(y),x)))$	*by* $T4$
$= T(S(N(x),x), S(N(x), S(N(y),x)))$	*by* (2.5)
$= T(1, S(N(x), S(x, N(y))))$	*by* (LEM) and $S1$
$= S(S(N(x),x), N(y))$	*by* $T4$ and $S2$
$= S(1, N(y))$	*by* (LEM)
$= 1$	*by* Remark 2.1.2.

■

Lemma 5.1.2 *If a QL-implication $I_{S,N,T}$ is generated by a strictly increasing in [0,1[t-conorm S, a t-norm T and a fuzzy negation N satisfies (1.4), then (S,N) satisfies (LEM) and $T = T_M$.*

PROOF: Assume that $I_{S,N,T}$ is a QL-implication which satisfies (1.4), then $I_{S,N,T}(1, I_{S,N,T}(y,1)) = 1$. Therefore, for any $y \in [0,1]$, $S(N(1), T(1, S(N(y), T(y,1)))) = 1$ and:

$S(N(1), T(1, S(N(y), T(y,1)))) =$	
$= S(0, T(1, S(N(y), T(1,y))))$	*by* $N1$, and $T1$
$= S(N(y), y))$	*by* $S4$ and $T4$.

Thus $S(N(y),y) = 1$. Hence (S,N) satisfies (LEM). Moreover since $I_{S,N,T}$ is a QL-implication which satisfies (1.4) then $I_{S,N,T}(x, I_{S,N,T}(1,x)) = 1$. Therefore, for any $x \in [0,1]$, $S(N(x), T(x, S(N(1), T(1,x)))) = 1$ and:

$S(N(x), T(x, S(N(1), T(1,x)))) =$	
$= S(N(x), T(x, S(0, T(1,x))))$	*by* $N1$
$= S(N(x), T(x, T(1,x)))$	*by* $S4$
$= S(N(x), T(x,x))$	*by* $T4$.

Thus $S(N(x), T(x,x)) = 1$. Since (S,N) satisfies (LEM), for any $x \in [0,1]$, $S(N(x), T(x,x)) = 1 = S(N(x),x)$. Case $x = 1$, so, $T(x,x) = x$. Case $x < 1$, since S is strictly increasing in [0,1[, then $S(N(x), T(x,x)) = S(N(x), x)$ implies $T(x,x) = x$. Therefore $T(x,x) = x$, i.e. T is an idempotent t-norm, for any $x \in [0,1]$ and T_M is the only idempotent t-norm. Hence, if I is a QL-implication — generated by a strictly increasing in [0,1[t-conorm S and a fuzzy negation N — which satisfies (1.4) then (S, N) satisfies (LEM) and so $T = T_M$. ∎

Theorem 5.1.3 *Let $I_{S,N,T}$ be a QL-implication generated by a strictly increasing in [0,1[t-conorm S, a fuzzy negation N and a t-norm T. I satisfies (1.4) iff (S, N) satisfies (LEM) and $T = T_M$.*

Proof: Straightforward from Lemmas 5.1.1 and 5.1.2. ∎

5.1.4 D-implications

Theorem 5.1.4 *Let $I_{S,T,N}$ be a D-implication generated by a strictly increasing in [0,1[t-conorm S, a t-norm T and a strong fuzzy negation N such that I satisfies the contraposition (I15). Then I satisfies (1.4) iff (S, N) satisfies (LEM) and $T = T_M$.*

Proof: See subsection 3.1.4 and Theorem 5.1.3. ∎

The result of the theorem above comes trivially. But there is another condition to guarantee the satisfiability of (1.4) for D-implications. The following lemmas and theorem show it.

Lemma 5.1.3 *Given a D-implication $I_{S,T,N}$ generated by a strictly increasing in [0,1[t-conorm S, a t-norm T and a fuzzy negation N. I satisfies (1.4), if (S, N) satisfies (LEM) and $T = T_M$.*

Proof: Straightforward from Theorem 3.1.2 and Theorem 5.1.3. ∎

Lemma 5.1.4 *Given a D-implication $I_{S,T,N}$ generated by a strictly increasing in [0,1[t-conorm S, a t-norm T and a continuous fuzzy negation N. If $I_{S,T,N}$ satisfies (1.4), then (S, N) satisfies (LEM) and $T = T_M$.*

Proof: Let $I_{S,T,N}$ be a D-implication which satisfies (1.4), so $I_{S,T,N}(0, I_{S,T,N}(y, 0)) = 1$. Therefore, $S(T(N(0), N(S(T(N(y), N(0)), 0))), S(T(N(y), N(0)), 0))$ $= 1$ for any $y \in [0,1]$. By N1, T4 and S4, $S(T(N(y), N(0)), 0) = N(y)$ $(*)$, so: $S(T(N(0), N(S(T(N(y), N(0)), 0))), S(T(N(y), N(0)), 0)) =$

$= S(T(N(0), N(N(y))), N(y))$ by $(*)$
$= S(N(N(y)), N(y))$ *by* $N1$ and $T4$.

Since N is continuous, then for all $y' \in [0,1]$ there exists $y \in [0,1]$ such that $N(y) = y'$. Therefore $S(N(y'), y') = S(N(N(y)), N(y)) = 1$. Hence (S, N) satisfies (LEM).

Moreover, again by (1.4), $I_{S,T,N}(x, I_{S,T,N}(1,x)) = 1$, and $I_{S,T,N}(1,x) = S(T(0, N(x)), x) = x$. Then $I_{S,T,N}(x,x) = 1 = S(T(N(x), N(x)), x)$. It is known that $T(N(x), N(x)) \leq N(x)$. If $T(N(x), N(x)) < N(x)$ and since (S, N) satisfies (LEM) and S is strictly increasing in [0,1[, then $S(T(N(x), N(x)), x) < S(N(x), x) = 1$. However, $S(T(N(x), N(x)), x) = 1$. So $T(N(x), N(x))$ must not be less than $N(x)$. Thus $T(N(x), N(x)) = N(x)$. Once that N is continuous, for all $x' \in [0,1]$ there exists $x \in [0,1]$ such that $N(x) = x'$. Therefore $T(x', x') = x'$, for all $x' \in [0,1]$, i.e. T is an idempotent t-norm. Hence $T = T_M$ [40, theorem 3.9]. ■

Theorem 5.1.5 *Let $I_{S,T,N}$ be a D-implication generated by a strictly increasing in [0,1[t-conorm S, a t-norm T and a continuous fuzzy negation N. $I_{S,T,N}$ satisfies (1.4) whenever (S, N) satisfies (LEM) and $T = T_M$.*

Proof: Straightforward from Lemmas 5.1.3 and 5.1.4. ■

5.1.5 (N,T)-implications

Theorem 5.1.6 *Let N be a strong negation and T a t-norm. A (N,T)-implication $I_{N,T}$ satisfies (1.4) iff there exists a N-dual t-conorm S of T such that (S, N) satisfies (LEM).*

Proof: Straightforward from Lemma 3.1.14. ■

5.1.6 h-implications

Theorem 5.1.7 *Let h be an h-generator w.r.t. a fixed $e \in]0,1[$, then I^h does not satisfy (1.4).*

Proof: Assume that $y = 1$, so $I^h(1,x) = x$, by (I9), and it is known that $I^h(x,x) = 1$ iff $x = 0$ or $x = 1$ (see [49, Theorem 5(v)]). Therefore $I^h(x, I^h(1,x)) = I^h(x,x) \neq 1$, for any $x \in]0,1[$. Hence I^h does not satisfies (1.4). ■

h-implications do not satisfy (1.4) only when $y = 1$. See the next proposition.

Proposition 5.1.1 *Let h be an h-generator w.r.t. a fixed $e \in]0,1[$. If $e \leq x > 0$ and $y > 0$, then I^h does not satisfies (1.4).*

PROOF: Assume $e \leq x > 0$ and $y > 0$, so, by Def. 3.11: if $I^h(y,x) \leq e$ then $I^h(x, I^h(y,x)) = h^{-1}(x \cdot y \cdot h(x))$; and if $I^h(y,x) > e$ then $I^h(x, I^h(y,x) = h^{-1}(\frac{y}{x} \cdot h(x))$. In both cases, $h(x) \in \mathbb{R}^- \cup \{0\}$ and, x and y are positive. Therefore $I^h(x, I^h(y,x)) \in [0,e]$. Hence I^h does not satisfies (1.4) whenever $e \leq x > 0$ and $y > 0$. ■

5.1.7 Φ-conjugate fuzzy implications

Theorem 5.1.8 *Let $\varphi \in \Phi$. A fuzzy implication I satisfies (1.4) iff I_φ satisfies (1.4).*

PROOF: ⇒: Assume that $\varphi \in \Phi$ and a fuzzy implication I satisfies (1.4). Now by (3.12) we have that $I_\varphi(x, I_\varphi(y,x)) = \varphi^{-1}(I(\varphi(x), \varphi \circ \varphi^{-1}(I(\varphi(y), \varphi(x)))))$ and trivially $\varphi^{-1}(I(\varphi(x), \varphi \circ \varphi^{-1}(I(\varphi(y), \varphi(x))))) = \varphi^{-1}(I(\varphi(x), I(\varphi(y), \varphi(x))))$. By (1.4), $I(\varphi(x), I(\varphi(y), \varphi(x))) = 1$, so $\varphi^{-1}(I(\varphi(x), I(\varphi(y), \varphi(x)))) = \varphi^{-1}(1) = 1$. Therefore $I_\varphi(x, I_\varphi(y,x)) = 1$. Hence I_φ satisfies (1.4). The converse follows straightforward from $\varphi^{-1}((I_\varphi)) = I$. ■

5.2 Final considerations

This chapter provided necessary and sufficient conditions under which the Boolean-like law $I(x, I(y,x)) = 1$, referred by (1.4), holds for (S,N)-, R-, QL-, D-, (N,T)- and h-implications.

The main results are displayed in Theorems 5.1.1, 5.1.2, 5.1.3. 5.1.4, 5.1.5, 5.1.6, 5.1.7 and 5.1.8. Such results show that there is no common set of properties that guarantees the validity of (1.4) for all of those classes. However, there is a strong relation between (1.4) and (LEM).

The results obtained in this chapter may imply that a generalization of a classical implication to a fuzzy context — an (S,N)-implication — satisfies (1.4) iff (S,N) satisfies (LEM). The dual implication of such generalization must obey similar necessary and sufficient conditions: A (N,T)-implication satisfies (1.4) iff

N is strong and (S,N) satisfies (LEM), such that S is the N-dual of T. A quantum implication in a fuzzy context — QL-implication — satisfies (1.4) iff (S, N) satisfies (LEM) and $T = T_M$, given that its underlying t-conorm must be strictly increasing in [0,1[. The contraposition of the quantum implication generalization — D-implication — has an additional condition: its underlying negation must be continuous. We have also proved that any R-implication satisfies (1.4) and h-implications do not satisfy (1.4).

