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On Fuzzy Ideals and Fuzzy Filters of Fuzzy Lattices

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IVAN MEZZOMO

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Abstract

In the literature there are several proposals of fuzzification of lattices and ideals concepts. Chon in (Korean J. Math 17 (2009), No. 4, 361-374), using the notion of fuzzy order relation defined by Zadeh, introduced a new notion of fuzzy lattice and studied the level sets of fuzzy lattices, but did not define a notion of fuzzy ideals for this type of fuzzy lattice.

In this thesis, using the fuzzy lattices defined by Chon, we define fuzzy homomorphism between fuzzy lattices, the operations of product, collapsed sum, lifting, opposite, interval and intuitionistic on bounded fuzzy lattices. They are conceived as extensions of their analogous operations on the classical theory by using this definition of fuzzy lattices and introduce new results from these operators.

In addition, we define ideals and filters of fuzzy lattices and concepts in the same way as in their characterization in terms of level and support sets. One of the results found here is the connection among ideals, supports and level sets. The reader will also find the definition of some kinds of ideals and filters as well as some results with respect to the intersection among their families.

Moreover, we introduce a new notion of fuzzy ideals and fuzzy filters for fuzzy lattices defined by Chon. We define types of fuzzy ideals and fuzzy filters that generalize usual types of ideals and filters of lattices, such as principal ideals, proper ideals, prime ideals and maximal ideals. The main idea is verifying that analogous properties in the classical theory on lattices are maintained in this new theory of fuzzy ideals. We also define, a fuzzy homomorphism h from fuzzy lattices $\mathcal{L}$ and $\mathcal{M}$ and prove some results involving fuzzy homomorphism

and fuzzy ideals as if h is a fuzzy monomorphism and the fuzzy image of a fuzzy set $\tilde{h}(I)$ is a fuzzy ideal, then I is a fuzzy ideal. Similarly, we prove for proper, prime and maximal fuzzy ideals. Finally, we prove that h is a fuzzy homomorphism from fuzzy lattices $\mathcal{L}$ into $\mathcal{M}$ if the inverse image of all principal fuzzy ideals of $\mathcal{M}$ is a fuzzy ideal of $\mathcal{L}$.

Lastly, we introduce the notion of α-ideals and α-filters of fuzzy lattices and characterize it by using its support and its level set. Moreover, we prove some similar properties in the classical theory of α-ideals and α-filters, such as, the class of α-ideals and α-filters are closed under intersection. We also define fuzzy α-ideals of fuzzy lattices, some properties analogous to the classical theory are also proved and characterize a fuzzy α-ideal on operation of product between bounded fuzzy lattices $\mathcal{L}$ and $\mathcal{M}$ and prove some results.

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Symbol List

$(L, \vee, \wedge)$ - Algebraic lattice
$(L, \leq)$ - Order lattice
(X, A_α) - Lattice of a α-level set
$\mathcal{L} = (X, A)$ - Fuzzy lattice
$\mathcal{L} \times \mathcal{M} = (X \times Y, C)$ - Product operator of the fuzzy lattices
$\mathcal{L} \oplus \mathcal{M} = (X \oplus Y, C)$ - Collapsed sum operator of the fuzzy lattices
$\mathcal{L}_\uparrow = (X_\uparrow, A_\uparrow)$ - Lifting operator of the fuzzy lattice
$\mathcal{L}^{op} = (X^{op}, A^{op})$ - Opposite operator of the fuzzy lattice
$\mathbb{L} = (\mathbb{X}, \mathbb{A})$ - Interval operator of the fuzzy lattice
$\mathcal{L}^*_\neg = (X^*_\neg, A^*)$ - Intuitionistic operator of the fuzzy lattice
$A(x, y)$ - Fuzzy relation
A_α - Fuzzy relation of a α-level set
A - Fuzzy relation
F_α - Filter of a lattice (X, A_α)
$F(\mathcal{L})$ - Family of all fuzzy filters of a fuzzy lattice $\mathcal{L}$
$F_p(\mathcal{L})$ - Family of all fuzzy proper filters of a fuzzy lattice $\mathcal{L}$
H_α - Fuzzy α-ideal of $\mathcal{L} \times \mathcal{M}$
$I(\mathcal{L})$ - Family of all fuzzy ideals of a fuzzy lattice $\mathcal{L}$
$I_p(\mathcal{L})$ - Family of all fuzzy proper ideals of a fuzzy lattice $\mathcal{L}$
I_α - Ideal of a lattice (X, A_α)
N - Negation
$S(A)$ - Support of a fuzzy relation A
S^l - Set of all lower bounds of S
S^u - Set of all upper bounds of S
$\mathcal{I}_\alpha$ - Fuzzy α-ideal of (X, A)