These observations lead us to conclude that the Boolean-like law (1.4) cannot be indiscriminately adopted in any computational system based on fuzzy concepts. Other properties must be regarded in addition to (1.4) for this one to be adopted by FRBSs.

Finally, we have also proved that if a new implication is defined through an automorphism, this new implication will satisfy (1.4) iff the original fuzzy implication also satisfies it — Theorem 5.1.8.

Chapter 6

On the Boolean-like Law $I(x,I(y,z)) = I(I(x,y),I(x,z))$

This chapter will investigate solutions of (1.5) for the six fuzzy implication classes, but it will, firstly, prove some relationships between (1.5) and other fuzzy implication properties.

Proposition 6.0.1 *A fuzzy implication I satisfies (I14), if I satisfies (1.5), (I10) and (I9).*

PROOF: Respectively by (1.5), (I10) and (I9), $I(x, I(x, y)) = I(I(x, x), I(x, y)) = I(1, I(x, y)) = I(x, y)$. ∎

Corollary 6.0.1 *If an R-implication I satisfies (1.5) then I satisfies (I14).*

PROOF: Straightforward from Proposition 6.0.1 and Lemma 3.1.3. ∎

Lemma 6.0.1 *If a fuzzy implication I satisfies (1.5) and (I10), then I satisfies (I8).*

PROOF: If I satisfies (1.5), then $I(x, I(t, t)) = I(I(x, t), I(x, t))$. By (I10), $I(I(x, t), I(x, t)) = 1$ and so $I(x, I(t, t)) = 1$. Thus, again by (I10), $I(x, 1) = I(x, I(t, t)) = 1$, for each $x \in [0, 1]$. Hence I satisfies (I8). ∎

Corollary 6.0.2 *If a fuzzy implication I satisfies (1.5) and (I10), then I satisfies (1.4).*

PROOF: By Lemma 6.0.1, if I satisfies (1.5) and (I10) then I satisfies (I8). Thus, by (1.5), (I10) and (I8), respectively: $I(x, I(y, x) = I(I(x, y), I(x, x)) = I(I(x, y), 1) = 1$. ■

The following section will present results about when the Boolean-like law (1.5) remain valid in fuzzy logics. Section 6.2 will analyse those results regarding aspects presented in the introduction of this thesis.

6.1 Solutions of I(x,I(y,z))=I(I(x,y),I(x,z)) for (S,N)-, R-, QL-, D-, (N,T)- and h-implications

Solutions of (1.5) will be studied separately for each fuzzy implication class in each following subsection.

6.1.1 (S,N)-implications

Theorem 6.1.1 *Given an (S,N)-implication $I_{S,N}$. $I_{S,N}$ satisfies (1.5) whenever (S, N) satisfies (LEM) and for some t-norm T, (T, S, N) is a triple that satisfies the Distributivity over De Morgan (Eq. 2.9).*

PROOF: Let T be a t-norm such that (T, S, N) is a triple that satisfies the Distributivity over De Morgan.

$$\begin{aligned}
&I_{S,N}(I_{S,N}(x, y), I_{S,N}(x, z)) = \\
&= S(N(S(N(x), y)), S(N(x), z)) && by\, Def.\, 3.1.1 \\
&= S(S(N(x), z), N(S(N(x), y))) && by\, S1 \\
&= T(S(S(N(x), z), N(N(x))), S(S(N(x), z), N(y))) && by\, (2.9) \\
&= T(S(S(N(N(x)), N(x)), z), S(N(x), S(N(y), z))) && by\, S1\, and\, S2 \\
&= T(S(1, z), S(N(x), S(N(y), z))) && by\, (LEM) \\
&= T(1, S(N(x), S(N(y), z))) && by\, Remark\, 2.1.2 \\
&= S(N(x), S(N(y), z)) && by\, (T4) \\
&= I_{S,N}(x, I_{S,N}(y, z)) && by\, Def.\, 3.1.1
\end{aligned}$$

■

Note that $(S_M, N_\top)$ satisfies (LEM) and $I_{S_M,N_\top}$ satisfies (1.5), but (T_P,S_M, $N_\top$) does not satisfy (2.9). Thus the converse of the Theorem 6.1.1 is not true and therefore a weaker version of (2.9) will be required. In this case, such weakening is the property (2.11) (notice that $(T_P, S_M, N_\top)$ satisfies (2.11) — see Example 2.2.1 for more details.

Theorem 6.1.2 *Let $I_{S,N}$ be an (S,N)-implication generated by a continuous t-conorm S and a continuous fuzzy negation N. If $I_{S,N}$ satisfies (1.5) then (S, N) satisfies (LEM) and for any t-norm T, (T, S, N) is a triple that satisfies (2.11).*

PROOF: Let $x \in [0, 1]$. Since, N is continuous, decreasing, $N(1) = 0$ and $N(0) = 1$ then, by the intermediate value theorem, exists $y \in [0, 1]$ such that $N(y) = x$. It is known that $I_{S,N}$ satisfy (I8) and (I1) [7, Prop. 2.4.3 (i)]. Assume that $I_{S,N}$ satisfies (1.5), so:

$$\begin{aligned}
S(N(x), x) &= S(N(N(y)), N(y)) & \\
&= I_{S,N}(N(y), N(y)) & \text{by Def. 3.1.1} \\
&= I_{S,N}(\mathtt{N}_{I_{S,N}}(y), \mathtt{N}_{I_{S,N}}(y)) & \text{by Prop. 3.1.1} \\
&= I_{S,N}(I_{S,N}(y, 0), I_{S,N}(y, 0)) & \text{by def. of } N_I \\
&= I_{S,N}(y, I(0, 0)) & \text{by (1.5)} \\
&= I_{S,N}(y, 1) & \text{by (I1)} \\
&= 1 & \text{by (I8)}
\end{aligned}$$

Let $x, y, z \in [0, 1]$ such that $0 \le y \le x \le 1$. Again, since N is continuous, decreasing, $N(0) = 1$ and $N(1) = 0$, then by the intermediate value theorem, there exists $y' \in [0, 1[$ such that $(*)$ $N(y') = y$. Analogously, since S is continuous, increasing, $S(y, 0) = y$ and $S(y, 1) = 1$, then by the intermediate value theorem there exist $x' \in [0, 1]$ such that $(**)$ $x = S(N(y'), x')$.

$$\begin{aligned}
&T(S(x, N(y)), S(x, N(z))) & \\
&= T(S(S(N(y'), x'), N(y)), S(S(N(y'), x'), N(z))) & by\, (**) \\
&= T(S(S(N(y'), x'), N(N(y'))), S(S(N(y'), x'), N(z))) & by\, (*) \\
&= T(S(S(N(N(y')), N(y')), x'), S(N(y'), S(N(z), x'))) & by\, S1\, \&\, S2 \\
&= T(S(1, x'), S(N(y'), S(N(z), x'))) & by\, (LEM) \\
&= S(N(y'), S(N(z), x')) & by\, Rem.\, 2.1.2\, \&\, (T4) \\
&= I_{S,N}(y', I_{S,N}(z, x')) & by\, Def.\, 3.1.1 \\
&= I_{S,N}(I_{S,N}(y', z), I_{S,N}(y', x')) & by\, (1.5) \\
&= S(N(S(N(y'), z)), S(N(y'), x')) & by\, Def.\, 3.1.1 \\
&= S(N(S(y, z)), x) & by\, (*)\, and\, (**) \\
&= S(x, N(S(y, z))) & by\, S1
\end{aligned}$$

Hence (T, S, N) satisfies (2.11). ∎

Proposition 6.1.1 *Let $I_{S,N}$ be an (S,N)-implication generated by a t-conorm S and a continuous fuzzy negation N. If $I_{S,N}$ satisfies (1.5), then $I_{S,N}$ satisfies (1.4).*

PROOF: By Theorem 6.1.2, if $I_{S,N}$ satisfies (1.5), (S, N) satisfies (LEM) and therefore $I_{S,N}$ satisfies (1.4) (by Theorem (5.1.1)). ■

The above theorems for (S,N)-implications are based on continuous negations and they relate the property (1.5) to (LEM), (2.9) and a weakening of the last. The next subsections will detail our investigation into the sub-classes of (S,N)-implications.

6.1.1.1 S-implications

Theorem 6.1.3 *There are not a t-norm T, a t-conorm S and a strict negation N such that (S, N) satisfies (LEM) and (T, S, N) satisfies (2.9).*

PROOF: By Lemma 3.1.2, N is a strict negation and (T, S, N) satisfies (2.9) iff $T = T_M$ and $S = S_M$. Only $(S_M, N_\top)$ satisfies (LEM) — see [7, Remark 2.3.10(i)] —, but $N_\top$ is not strict. ■

The next examples show two cases where S-implication does not satisfy (1.5). In the first case, (2.9) is also unsatisfied and in the second one, it (LEM) is also unsatisfied.

Example 6.1.1 *Consider the strong fuzzy negation N_Z and the nilpotent t-conorm*

$$S_{nM}(x, y) = \begin{cases} 1 & , \textit{if } x + y \geq 1 \\ max(x, y) & , \textit{otherwise.} \end{cases}$$

S_{nM} is N_Z-dual of the nilpotent t-norm T_{nM} in which:

$$T_{nM}(x, y) = \begin{cases} 0 & , \textit{if } x + y \leq 1 \\ \min(x, y) & , \textit{otherwise.} \end{cases}$$

Indeed, S_{nM} satisfies (LEM). The S-implication generated by N_Z and S_{nM} is the Fodor implication

$$I_{FD}(x, y) = \begin{cases} 1 & , \textit{if } x \leq y \\ max(1 - x, y) & , \textit{if } x > y \end{cases}$$

S_{nM} is not distributive over T_{nM} (see (6.1)) and consequently (1.5) does not hold (see (6.2)).

$$\begin{aligned} &S_{nM}(0.5, T_{nM}(0.5, 0.5)) = S_{nM}(0.5, 0) = 0.5. \\ &T_{nM}(S_{nM}(0.5, 0.5), S_{nM}(0.5, 0.5)) = T_{nM}(1, 1) = 1. \end{aligned} \tag{6.1}$$

$$\begin{aligned} &I_{FD}(0.3, I_{FD}(0.4, 0.2)) = I_{FD}(0.3, 0.6) = 1. \\ &I_{FD}(I_{FD}(0.3, 0.4), I_{FD}(0.3, 0.2)) = I_{FD}(1, 0.7) = 0.7. \end{aligned} \tag{6.2}$$

Example 6.1.2 *Consider the strong fuzzy negation N_Z and the t-conorm S_M which is the N_Z-dual of T_M. S_M is distributive over T_M (see Proposition 2.2.1). The S-implication generated by N_Z and S_M is the Kleene-Dienes implication $I_{KD}(x, y) = max(1 - x, y)$ [36], but I_{KD} does not satisfy (1.5) and (S_M, N_Z) does not satisfy (LEM).*

6.1.1.2 nS-implications

Let $N_\alpha(x)$ be the following family of non-continuous negations and therefore $N_\alpha(x)$ is also a family of non-strong negations:

$$N_\alpha(x) = \begin{cases} 1 & \text{, if } x = 0 \\ 0 & \text{, if } x = 1 \\ \alpha & \text{, otherwise.} \end{cases}$$

Theorem 6.1.4 *Let S be a positive t-conorm and $\alpha \in [0, 1]$. Then I_{S,N_α} satisfies (1.5) iff $\alpha = 0$ or $\alpha = 1$.*

Proof: Case $\alpha = 0$, then $I_{S,N_\alpha} = I_{S,N_\perp}$, i.e.

$$I_{S,N_\perp}(x, y) = \begin{cases} 1 & \text{, if } x = 0 \\ y & \text{, if } x > 0. \end{cases}$$

By the definition of $I_{S,N_\perp}$, if $I_{S,N_\perp}(x, I_{S,N_\perp}(y, z)) = 1$, then $x = 0$ (case 1) or $x > 0$ and $I(y, z) = 1$. If $I(y, z) = 1$ then $y = 0$ (case 2) or $y = 1$ and $z = 1$ (case 3). On the other hand:

Case 1: If $x = 0$ then $I_{S,N_\perp}(I_{S,N_\perp}(x, y), I_{S,N_\perp}(x, z)) = I_{S,N_\perp}(1, 1) = 1$;

Case 2: If $x > 0$ and $y = 0$ then $I_{S,N_\perp}(I_{S,N_\perp}(x, 0), I_{S,N_\perp}(x, z)) = I_{S,N_\perp}(0, z) = 1$;

Case 3: If $x > 0$ and $y = z = 1$ then $I_{S,N_\perp}(I_{S,N_\perp}(x, 1), I_{S,N_\perp}(x, 1)) = 1$.

Again by the definition of $I_{S,N_\perp}$, if $I_{S,N_\perp}(x, I_{S,N_\perp}(y,z)) < 1$ then $x > 0$ and $I_{S,N_\perp}(y,z) < 1$. Thus $x > 0$, $y > 0$ and $z < 1$, and so $I_{S,N_\perp}(x, I_{S,N_\perp}(y,z)) = z$. On the other side, $I_{S,N_\perp}(I_{S,N_\perp}(x,y), I_{S,N_\perp}(x,z)) = I_{S,N_\perp}(y,z) = z$.

Case $\alpha = 1$, then $I_{S,N_\alpha} = I_{S,N_\top}$, i.e.

$$I_{S,N_\top}(x,y) = \begin{cases} 1 & , \text{ if } x < 1 \\ y & , \text{ if } x = 1. \end{cases}$$

By the definition of $I_{S,N_\top}$, if $I_{S,N_\top}(x, I_{S,N_\top}(y,z)) = 1$, then $x < 1$ (case 1) or $x = 1$ and $I(y,z) = 1$. If $I(y,z) = 1$ then $y < 1$ (case 2) or $y = 1$ and $z = 1$ (case 3). On the other side:

Case 1: If $x < 1$ then $I_{S,N_\top}(I_{S,N_\top}(x,y), I_{S,N_\top}(x,z)) = I_{S,N_\top}(1,1) = 1$;

Case 2: If $x = 1$ and $y < 1$ then $I_{S,N_\top}(I_{S,N_\top}(1,y), I_{S,N_\top}(1,z)) = I_{S,N_\top}(y,z) = 1$;

Case 3: If $x = y = z = 1$ then $I_{S,N_\top}(I_{S,N_\top}(1,1), I_{S,N_\top}(1,1)) = 1$.

Again by the definition of $I_{S,N_\top}$, if $I_{S,N_\top}(x, I_{S,N_\top}(y,z)) < 1$ then $x = 1$ and $I_{S,N_\top}(y,z) < 1$. Thus $x = y = 1$ and $z < 1$, consequently $I_{S,N_\top}(x, I_{S,N_\top}(y,z)) = z$. On the other side, $I_{S,N_\top}(I_{S,N_\top}(x,y), I_{S,N_\top}(x,z)) = I_{S,N_\top}(I_{S,N_\top}(1,1), I_{S,N_\top}(1,z)) = I_{S,N_\top}(1,z) = z$.

Case $\alpha \in]0,1[$, then

$$I_{S,N_\alpha}(x,y) = \begin{cases} 1 & , \text{ if } x = 0 \\ y & , \text{ if } x = 1 \\ S(\alpha, y) & , \text{ if } x \in]0,1[\end{cases}$$

But, on the one hand, $I_{S,N_\alpha}(0.5, I_{S,N_\alpha}(0,0)) = S(N_\alpha(0.5), 1) = S(\alpha, 1) = 1$ and on the other hand, $I_{S,N_\alpha}(I_{S,N_\alpha}(0.5,0), I_{S,N_\alpha}(0.5,0)) = I_{S,N_\alpha}(S(\alpha,0), S(\alpha,0)) = S(\alpha,\alpha)$. Since S is positive, $S(\alpha,\alpha) \neq 1$. ∎

$I_{S,N_\top}$ is the known Weber implication I_W (defined in [75]) and it is the greatest (S,N)-implication [7]. While $I_{S,N_\perp}$ is the least one.

Note that the problem when $\alpha \in]0,1[$ can be generalized for any negation with an equilibrium point.

Theorem 6.1.5 *If N is a fuzzy negation with equilibrium point and S a positive t-conorm then $I_{S,N}$ does not satisfy (1.5).*

PROOF: Analogous to the Theorem 6.1.4. ■

Remark 6.1.1 *Now, we present some examples of the family of I_{S,N_α} in which S is a non-positive t-conorm.*

1) $I_{S,N_\perp}$ and $I_{S,N_\top}$ remain satisfying (1.5).

2) I_{S_D,N_α} satisfies (1.5).

3) I_{S_{LK},N_α} does not satisfy (1.5).

Example 6.1.3 *By Corollary 2.2.1, if a triple (T,S,N) satisfies (2.9) then $T = T_M$. And, by Proposition 2.2.5, $(T_M, S, N_\top)$ satisfies (2.9), for any S. Moreover, for any S, $(S, N_\top)$ satisfies (LEM). Finally, it is known that $I_{S,N_\top} = I_W$ (defined below) [7] and I_W satisfies (1.5).*

$$I_{S,N_\top}(x,y) = I_W(x,y) = \begin{cases} 1 & , \textit{if } x < 1 \\ y & , \textit{if } x = 1 \end{cases} = I_W(x,y)$$

Proposition 6.1.2 *$I_{S,N}$ satisfies (1.5) does not imply that (S,N) satisfies (LEM).*

PROOF: For any t-conorm S, $(S, N_\perp)$ does not satisfies (LEM). However, by Theorem 6.1.4, if $\mathcal{S}$ is a positive t-conorm, $(\mathcal{S}, N_\perp)$-implication satisfies (1.5). Hence the satisfiability of (1.5) by an (S,N)-implication $I_{S,N}$ is not a sufficient condition for (S,N) to satisfy (LEM). ■

6.1.1.3 Final considerations w.r.t. (S,N)-implications and (1.5)

We proved in this subsection that $I_{S,N}$ satisfies (1.5) if the pair (S,N) satisfies (LEM) and the triple (T,S,N) satisfies (2.9) (Theorem 6.1.1) and if $I_{S,N}$ satisfies (1.5) then (S,N) satisfies (LEM) and the triple (T,S,N) satisfies (2.11) (Theorem 6.1.2). To exploit these results, we show that if N is a strong fuzzy negation then there is not a pair (S,N) that satisfies (LEM) and a triple (T,S,N) that satisfies (2.9) (Theorem 6.1.3). Moreover, if N_α is a family of non-strong fuzzy negations and $\mathbf{S}$ is a positive t-conorm then $I_{\mathbf{S},\mathbf{N}_\alpha}$ satisfies (1.5) iff $\alpha = 0$ or $\alpha = 1$ (Theorem 6.1.4). Also, if $\mathbf{N}$ is a fuzzy negation with equilibrium point then $I_{\mathbf{S},\mathbf{N}}$ does not satisfy (1.5) (Theorem 6.1.5). Now, if S is not positive then $I_{S,N}$ may (or may not) satisfies (1.5) (remark 6.1.1).

Some questions are still open w.r.t. the characterization of (S,N)-implications satisfying property (1.5). For example, it is not known if there is some S-implication which satisfies (1.5). It is not known which class of $I_{S,N}$, whose underlying t-conorm S is not positive, satisfies (1.5). Finally, it is not known if the set of (S,N)-implications that satisfies the conditions of Theorems 6.1.1 and 6.1.2 is empty.

Finally, by Proposition 6.1.1, we also showed that if an (S,N)-implication I satisfies (1.5) then I also satisfies (1.4).

6.1.2 R-implications

Theorem 6.1.6 *An R-implication I satisfies (I6), (I11), (I12), right-continuity in the second argument and (1.5) iff I is the Gödel implication I_G, i.e.,*

$$I(x,y) = I_G(x,y) = \begin{cases} 1 & , \textit{if } x \leq y \\ y & \textit{otherwise} \end{cases}$$

PROOF:
($\Rightarrow$) For all $x, t \in [0,1]$, $I(x,t) = 1$ (and so $I(x,t) \geq x$) iff $x \leq t$ (by (I12)). Now assume there is $t' \in [0, x[$ such that $I(x,t') \geq x$, then, by (I12), $I(x, I(x,t')) = 1$. From this and from (1.5), we have $I(I(x,x), I(x,t')) = I(x, I(x,t')) = 1$. By (I10), $I(I(x,x), I(x,t')) = I(1, I(x,t'))$, and $I(1, I(x,t')) = I(x,t')$ by (I9). Therefore $I(x,t') = 1$ and so $x \leq t'$, by (I12), what is a contradiction. Hence for all $t' \in [0, x[$, $I(x,t') < x$ and so, if $I(x,t') \geq x$ then $t' \geq x$.