$\Downarrow I$ - Fuzzy set generated by the fuzzy set I

$\Downarrow \widetilde{x}$ - Principal fuzzy ideal of (X, A)

$\Uparrow F$ - Fuzzy set generated by the fuzzy set F

$\Uparrow \widetilde{x}$ - Principal fuzzy filter of (X, A)

$\perp$ - Bottom

χ - Universe set

$\downarrow S$ - Down-set of a ordered set S

$\downarrow Y$ - Set generated by the set Y

$\downarrow x$ - Principal ideal of (X, A)

$\downarrow x_\alpha$ - Principal α-ideal of (X, A)

$\mathcal{F}(X)$ - Set of the fuzzy sets of X

$\mathcal{I}_\alpha \times \mathcal{J}_\alpha$ - Cartesian Product of the fuzzy sets

$\mathcal{I}_p(\mathcal{L})$ - Family of all proper fuzzy ideals of a fuzzy lattice

$\mathcal{F}_p(\mathcal{L})$ - Family of all proper fuzzy filters of a fuzzy lattice

$\mathcal{P}$ - Ordered set

$\mathcal{P}(X)$ - Family of parts of X

$\mathcal{P}^d$ - Dual of the ordered set $\mathcal{P}$

$\mathcal{P}_\perp$ - Lifted of $\mathcal{P}$

$\mu_{\widetilde{X}}$ - Relationship degree of the fuzzy set $\widetilde{X}$

$\overleftarrow{\widetilde{h}}(W)$ - Fuzzy inverse image of W

$\widetilde{h}$ - Fuzzy funtion

$\top$ - Top

$\uparrow S$ - Up-set of a ordered set S

$\uparrow x$ - Principal filter of (X, A)

$\uparrow Y$ - Set generated by the set Y

$\wedge$ - Infimum

$\vee$ - Supremum

$\widetilde{X}$ - Fuzzy set

$\widetilde{X}_\alpha$ - Fuzzy α-set

$\mathbf{L}$ - Lattice

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Chapter 1

Introduction

According to Novak et al. [1999], fuzzy logic is a form of many-valued logic or probabilistic logic; it deals with reasoning which is approximate rather than fixed and exact. In contrast with traditional logic, they can have varying values, while binary sets have two-valued logic, true or false, fuzzy logic variables may have a truth value that ranges in degree between 0 and 1.

Fuzzy logic has been extended to handle the concept of partial truth, where the truth value may range between completely true and completely false. Furthermore, when linguistic variables are used, these degrees may be managed by specific functions.

1.1 Fuzzy Logic

The term "fuzzy logic" was introduced with the 1965 proposal of fuzzy set theory by Zadeh [1965]. Fuzzy logic has been applied to many fields such as control theory, artificial intelligence, medicine, economy, etc. Fuzzy logics, however, had been studied since the 1920s as infinite-valued logics notably by Łukasiewicz and Tarski; it is a popular misconception that they were invented by Zadeh.

A fuzzy concept generally means the concept is vague, lacking a precise meaning, without however being meaningless altogether. It has a meaning, or multiple meanings (it has different semantic associations). But these can become clearer only through further elaboration and specification, including a closer definition

of the context in which they are used.

In logic, fuzzy concepts are often regarded as concepts in which their application, formally speaking, are neither completely true nor completely false, and are, therefore, partially true and partially false. These are ideas that require further elaboration, specification or qualification to understand their applicability (the conditions under which they truly make sense).

In mathematics and statistics, a fuzzy variable (such as "the temperature", "velocity" or "age") is a value that could lie in a probable range defined by quantitative limits or parameters, and can be usefully described with imprecise categories (such as "high", "medium" or "low").