If I is right continuous w.r.t. the second variable, then T_I can be defined by (3.7) — subsection 3.1.2. And from Lemma 3.1.4 and Theorem 3.1.1, I satisfies (I6), (I11), (I12) and it is right-continuous in the second argument iff I is based on a left-continuous T such that, $T = T_I$ and $I = I_{T_I}$.Therefore $T_I(x,x) = \min\{t \in [0,1] \mid I(x,t) \geq x\}$ and since $I(x,t) \geq x$ so $t \geq x$, then $T_I(x,x) = x$, for all $x \in [0,1]$. Hence $T_I = T_M$.

$$I(x,y) = sup\{t \in [0,1] \mid \min(x,t) \leq y\} = \begin{cases} 1 & , \text{if } x \leq y \\ y & \text{otherwise} \end{cases} = I_G(x,y).$$

($\Leftarrow$) It is easy to see that I_G satisfies (I6), (I11), (I12), right-continuity in the second argument and (1.5). ∎

Corollary 6.1.1 *I_G is the only right continuous R-implication which satisfies (I14).*

PROOF: Straightforward from Theorem 6.1.6 and Proposition 6.0.1. ∎

6.1.3 QL-implications

Theorem 6.1.7 *Given a QL-implication $I_{S,N,T}$ generated by a t-norm T, a positive t-conorm S and a fuzzy negation N. $I_{S,N,T}$ satisfies (1.5) and (I10) iff $N = N_{\top}$.*

Proof: $\Rightarrow$: By Lemma 6.0.1, if a fuzzy implication I satisfies (1.5) and (I10) so I satisfies (RB). Therefore $I_{S,N,T}(x, 1) = 1$ for each $x \in [0, 1]$. By Definition 3.1.4 and (T4), $I_{S,N,T}(x, 1) = S(N(x), T(x, 1)) = S(N(x), x)$. Since S is positive and $S(N(x), x) = 1$, by Proposition 2.2.11, $N = N_{\top}$. For any t-norm T and any t-conorm $\mathbf{S}$, $I_{\mathbf{S},\mathbf{N}_{\top},\mathbf{T}} = I_W$ [7, Table 2.8].

$\Leftarrow$: If $N = N_{\top}$ then for any t-norm T and any t-conorm $\mathbf{S}$, $I_{\mathbf{S},\mathbf{N}_{\top},\mathbf{T}} = I_W$ [7, Table 2.8] and therefore I_W trivially satisfies (I10) and, by Example 6.1.3, I_W satisfies (1.5). ∎

Corollary 6.1.2 *Given a QL-implication $I_{S,N,T}$ generated by a positive t-conorm S, a fuzzy negation N and a t-norm T. $I_{S,N,T}$ satisfies (1.5) and (I10) iff $I_{S,N,T} = I_{S,N_{\top},T} = I_W$.*

Proof: Straightforward from Theorem 6.1.7. ∎

But what happens if the t-conorm is not positive? We present some examples to answer this question as follows: Let S_D be the drastic t-conorm defined below.

$$S_D(x, y) = \begin{cases} 1 & \text{, if } x, y > 0 \\ \max(x, y) & \textit{otherwise.} \end{cases}$$

So, 1) For any t-norm T and t-conorm S, $I_{S,N_{\top},T} = I_W$, so for any non-positive t-conorm $\mathtt{S}$, $I_{\mathtt{S},N_{\top},T}$ remain satisfying (1.5) and (I10).

2) For any t-norm T and t-conorm S,

$$I_{S,N_{\perp},T} = \begin{cases} T(x, y) & \text{, if } x > 0 \\ 1 & \text{, if } x = 0 \end{cases}$$

and $I_{T,S,N_{\perp}}$ does not satisfy (1.5);

3) $I_{S_D,N_Z,T_M} = I_{S_D,N_Z,T_P}$ satisfy (1.5) and (I10);

4) I_{S_{LK},N_Z,T_M}, I_{S_{LK},N_Z,T_P} and I_{T_{LK},S_{LK},N_Z} are, respectively, the well-known Lukasiewics implication [46], Reichenbach implication [55] and Kleene-Dienes im-

plication , and they do not satisfy (1.5). Of the three, only Łukasiewics implication satisfies (I10).

To expand the results w.r.t. QL-implications, we present the following theorems.

Theorem 6.1.8 *Let T be a t-norm, S a t-conorm and N a continuous fuzzy negation. If a QL-implication $I_{S,N,T}$ satisfies (2.9) and (LEM) then $I_{S,N,T}$ satisfies (1.5).*

PROOF: $I_{S,N,T} = I_{S,N}$ whenever (S, N) satisfies (LEM) and (T, S, N) satisfies (2.4) (by Lemma 3.1.12). Therefore, if $I_{S,N,T}$ is a QL-implication which satisfies (2.9) and (LEM) then $I_{S,N,T}$ satisfies (1.5) — by Proposition 2.2.4 and Theorem 6.1.1. ■

Theorem 6.1.9 *Let T be a t-norm, S a continuous t-conorm and N a continuous fuzzy negation. If $T = T_M$, (S, N) satisfies (LEM) and $I_{S,N,T}$ satisfies (1.5), then $I_{S,N,T}$ satisfies (2.11).*

PROOF: Straightforward from Lemma 3.1.12 and Theorem 6.1.2. ■

6.1.4 D-implications

From the equivalences between D- and QL-implications showed by [48] and the Lemma 3.1.13, we prove the following theorems.

Theorem 6.1.10 *Given a D-implication $I_{S,T,N}$ generated by a t-conorm S, a t-norm T and a strong fuzzy negation N. If $I_{S,T,N}$ satisfies (I15), (1.5) and (I10) then $N = N_\top$.*

PROOF: Straightforward from the fact that if $I_{S,T,N}$ satisfies (I15) and N is strong then $I_{S,T,N} = I_{S,N,T}$ [47] and Theorem 6.1.7. ■

Theorem 6.1.11 *Given a D-implication $I_{S,T,N}$ generated by a t-norm T, a positive t-conorm S and a fuzzy negation N. If (S, N) satisfies (LEM), $T = T_M$ and, $I_{S,T,N}$ satisfies (1.5) and (I10) then $N = N_\top$.*

PROOF: Straightforward from Proposition 3.1.2 and Theorem 6.1.7. ■

Corollary 6.1.3 *Given a D-implication $I_{S,T,N}$ generated by a t-norm T, a positive t-conorm S and a fuzzy negation N. If (S, N) satisfies (LEM), $T = T_M$ and, $I_{S,T,N}$ satisfies (1.5) and (I10) then $I_{S,T,N} = I_{S,T,N_\top} = I_W$.*

PROOF: Straightforward from Theorem 6.1.11. ∎

Theorem 6.1.12 *Let N be a continuous fuzzy negation. If a D-implication $I_{S,T,N}$ satisfies (2.9) and (LEM) then $I_{S,T,N}$ satisfies (1.5).*

PROOF: Analogous to the proof of the Theorem 6.1.8 regarding Lemma 3.1.13 instead of the Lemma 3.1.12. ∎

Theorem 6.1.13 *Let T be a t-norm, S a continuous t-conorm and N a continuous fuzzy negation. If $T = T_M$, (S, N) satisfies (LEM) and $I_{S,T,N}$ satisfies (1.5), then (T, S, N) satisfies (2.11).*

PROOF: Analogous to proof of Theorem 6.1.9 regarding Lemma 3.1.13 instead Lemma 3.1.12. ∎

6.1.5 (N,T)-implications

From the equivalence between (S,N) and (N,T)-implications shown in the Remark 3.1.14, we cannot demonstrate new theorems since N must be strong and there are not (S, N) and (T, S, N) which satisfy (LEM) and (2.9), respectively. The following theorems will be proved by dual properties of (LEM), (2.9) and (2.11), i.e. they will be proved by (2.13), (2.10) and (2.12), respectively.

Theorem 6.1.14 *Let T be a t-norm, N a strong negation and S a t-conorm. A (N,T)-implication $I_{N,T}$ satisfies (1.5) whenever (T, N) satisfies (2.13) and for some t-norm T, (T, S, N) satisfies the Distributivity over De Morgan (Eq. 2.10).*

PROOF:

$$
\begin{aligned}
& I_{N,T}(I_{N,T}(x,y), I_{N,T}(x,z)) = \\
&= N(T(N(T(x,N(y))), N(N(T(x,N(z)))))) && \textit{by Def. } 3.1.6 \\
&= N(T(N(T(x,N(y))), T(x,N(z)))) && \textit{by } N5 \\
&= N(T(T(x,N(T(x,N(y)))), N(z))) && \textit{by } T1 \\
&= N(T(S(T(x,N(x)), T(x,N(N(y)))), N(z))) && \textit{by } 2.10 \\
&= N(T(S(0,T(x,y)), N(z))) && \textit{by } (2.13) \text{ and } N5 \\
&= N(T(T(x,y), N(z))) && \textit{by } S4 \\
&= N(T(x,T(y,N(z)))) && \textit{by } T1 \\
&= N(T(x,N(N(T(y,N(z)))))) && \textit{by } N5 \\
&= I_{N,T}(x, I_{N,T}(y,z)) && \textit{by Def. } 3.1.6
\end{aligned}
$$

■

Theorem 6.1.15 *Let $I_{N,T}$ be a (N,T)-implication generated by a strong fuzzy negation N and a continuous t-norm T. If $I_{N,T}$ satisfies (1.5) then (N,T) satisfies (2.13) and for any t-conorm S, (T,S,N) is a triple that satisfies (2.12).*

Proof: Let $x \in [0,1]$. Since, N is continuous, decreasing, $N(1) = 0$ and $N(0) = 1$, then there exists $y \in [0,1]$ such that $N(y) = x$. It is known that $I_{N,T}$ satisfy (RB) and (I1) [14, Prop. 2.6]. Assume that $I_{N,T}$ satisfies (1.5), so:

$$
\begin{aligned}
N(T(N(x),x)) &= N(T(N(N(y)),N(y))) && \\
&= I_{N,T}(N(y),N(y)) && \text{by Def. } 3.1.1 \\
&= I_{N,T}(\mathtt{N}_{I_{S,N}}(y), \mathtt{N}_{I_{S,N}}(y)) && \text{by Prop. } 3.1.1 \\
&= I_{N,T}(I_{N,T}(y,0), I_{N,T}(y,0)) && \text{by Def. } 2.1.5 \\
&= I_{N,T}(y, I_{N,T}(0,0)) && \text{by } (1.5) \\
&= I_{N,T}(y,1) && \text{by (I1)} \\
&= 1 && \text{by (RB)}
\end{aligned}
$$

Since N is strong and therefore strict, hence $T(N(x),x) = 0$.

Now, let $x,y,z \in [0,1]$ such that $0 \leq y \leq x \leq 1$. By the intermediate value theorem, there exists $y' \in [0,1[$ such that $(*)$ $N(y') = y$. Analogously, since T is continuous, increasing, $T(y,0) = 0$ and $T(y,1) = y$, then by the intermediate value theorem there exist $x' \in [0,1]$ such that $(**)$ $x = T(N(y'),x')$.

$$
\begin{aligned}
&S(T(x,N(y)),T(x,N(z))) \\
&= S(T(T(N(y'),x'),N(y)),T(T(N(y'),x'),N(z))) && \textit{by } (**) \\
&= S(T(T(N(y'),x'),N(N(y'))),T(T(N(y'),x'),N(z))) && \textit{by } (*) \\
&= S(T(T(N(N(y')),N(y')),x'),T(N(y'),T(x',N(z)))) && \textit{by } T1 \\
&= S(T(0,x'),T(N(y'),T(x',N(z)))) && \textit{by } (2.13) \\
&= T(N(y'),T(x',N(z))) && \textit{by } \text{Rem.}2.1.1 \ \& \ S4 \\
&= N(N(T(N(y'),N(N(T(N(z),N(N(x')))))))) && \textit{by } T1 \text{ and } N5 \\
&= N(I_{N,T}(N(y'),I_{N,T}(N(z),N(x')))) && \textit{by } \text{Def. } 3.1.6 \\
&= N(I_{N,T}(I_{N,T}(N(y'),N(z)),I_{N,T}(N(y'),N(x')))) && \textit{by } (1.5) \\
&= N(N(T(N(T(N(y'),N(N(z)))), \\
&\quad N(N(T(N(y'),N(N(x')))))))) && \textit{by } \text{Def. } 3.1.6 \\
&= T(N(T(N(y'),z)),T(N(y'),x')) && \textit{by } N5 \\
&= T(N(T(y,z)),x) && \textit{by } (*) \textit{ and } (**) \\
&= T(x,N(T(y,z))) && \textit{by } (T1)
\end{aligned}
$$

Hence (T,S,N) satisfies (2.11). ∎

6.1.6 h-implications

Theorem 6.1.16 *Let h be an h-generator w.r.t. a fixed $e \in]0,1[$. If I^h satisfies (1.5) then $y = h^{-1}(x \cdot h(y))$ when $y \leq e$.*

Proof: By (I11), $I^h(x, I^h(y,z)) = I^h(y, I^h(x,z))$. Moreover

$$
I^h(y, I^h(x,z)) = \begin{cases} 1 & \text{, if } y = 0 \\ h^{-1}(y \cdot h(I^h(x,z))) & \text{, if } y > 0 \text{ and } I^h(x,z) \leq e \\ h^{-1}(\frac{1}{y} \cdot h(I^h(x,z))) & \text{, if } y > 0 \text{ and } I^h(x,z) > e. \end{cases}
$$

On the other hand,

$$
I^h(I^h(x,y), I^h(x,z)) = \begin{cases} 1 & \text{, if } I^h(x,y) = 0 \\ h^{-1}(I^h(x,y) \cdot h(I^h(x,z)) & \text{, if } I^h(x,y) > 0 \text{ and} \\ & \quad I^h(x,z) \leq e \\ h^{-1}(\frac{1}{I^h(x,y)} \cdot h(I^h(x,z)) & \text{, if } I^h(x,y) > 0 \text{ and} \\ & \quad I^h(x,z) > e \end{cases}
$$

$$= \begin{cases} 1 & \text{, if } I^h(x,y)=0 \\ h^{-1}(h^{-1}(x\cdot h(y))\cdot h(I^h(x,z))) & \text{, if } x>0,\ y\le e \text{ and } I^h(x,z)\le e \\ h^{-1}(h^{-1}(\frac{1}{x}\cdot h(y))\cdot h(I^h(x,z))) & \text{, if } x>0,\ y>e \text{ and } I^h(x,z)\le e \\ h^{-1}(\frac{1}{h^{-1}(x\cdot h(y))}\cdot h(I^h(x,z))) & \text{, if } x>0,\ y\le e \text{ and } I^h(x,z)>e \\ h^{-1}(\frac{1}{h^{-1}(\frac{1}{x}\cdot h(y))}\cdot h(I^h(x,z))) & \text{, if } x>0,\ y>e \text{ and } I^h(x,z)>e. \end{cases}$$

Therefore, since I^h satisfies (1.5) and h (and h^{-1}) is strict increasing then:

Case1: $0<y\le e$ and $I^h(x,z)\le e$, so $I^h(y,I^h(x,z))=h^{-1}(y\cdot h(I^h(x,z))$ and $I^h(I^h(x,y),I^h(x,z))=h^{-1}(h^{-1}(x\cdot h(y))\cdot h(I^h(x,z)))$. Therefore $y=h^{-1}(x\cdot h(y))$.

Case2: $0<y\le e$ and $I^h(x,z)>e$, so $I^h(y,I^h(x,z))=h^{-1}(\frac{1}{y}\cdot h(I^h(x,z)))$ and $I^h(I^h(x,y),I^h(x,z))=h^{-1}(\frac{1}{h^{-1}(x\cdot h(y))}\cdot h(I^h(x,z)))$. Therefore $y=h^{-1}(x\cdot h(y))$ again.

Hence, if I^h satisfies (1.5) then $y=h^{-1}(x\cdot h(y))$ when $y\le e$. ■

Corollary 6.1.4 *No h-implication satisfies (1.5).*

Proof: Let $y<e$ and $x_1\neq x_2$ in which $x_1,x_2>0$. Assume that I^h satisfies (1.5), so $h^{-1}(x_1\cdot h(y))=y=h^{-1}(x_2\cdot h(y))$ (by Theorem 6.1.16). By the strictness of function h, $x_1=x_2$ (contradiction). Hence no h-implication satisfies (1.5). ■

Remark 6.1.2 *By proof of the Theorem 6.1.16 also can be concluded that if I^h satisfies (1.5) then $y=h^{-1}(\frac{1}{x}\cdot h(y))$ when $y>e$:*

1. *When $y>e$ and $I^h(x,z)\le e$, so $I^h(y,I^h(x,z))=h^{-1}(y\cdot h(I^h(x,z))$ and $I^h(I^h(x,y),I^h(x,z))=h^{-1}(h^{-1}(\frac{1}{x}\cdot h(y))\cdot h(I^h(x,z)))$. Therefore $y=h^{-1}(\frac{1}{x}\cdot h(y))$.*

2. *When $y>e$ and $I^h(x,z)>e$, so $I^h(y,I^h(x,z))=h^{-1}(\frac{1}{y}\cdot h(I^h(x,z)))$ and $I^h(I^h(x,y),I^h(x,z))=h^{-1}(\frac{1}{h^{-1}(\frac{1}{x}\cdot h(y))}\cdot h(I^h(x,z)))$. Therefore $y=h^{-1}(\frac{1}{x}\cdot h(y))$.*

Hence, if I^h satisfies (1.5) then $y=h^{-1}(\frac{1}{x}\cdot h(y))$ when $y>e$ (and $y=h^{-1}(x\cdot h(y))$ when $y\le e$ by Theorem 6.1.16).

The converse of this statement is also true: since $y=h^{-1}(x\cdot h(y))$ when $y\le e$ and $y=h^{-1}(\frac{1}{x}\cdot h(y))$ when $y>e$ so $y=I^h(x,y)$. Thus $I^h(y,I^h(x,z))=I^h(I^h(x,y),I^h(x,z))$. By (I11), $I^h(y,I^h(x,z))=I^h(x,I^h(y,z))$, then I^h satisfies

(1.5). Hence I^h satisfies (1.5) iff $y = h^{-1}(x \cdot h(y))$ (case $y \leq e$) and $y = h^{-1}(\frac{1}{x} \cdot h(y))$ (case $y > e$). However, by Corollary 6.1.4, such conditions cannot be satisfied for all $x \in [0,1]$.

6.1.7 Φ-conjugate fuzzy implications

Theorem 6.1.17 *Let $\varphi \in \Phi$. A fuzzy implication I satisfies (1.5) iff I_φ satisfies (1.5).*

PROOF: $\Rightarrow$: Assume that $\varphi \in \Phi$ and a fuzzy implication I satisfies (1.5). So,

$$
\begin{aligned}
& I_\varphi(x, I_\varphi(y,z)) = \\
& = \varphi^{-1}(I(\varphi(x), \varphi \circ \varphi^{-1}(I(\varphi(y), \varphi(z))))) && \textit{by } (3.12) \\
& = \varphi^{-1}(I(\varphi(x), I(\varphi(y), \varphi(z)))) && \textit{trivial} \\
& = \varphi^{-1}(I(I(\varphi(x), \varphi(y)), I(\varphi(x), \varphi(z)))) && \textit{by } (1.5) \\
& = \varphi^{-1}(I(\varphi \circ \varphi^{-1}(I(\varphi(x), \varphi(y))), \varphi \circ \varphi^{-1}(I(\varphi(x), \varphi(z))))) && \textit{trivial} \\
& = I_\varphi(I_\varphi(x,y), I_\varphi(x,z)) && \textit{by } (3.12).
\end{aligned}
$$

Hence I_φ satisfies (1.5).

$\Leftarrow$: Since $\varphi \in \Phi$ and a fuzzy implication I_φ satisfies (1.5). So,

$I_\varphi(x, I_\varphi(y,z)) = I_\varphi(I_\varphi(x,y), I_\varphi(x,z))$. By (3.12) $\varphi^{-1}(I(\varphi(x), \varphi \circ \varphi^{-1}(I(\varphi(y), \varphi(z))))) = \varphi^{-1}(I(\varphi \circ \varphi^{-1}(I(\varphi(x), \varphi(y))), \varphi \circ \varphi^{-1}(I(\varphi(x), \varphi(z)))))$ Thus, $\varphi \circ \varphi^{-1}(I(\varphi(x), (I(\varphi(y), \varphi(z))))) = \varphi \circ \varphi^{-1}(I((I(\varphi(x), \varphi(y))), I(\varphi(x), \varphi(z))))$. And by (3.12), $I(\varphi(x), (I(\varphi(y), \varphi(z)))) = I((I(\varphi(x), \varphi(y))), I(\varphi(x), \varphi(z)))$. Hence I satisfies (1.5). ■

6.2 Final considerations

In this chapter, we investigated a strong version of a classical axiom fuzzy generalization. In more detail, we determined under which necessary and sufficient conditions the Boolean-like law $I(xI(y,z)) = I(I(x,y), I(x,z))$ (refered by (1.5)) holds, for (S,N)-, R-, QL-, D-, (N,T)- and h-implications. Therefore the main results are given by the Theorems 6.1.1, 6.1.2, 6.1.6, 6.1.7, 6.1.11, 6.1.14, 6.1.15 and 6.1.17 and the Corollary 6.1.4. But other results are found in this chapter such as the relations between (1.5) and other fuzzy implication properties.