In mathematics and computer science, the various gradations of applicable meaning of a fuzzy concept are conceptualized and described in terms of quantitative relationships defined by logical operators. Such approach is sometimes called "degree-theoretic semantics" by logicians and philosophers, but the more usual term is fuzzy logic or many-valued logic. The basic idea is that a real number is assigned to each statement written in a language, within a range from 0 to 1, where 1 means that the statement is completely true and 0 means that the statement is completely false while intermediate values represent that the statements are "partially true". This makes it possible to analyze a distribution of statements for their truth-content, identify data patterns, make inferences and predictions and model how processes operate.

Fuzzy reasoning (i.e. reasoning with graded concepts) has many practical uses. It is nowadays widely used in the programming of vehicle and electronic transport, household appliances, video games, language filters, robotics, and various kinds of electronic equipment used for pattern recognition, surveying and monitoring such as radars. Fuzzy reasoning is also used in artificial intelligence and virtual intelligence research.

Fuzzy concepts can generate uncertainty because they are imprecise, especially if they refer to a process in motion, or a process of transformation where something is "in the process of turning into something else". In that case, they do not provide a clear orientation for action or decision-making "what does X really mean or imply?". Reducing fuzziness by applying fuzzy logic, would generate perhaps more certainty.

However, this is not necessarily always like this and we can not suggest that we can stop looking for additional tools. A concept, even though it is not fuzzy at all and even if it is very exact, could equally fail to capture the meaning of something adequately. That is, a concept can be very precise and exact, but may not be, or insufficiently be, applicable or relevant to the situation to which it refers. In this sense, a definition can be "very precise", but it "misses the point" altogether.

A fuzzy concept may indeed provide more security because it provides a meaning for something when an exact concept is unavailable, which is better than not being able to denote it at all. A concept such as God, although it is not easily definable, it can provide security for the believer for instance.

Fuzzy concepts often play a role in the creative process of forming new concepts to understand something. In the most primitive sense, this can be observed in children who, through practical experience, learn to identify, distinguish and generalize the correct application of a concept, and relate it to other concepts.

However, fuzzy concepts may also occur in scientific, journalistic, programming and philosophical activities, when a thinker is in the process of clarifying and defining a newly emerging concept, which is based on distinctions that for one reason or another, cannot (yet) be more exactly specified or validated. Fuzzy concepts are often used to denote complex phenomena, or to describe something which is developing and changing, which might involve shedding some old meanings and acquiring new ones.

The fuzzy logic is widely used in computer science and mathematics. In mathematics it is widely used in the fuzzification of mathematics concepts as groups, rings, lattices, ideals, filters and many others. According to Pedrycz and Gomide [1998], fuzzy sets notions constitute one of the most fundamental and influential tools of computational intelligence. The concept of fuzzy sets is rather new and intellectually stimulating and their applications are diverse and advanced.

1.2 Fuzzy Mathematics

Order theory is a branch of mathematics that investigates our intuitive notion of order using binary relations. It provides a formal framework for describing statements such as "this is less than that" or "this precedes that". The properties common to orders we see in our daily lives have been extracted and are used to characterize the concepts of order. Cars waiting for the signal to change at an intersection, natural numbers ordered in the increasing order of their magnitude are just a few examples of order we encounter in our daily lives. This intuitive concept can be extended to orders on other sets of numbers, such as the integers and the reals. The order relations we are going to study here are an abstraction of those relations.

According to Zadeh [1965], a fuzzy relation, which is a generalization of a function, has a natural extension to a fuzzy set and plays an important role in the theory of such sets and their applications. Similarly to an ordinary relation, a fuzzy relation in a set X is a fuzzy set in the product space $X \times X$. Thus, fuzzy binary relations R are fuzzy sets of $X \times Y$ defined as a fuzzy collection of ordered pairs characterized by a membership function μ_R which associates with each pair (x, y) a membership degree $\mu_R(x, y)$.

Orders are everywhere in mathematics and related fields like computer science. However, each element can not always be compared to any other. For example, consider the inclusion ordering of sets. If a set A contains all the elements of a set B, then B is said to be smaller than or equal to A. However, there are some sets that cannot be related in this way. Whenever both contain some elements that are not in the other one, the two sets are not related by inclusion. Hence, inclusion is only a partial order, as opposed to the total orders given before.