We did not find the same necessary and sufficient condition in order to (1.5) hold for (S,N)-implications. By Theorem 6.1.1, if (S, N) satisfies (LEM) and

(T, S, N) satisfies (2.9) then $I_{S,N}$ satisfies (1.5), and let S and N be continuous t-conorm and negation (respectively), if $I_{S,N}$ satisfies (1.5) then (S, N) satisfies (LEM) and (T, S, N) satisfies (2.11). We found out more details about solutions of (1.5) for (S,N)-implications. A summary of this was presented in subsection 6.1.1.3. Dual results were found for (N,T)-implications.

W.r.t. R-implications, only I_G satisfies (1.5), (I6), (I11), (I12) and right continuity (Theorem 6.1.6). Regarding a positive t-conorm, I_W is the unique QL-implication which satisfies (1.5) and (I10). D-implication results were obtained from QL-implication results. Finally, h-implications do not satisfy (1.5).

In a practical point of view, from those results, we conclude that (1.5) is a typical example of a hard property to be adopted in FRBSs, since some non-trivial additional properties must be regarded in order to (1.5) be applied in the system. Moreover, in a theoretical sense, the investigation of (1.5) shows that fuzzy semantics can differ considerably from Boolean semantics.

This chapter also proves that a fuzzy implication I satisfies (1.5) iff I, up to a Φ-conjugation, also satisfies (1.5).

Chapter 7

Towards Extensions of Fuzzy Rule-Based Systems

This chapter introduces practical contributions whose application depends on a good knowledge of fuzzy operators. Firstly we will define a Deductive System and show its relation with a FRBS. Furthermore, we will analyse the state of art of the use of production rules originated from t-norms and from fuzzy implications and finally propose a solution w.r.t. FRBS. We will also see that this solution can be extended for other Fuzzy Systems.

7.1 Deductive systems and FRBSs

Deductive systems are defined by a formal language, a set of axioms and a set of inference rules. The first one is composed by a countable set of symbols and a finite set of derivative rules to state how the symbols must be grouped to form well-formed formulas[1] (also called sentences of the formal theory). Given a set S of wffs, an n-ary inference rule $\mathtt{R}$ on S is a subset of $S^n \times S$ and it is usually denoted by:

$$\frac{p_1, \ldots, p_n}{p} \ (\mathtt{R})$$

If $\mathtt{R}$ is a nullary inference rule, i.e. $\mathtt{R} \subseteq Id \times S \approx S$ then p is said to be an axiom.

[1]wffs for short.

Let R be a set of inference rules over S. Given a subset Γ of S (called premisses) and an element p of S (called conclusion), we say "p is provable from Γ using R" if there is a finite sequence $p_1, ..., p_k$ of elements of S, where p_k is p and, for each $1 \leq i \leq k$, p_i is an axiom, or is a premiss or there is a u-ary inference rule $\mathtt{R} \in R$ for some $u \geq 1$ and $j_1, ..., j_u < i$ such that $(p_{j_1}, ..., p_{j_u}, p_i) \in \mathtt{R}$. The sequence $p_1, ..., p_k$ is called a *formal proof* of p from Γ and denoted by $\Gamma \vdash_R p$.

Definition 7.1.1 *Let S be a set of wffs and R be a set of inference rules on S. The triple $(S, R, \vdash)$ is called formal proof system or just deductive system.*

Taking the Propositional Classical Logic as an example, we can choose the system of Łukasiewicz whose signature consists of the unary operation symbol '$\neg$' and the binary operation symbol '$\to$', the inference rule is the well-known Modus Ponens: p, $p \to q \vdash q$; and the minimal set of axiom schemes is:

A1 $p \to (q \to p)$;

A2 $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$;

A3 $(\neg p \to \neg q) \to (q \to p)$.

A set Γ of wffs is said to be closed (w.r.t. formal proofs) if p belongs to Γ whenever p is provable from Γ.

An important result for the Łukasiewicz system is that an inference rule $\mathtt{R}$ and the symbol $\to$ are related through Deduction Theorem:

Theorem 7.1.1 *Let p, q be elements of S and Γ a subset of S. $\Gamma, p \vdash q$ iff $\Gamma \vdash p \to q$.*

This theorem is not valid for every formal system involving implications. In fact, the deduction theorem requires the structure of the Heyting algebras. By Heyting algebras, we mean a complete lattice which satisfies $p \wedge \bigvee_i q_i = \bigvee_i (p \wedge q_i)$. In such structure the following relation is valid: $a \wedge b \leq c$ iff $a \leq b \Rightarrow c$. The operation "$\Rightarrow$" interprets the conditional symbol and is called residual implication since it originates a family of residuated mappings. The generalization of the residual implication in the fuzzy context gives rise to the class of R-implications.

A FRBS is a deductive system in which the production rules are instances of axioms (postulates) and there is a set of inference rules (often the generalized Modus Ponens (GMP)). Thus, FRBSs also have a fuzzy semantics to interpret conjunction, disjunction, negation, implication, defuzzification operation, as well as, aggregation operator and compositional relation.[1]

Example 7.1.1 *Given a FRBS architecture similar to figure 7.1.*

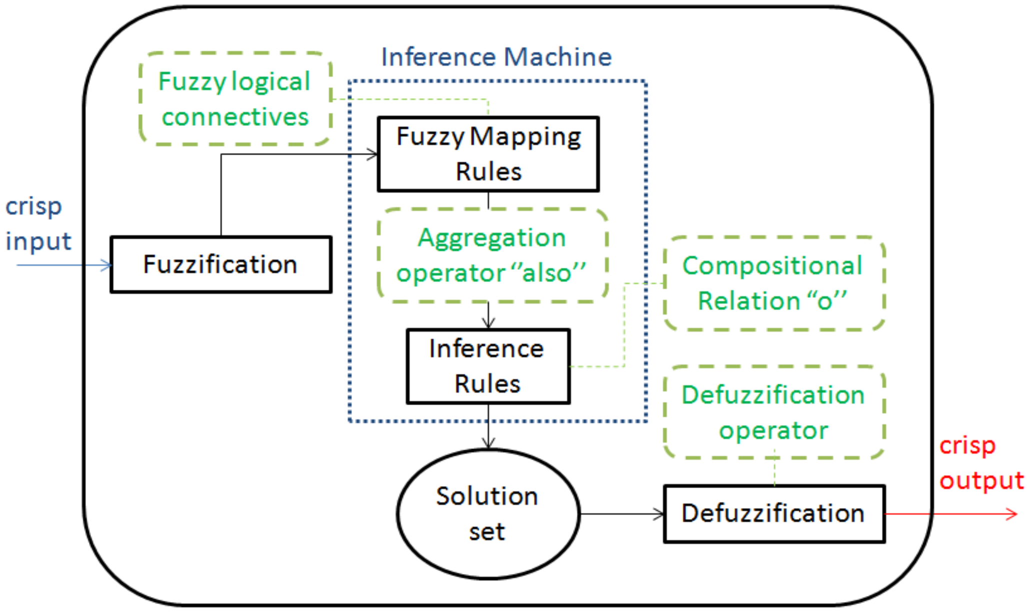

Figure 7.1: FRBS architecture detailed with fuzzy operators.

Assume (1.1) and (7.1) are the production rules, X is the set of input variables, Y is the set of output variables and A, B, C_1, C_2 are fuzzy sets. In addition, "not", "and", "or", "if-then", defuzzification operation, "also" (aggregation operator) and "$\circ$" (compositional relation) are respectively interpreted by $N_Z, T_{LK}, S_{LK}, I_{LK}, \frac{\int \mu_{Sol}(y)\cdot y\, dy}{\int \mu_{Sol}(y)\, dy}, \min$ and $\max$; where Sol is the solution set originated by a generalized kind of GMP [52] with the production rules combined by "also" (defined below).

$$\textit{If } A \textit{ and not } B \textit{ then } C_1. \tag{1.1}$$

$$\textit{If } A \textit{ or } B \textit{ then } C_2. \tag{7.1}$$

[1] Note that the aggregation and the compositional relation can be given by a different fuzzy operator than the selected to represent conjunction or disjunction.

Production Rule 1:	*if x is A and not B then y is C_1 also*
Production Rule 2:	*if x is A or B then y is C_2*
Fact:	*x is A' and B'*
Conclusion:	*y is C'*

Applying the semantics to each FRBS operation, If-then rules (1.1) and (7.1) are interpreted as (7.2) and (7.3), respectively.

$$I_{LK}((T_{LK}(A(x), N_Z(B(x))), C_1(y)) \tag{7.2}$$

$$I_{LK}((S_{LK}(A(x), B(x)), C_2(y)) \tag{7.3}$$

Thus, "(1.1) also (7.1)" is interpreted by:

$$min[I_{LK}((T_{LK}(A(x), N_Z(B(x))), C_1(y)), I_{LK}((S_{LK}(A(x), B(x)), C_2(y))].$$

Consequently, the solution is given by

$$Sol = \max\left\{ T_{LK}(A'(x), B'(x)), min\left[\begin{array}{l} I_{LK}((T_{LK}(A(x), N_Z(B(x))), C_1(y)), \\ I_{LK}((S_{LK}(A(x), B(x)), C_2(y)) \end{array} \right] \right\}$$

In more details Sol would be given by the following expression.

$$Sol = \max\left\{ \begin{array}{l} \max(\mu_{A'}(x) + \mu_{B'}(x) - 1, 0), \\ \min\left[\begin{array}{l} \min(1, 1 - (\max(\mu_A(x) + (1 - \mu_B(x)) - 1, 0)) + \mu_{C_1}(y)) \\ \min(1, 1 - (\min(\mu_A(x) + \mu_B(x), 1)) + \mu_{C_2}(y)) \end{array} \right] \end{array} \right\}$$

Finally, applying the defuzzification operation, we obtain the crisp output of FRBS, y^*.

$$y^* = \frac{\int \mu_{Sol}(y) \cdot y\, dy}{\int \mu_{Sol}(y)\, dy}. \tag{7.4}$$

7.1.1 Applicability of fuzzy operators in the design of FRBS

The selection of the best fuzzy semantics to be applied in a Fuzzy System is a fundamental problem of these computational type of systems. In 80's and 90's many investigations arose about which fuzzy operators could provide good performances to fuzzy models, mainly regarding operators to interpret production rules. Among them, we highlight the following papers:

- Kiska et al. study the influence of production rules and connective "also" in the accuracy of a fuzzy model of a d.c. series motor in [34]. The authors test several fuzzy implications and t-norms as production rules and the operators min and max as the connective "also"[1]. While production rules and "also" are tested, other FRBS' operations are maintained static. The accuracy was verified by the root-mean-square-error. This work continues in [35] where the authors complement the investigation introducing an additional criterion of evaluation to estimate a maximal error. The results of [34] are inconclusive and its authors state that further investigations should be concerned with the adequate selection of the semantics of production rule accompanying the connectives "and" and "also". In [35], they suggest the use of I_G and min or I_R and min as the best interpretation to production rules and "also", respectively.

- In [52], Mizumoto introduces some operators to be interpreted as production rules and selects min as compositional relation. Then the author tests the accuracy of a plant model, represented by a differential equation[2] $T\frac{dh}{dt} + h = q$ with Yamazaki and Sugeno fuzzy control rules [79]. Mizumoto also investigates the influence of fuzzy sets in control results maintaining min to interpret production rules. The conclusion of this paper is that methods based on t-norms get better results than methods based on implications of many-valued logics.

- Park and Cao, in [53], select some operators used to interpret production rules that obtained good performances in previous works and so investigate their performance in the function relation $Y = X$. The authors conclude that min and I_{LK} have the best performances.

- Cordón and colleagues also investigate t-norms and fuzzy implications as implication operators in fuzzy control in [21] and [22]. In particular, [22] brings a comparison of the accuracy of two fuzzy control applications: inverted pendulum problem and the fuzzy modelling of the real curve $Y = X$.

[1] Connective "also" links the solution sets and it is also known as aggregation operator
[2] Let T be a time constant.

They use different parameters to implement production rules, aggregation operation, the connective "and" and the defuzzification operation. The authors conclude that t-norms present better behaviour than implications.

These studies show that there is a fundamental concern to determine the best fuzzy semantics for fuzzy models, mainly w.r.t. t-norms versus fuzzy implications selection, in order to improve the accuracy of those models. There is no doubt about the strong influence of those operators on the accuracy. But there are different conclusions about the most accurate operator for production rules in fuzzy models: several works state that t-norms present better behaviour than implications, while some works also affirm that some fuzzy implications perform as well.

Such differences (on the paper results) are provoked by the prerogatives assumed to model the problem. This means that these empirical attempts to determine the best semantics for fuzzy models always bring a punctual (very specific) study for the theme because they cannot contemplate all of the system's variables. I.e., it is impossible to simulate all the combinations among the conjunctions, disjunctions, negations and defuzzification operations, production rules and compositional rules[1]. Besides that, different specialists always will specify different fuzzy sets and sets of production rules.

From this we conclude:

1. There is not one semantics that is the best for every fuzzy model;
2. For any specific fuzzy model, it is not empirically feasible to determine the best semantics for it.

Note that the problem of obtaining an appropriate semantics for a fuzzy system is closely related to another fundamental problem encountered in themathematical modelling of technological systems: adequacy of the system's model in relation to the real system. Therefore, it is reasonable to specify the semantics w.r.t. the real problem in which the Fuzzy System is supposed to solve. For example, when Mizumoto establishes that "**if** e *is NB* and Δe *is ZO* **then** Δq *is PB*" is one of production rules for a system with first order delay [52], which

[1] Nowadays there is an uncountable set of examples of each fuzzy logical connective.

properties this *if-then operator* should satisfy? What does the specialist of the system have in mind when defining such rule?

Hence, instead of testing semantics to identify the best one for a specific fuzzy model (it is impossible), we propose the selection of semantics according to the adequacy of fuzzy model w.r.t. the real system. Again, to select such an appropriate semantics, a good knowledge about fuzzy operators must be held. Our research contributes to this.

7.2 Towards extensions of fuzzy systems

In this work a Fuzzy System is considered to be any computational system that uses fuzzy sets theory to reach a solution; e.g. FRBS, systems that use fuzzy similarity measures — Saracoglu et al., in [62], describe a system that determines the similarity between documents through a fuzzy similarity measure — Decision Making Fuzzy System (such as fuzzy Bayesian decision methods), Fuzzy Classifiers (such as Fuzzy c-Means), etc. [59]. Environments and languages whose purpose are to build fuzzy systems (such as Guaje [2] and Fuzzy Prolog) also shall be considered Fuzzy Systems.

In the next subsection we are going to propose an extension for a specific kind of Fuzzy Systems, namely FRBS.

7.2.1 Extended FRBS

In the following, we propose a definition to FRBSs.

Definition 7.2.1 *A Fuzzy Rule-Based System (FRBS) is a tuple* $FRBS = \langle S, R, P, \Sigma_S \rangle$*, where*

- $S = S_I \cup S_O$ *is a countable set composed by fuzzy sets of input* (S_I) *and output* (S_O) *variables;*
- R *is the set of inference rules;*
- P *is the set of production rules (or fuzzy mapping rules or if-then rules);*
- $\Sigma_S = \{s, t, n, i, also, cr, out\}$ *is the set of operators which interpret disjunction, conjunction, negation, production rules ("if-then" operation), aggregation operation, compositional relation and defuzzification operation.*

Remark 7.2.1 *There are FRBSs that do not defuzzify their solutions. Due to this, the defuzzification operator "out" is defined by*

$$out = \begin{cases} id & \text{, if there is not defuzzification process} \\ y^* & \text{, otherwise.} \end{cases}$$

Whereby y^* is a function involving the solution set.

By definition 7.2.1 we can note that FRBS is specified by $R \cup P \cup S$, and the fuzzy semantics, Σ_S, determines the implementation methodology.

Although the use of implications to interpret production rules is more intuitive than the use of conjunctions, many well-succeeded fuzzy controllers have been implemented with a conjunction to interpret production rules (e.g. Mamdani min operator). Some investigation have also pointed out that t-norms present better behaviour than implications to interpret production rules in some situations. On the other hand, such investigations present the analytical problem mentioned previously, as well as the intention that the model of the system be adaptable to the real system: it is reasonable that a sentence "if-then" is interpreted by an implication in a real system. Therefore, instead of the aim of discarding the inference machine composed by fuzzy mapping rules and its inference rules, we propose to add an inference machine composed by "if-then" rules and inference rules. It would be a FRBS with two inference machines, one implemented by a conjunction and another one implemented by a fuzzy implication, both to interpret sets of production rules (each machine would have its own inference rules).

Besides that, more complex FRBSs might require more than one interpretation for production rules. For example, a subset of production rules are interpreted as (S,N)-implication and another disjunct subset as an R-implication. The same can happen for the connectives "and", "or", "not" and "also". Then two or more interpretations for each operation would be required.

Therefore, there are two extensions for FRBS:

1. To extend the quantity of inference machines: one composed of fuzzy mapping rules and a set of inference rules and another composed of "if-then" rules and a set of inference rules.

2. To extend the quantity of interpretations for each fuzzy operator.

In order to implement both extensions, we propose the *Polymorphous FRBS*.

Definition 7.2.2 *A Polymorphous*[1] *Fuzzy Rule-Based System ($\underline{P}$FRBS, for short) is a tuple* $\underline{P}FRBS = \langle S, R, P, \Sigma_S, comb \rangle$, *where*

- $S = S_I \cup S_O$ *is a countable set composed by fuzzy sets of input* (S_I) *and output* (S_O) *variables;*
- $R = R_\wedge \cup R_\Rightarrow$ *is the set of inference rules, in which* $R_\wedge$ *is the set of inference rules of an inference machine implemented by a conjunction and* $R_\Rightarrow$ *is the same, but implemented by an implication.*
- $P = P_\wedge \cup P_\Rightarrow$ *is the set of production rules composed by a set of fuzzy mapping rules* $(P_\wedge)$ *and a set of if-then rules* $(P_\Rightarrow)$*;*
- $\Sigma_S = S \cup T \cup N \cup I \cup ALSO \cup CR \cup OUT$ *is a family of non-empty sets of operators which interprets disjunction(s), conjunction(s), negation(s), production rules(s), aggregation operation(s), compositional relation(s) and defuzzification operation(s) written in the syntax.*
- *comb is the operator which combines the solution sets.*

Remark 7.2.2 *Observe the following aspects:*

- *Since $\underline{P}$FRBS is an extension, any usual FRBS can be implemented by a $\underline{P}$FRBS.*
- *It is still possible to extend $\underline{P}$FRBSs allowing more than two inference machines through the definition of P and R as follows:*
  - $R = R^i_\wedge \cup R^i_\Rightarrow$;
  - $P = P^i_\wedge \cup P^i_\Rightarrow$.

  Where i identifies the inference machine.

Note that every Fuzzy System has at least one fuzzy operator that requires a semantics, see Example 7.2.1.

Example 7.2.1 *Saracoglu et al., in [62], use a fuzzy similarity measure, membership functions are used in Fuzzy Classifier and Clusters, and the fuzzy logical connectives are used to implement FRBSs. All of those operations must be interpreted by a specific fuzzy operator or fuzzy set.*

[1]The extension is called "Polymorphous" since it will allow more than one interpretation for each FRBS operation.

Thus, mathematically, it is possible to sign at least one fuzzy operator for each operation in the fuzzy model, i.e. a similar extension defined for FRBSs can be defined to implement Polymorphous Fuzzy Systems.

7.3 Final considerations

This chapter proposes two possible extensions to FRBSs. Both can be implemented by the Polymorphous FRBS. We will extend the Example 7.1.1 in order to better understand a Polymorphous FRBS behaviour.

Example 7.3.1 *Given a FRBS architecture similar to figure 7.2.*

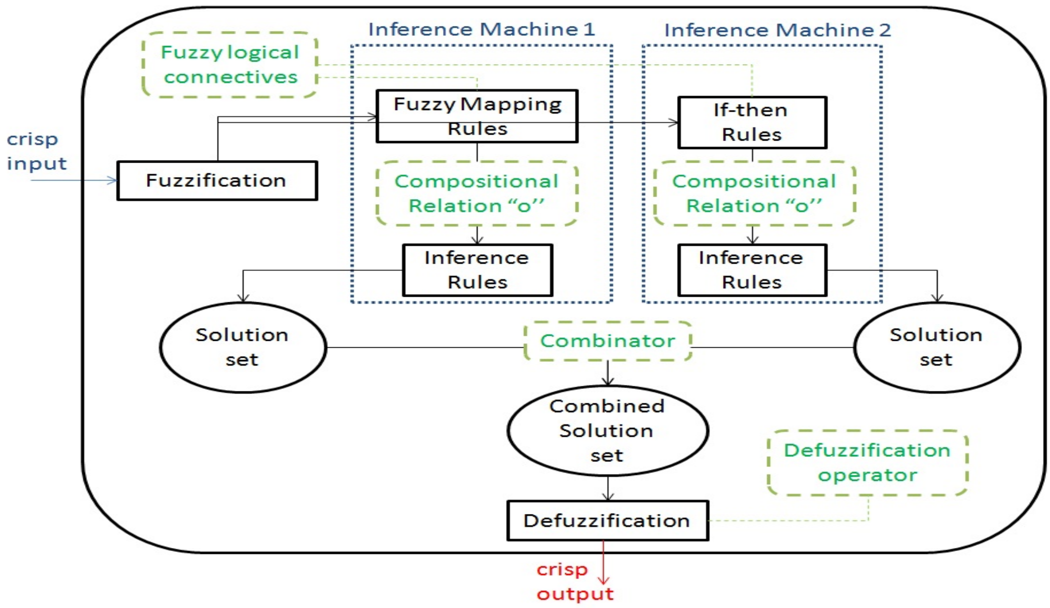

Figure 7.2: PFRBS architecture detailed with fuzzy operators.

Assume (1.1) is the unique fuzzy mapping rule, (7.1) is is the unique if-then rule, X is the set of input variable, Y is the set of output variable and A, B, C_1, C_2 are fuzzy sets. To simplify, we have the same inference rule, GMP, and the same interpretation for "not", "and", "or", combinator, "$\circ$" (compositional relation) and defuzzification operator. They are interpreted, respectively, by: $N_Z, T_{LK}, S_{LK}, \max, \min$ *and* $\frac{\int \mu_{Sol}(y) \cdot y \, dy}{\int \mu_{Sol}(y) \, dy}$[1]. *Fuzzy mapping rule is interpreted by* $\min$ *and if-then rule by* I_{LK}.

[1]Since there is only one production rule in each inference machine, the aggregation operator ("also") is not required.

If A and not B then C_1. (1.1)

If A or B then C_2. (7.1)

Applying the semantics to each FRBS operation, If-then rules (1.1) and (7.1) are interpreted as (7.5) and (7.6), respectively.

$$\min((T_{LK}(A(x), N_Z(B(x))), C_1(y)) \tag{7.5}$$

$$I_{LK}((S_{LK}(A(x), B(x)), C_2(y)) \tag{7.6}$$

Since the inference rule of both inference machines is GMP, so this PFRBS will reason on the above production rules as follows.

Production Rule 1:	*if x is A and not B then y is C_1 also*
Fact:	*x is A' and B'*
Conclusion:	*y is C'_1*

Production Rule 1:	*if x is A or B then y is C_2*
Fact:	*x is A' and B'*
Conclusion:	*y is C'_2*

Thus, applying GMP on such production rules, we have:

$$\begin{aligned} C'_1(y) &= T_{LK}(A'(x), B'(x)) \circ \min(T_{LK}(A(x), N_Z(B(x))), C_1(y)) \\ &= \min(T_{LK}(A'(x), B'(x)), \min(T_{LK}(A(x), N_Z(B(x))), C_1(y))); \\ C'_2(y) &= T_{LK}(A'(x), B'(x)) \circ I_{LK}(S_{LK}(A(x), B(x)), C_2(y)) \\ &= \min(T_{LK}(A'(x), B'(x)), I_{LK}(S_{LK}(A(x), B(x)), C_2(y))). \end{aligned}$$

Combining the solutions, we obtained the following:

$$Sol = \max \begin{pmatrix} \min(T_{LK}(A'(x), B'(x)), \min(T_{LK}(A(x), N_Z(B(x))), C_1(y))), \\ \min(T_{LK}(A'(x), B'(x)), I_{LK}(S_{LK}(A(x), B(x)), C_2(y))). \end{pmatrix}.$$

Finally, applying the defuzzification operation, we obtain the crisp output of FRBS, y^*, by the function (7.4).

The following chapter will show that future work can emerge from the Polymorphous FRBS.

Chapter 8

Final Remarks and Future Work

This chapter will present final considerations about the thesis of this dissertation and it lists possible future works.

This thesis characterized three fuzzy implication classes satisfying the following Boolean-like laws:

- $x \leq I(y, x)$, refereed by (1.3);
- $I(x, I(x, y)) = I(x, y)$, refereed by (1.4); and
- $I(x, I(y, z)) = I(I(x, y), I(x, z))$ refereed by (1.5).

Thus, the original results of this thesis can be listed as follows:

- results related to Distributivity over De Morgan since this property was firstly defined and discussed in this dissertation (Chapter 2);
- results that are closely related to the conditions under which those Boolean-like laws hold for fuzzy implication classes (concentrated in Chapters 4, 5 and 6); and
- discussion about extensions of FRBSs and the definition of the $\underline{\text{P}}$FRBS (Chapter 7).

For a better investigation of the above Boolean-like laws, we first studied their relation with other fuzzy properties. Furthermore, we studied under which sufficient and necessary conditions (1.3), (1.4) and (1.5) hold for (S,N)-, R-, QL-, D-, (N,T)- and h-implications. From this study, the main results were summarized in the final remarks of the Chapters 4, 5 and 6.

In accordance with those results, we noted that (1.3) has a particular similarity with (1.4) as we can see in the Corollary 4.0.1. It entails common properties to be held for fuzzy implication classes: every R-implications satisfies (1.3) and (1.4); and (S, N) satisfies (LEM) and $T = T_M$ iff $I_{S,N,T}$ satisfies (1.3) and (1.4). Boolean-like laws (1.4) and (1.5) are generalizations of classical axioms, , besides being the relation stated in Corollary 6.0.2. Such corollary added to the Corollary 6.1.7 imply that if S is a positive t-conorm then I_W is the only intersection between QL-implications that satisfies (1.4) and (1.5).

In addition to that, there is close relationships between (I10) and (1.3) and between (1.4) and (LEM).

From a theoretical point of view, this proposal improves on the knowledge about what a logical implication means, especially fuzzy implications and the six classes analyzed in this doctoral dissertation. From a practical point of view, it contributes to show which fuzzy operators a software engineer must choose in order to consider Boolean-like laws as true in his fuzzy system.

In this scenario, we highlight FRBSs whose semantics includes an (S,N)-, R-, QL-implication as well as D-, (N,T)- or h-implication, and the validity of (1.3), (1.4) or (1.5).

FRBSs whose implication operator is an (S,N)-, D- or h-implication, then (1.3) is automatically satisfied. Also, if it is an R-implication, then both (1.3) and (1.4) will already be satisfied. However, if a FRBS must be implemented with a QL-implication and (1.3) being valid, then a specific structure $([0, 1], T, S, N, I)$ must be defined: (S, N) must satisfy (LEM) and $T = T_M$. Moreover, if a (N,T)-implication must be selected to interpret production rules, N must be strong in order to hold (1.3).

According to the solutions of (1.4), if a FRBS must be implemented with an (S,N)-implication and regarding also (1.4), the software engineer must regard a specific structure (T, S, N, I) such that (S, N) satisfies (LEM). Case it is a QL-implication, S must be strictly increasing in [0,1[and he must also choose T_M as t-norm of the system. While its contrapositive has an additional condition: its underlying negation must be continuous. Now, if a FRBS must be implemented with an $I_{N,T}$ regarding also (1.4) then N must be strong and there must be an S that is N-dual of T.

From the results of (1.5), if a FRBS must be implemented with features of (S,N)-implications added to (1.5) validity, a pair (S,N) that satisfies (LEM) and a triple (T,S,N) that satisfies (2.9) must also be regarded. Dual properties are required for (N,T)-implications. In case it is an R-implication, so such operator must be I_G, and in case it is a QL-implication, then $I = I_W$.

In addition, it is impossible to implement FRBSs whose implication operator must have features of an h-implications and also satisfy (1.4) or (1.5).

Clearly, the Boolean-like laws studied here cannot be indiscriminately adopted in any computational system based on fuzzy concepts. That is, in case the application domain *asks* for (1.3), (1.4) or (1.5) validity, then the software engineer has to select a structure (T,S,N,I) that obeys the conditions announced in the previous three paragraphs.

An alternative way to define new fuzzy implications is by using automorphisms φ. In this scenario, we also proved in Chapters 4, 5 and 6 that a fuzzy implication I satisfies (1.3), (1.4) or (1.5) iff I_φ also satisfies it.

If we look at this work as an extension of [16], we will see another motivation to study (1.4) and (1.5) Boolean-like laws: (1.4), ($\bigstar$) $I(I(x,I(y,z)),I(I(x,y),I(x,z))) = 1$ — weaker version of (1.5) — and (I15b)[1] are fuzzy generalizations of the minimal set of axioms of the propositional classical deduction system [25].

In this dissertation we also presented an extension to FRBSs, called PFRBSs, which allows two or more interpretations to each FRBS' operation. Moreover, we see that such an extension can be achieved in other fuzzy systems. They are so-called Polymorphous Fuzzy Systems.

This dissertation points to some future works:

- Analysis of (1.3), (1.4) and (1.5) for interval-valued fuzzy implications (see [1; 17; 58]).

- Analysis of (1.3), (1.4) and (1.5) for other classes of fuzzy implications, such as: g-implications [77], Xor- and E-implications [18], and T-implications

[1] The third Łukasiewicz axiom was originally written as "$(\neg y \to \neg x) \to ((\neg y \to x) \to y)$", but it is equivalent to "$(\neg y \to \neg x) \to (x \to y)$" since $x \to y = \neg x \vee y$.

[42].

- Investigation of other Boolean-like laws, such as (I15b) and (★) in order to extend [16].

- Implementation of $\underline{\mathrm{P}}$FRBSs as well as definition and implementation of other Polymorphous Fuzzy Systems.

- Implementation of an environment for building $\underline{\mathrm{P}}$FRBSs with focusing on *consistent fuzzy semantics*[1].

- Implementation of an "Extended FRBS" (E-FRBS) in a similar way EML extends ML [33], in order to obtain an specification language.

[1]There are other environments to implement FRBSs, e.g. Guaje [2], but they have other focuses and a poor set of semantics for the fuzzy operations.

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