The lattice theory has several applications in the areas of information engineering, physical sciences, communication systems, and information analysis, especially in coding theory and cryptography. Cryptographic algorithms based on lattice theory have gained notoriety due to the advent of quantum computers because they become unsafe methods of public key based on number theory as well as the constructions of codes and lattices through rings of semigroup and decoding algorithms for codes obtained. Another application of the theory of

lattices is on image processing techniques such as magnetic resonance imaging, nuclear magnetic resonance or computed tomography, which is expected to get good results using codes constructed from lattices.

These ideas can be extended to fuzzy theory. In 1971, Zadeh [1971] defined a fuzzy ordering as a generalization of the concept of ordering, that is, a fuzzy ordering is a fuzzy relation that is transitive. In particular, a fuzzy partial ordering is a fuzzy ordering that is reflexive and antisymmetric. In 1990, Yuan and Wu [1990] introduced the concepts of fuzzy sublattices not based on the notion of fuzzy orders and, in 1994, Ajmal and Thomas [1994] defined a fuzzy lattice as a fuzzy algebra and characterized fuzzy sublattices. In 2001, Tepavcevic and Trajkovcki [2001] defined that a fuzzy lattice is a fuzzy set L-valued under a lattice M such that all α-cut is a sublattice of M and, therefore, could not associate a fuzzy order for this notion as well. In 2009, Zhang et al. [2009] defined a fuzzy complete lattice as a set X with a fuzzy order L-valued where L is a classic lattice such that all fuzzy set L-valued under X has supremum and infimum. More recently, Chon [2009], considering the notion of fuzzy order of Zadeh [1971], introduced a new notion of fuzzy lattice and studied the level sets of fuzzy lattices. He also introduced the notions of distributive and modular fuzzy lattices and considered some basic properties of fuzzy lattices.

The applications of fuzzy lattices theory are very similar to lattices theory as in coding theory, cryptography, image processing techniques, decoding algorithms for codes and neurocomputing of fuzzy lattice that emerges as a connectionist paradigm in the framework of fuzzy lattices whose advantages include the capacity to deal rigorously with disparate types of data such as numeric and linguistic data.

Although several different notions of fuzzy order relations have been presented, for example Belohlávek [2004]; Bodenhofer and Kung [2004]; Fodor and Roubens [1994]; Gerla [2004]; Yao and Lu [2009]. The notion introduced by Zadeh [1971] has been widely considered in recent years as we can find in Amroune and Davvaz [2011]; Beg [2012]; Chon [2009]; Mezzomo et al. [2012b]; Seselja and Tepavcevic [2010].

In mathematical order theory, an ideal is a special subset of a partially ordered set (poset). Although this term was historically derived from the notion of a ring ideal of abstract algebra, it has subsequently been generalized to a different

notion. Ideals are of great importance for many constructions in order and lattice theory. The dual notion of an ideal is known as filter. An important special case of an ideal is constituted by those ideals whose set-theoretic complements are filters, i.e., ideals in the inverse order. Such ideals are called prime ideals. Also notice that, since we require ideals and filters to be nonempty, every prime filter is necessarily proper.

The construction of ideals and filters is an important tool in many applications of order theory. Likewise, there are several applications for fuzzy ideals in the areas as mathematics, physics, science computing and medicine which fuzzy ideals are used to medical diagnosis system.

The notion of fuzzy ideals began in 1982, when Liu [1982] defined fuzzy ideals of fuzzy invariant subgroups. From that time, several papers have been based on this notion, for example, Majumdar and Sultana [1996] and Navarro et al. [2012]. In 1990, Yuan and Wu [1990] defined fuzzy ideal as a kind of fuzzy set under a conventional distributive lattice and this approach has been followed by several authors, including Attallah [2000], Koguep et al. [2008] and, more recently, by Davvaz and Kazanci [2011]. In 2000, Attallah [2000] characterized a fuzzy ideal and fuzzy filter of lattice in terms of meet and join operations. In 2008, Koguep et al. [2008] studied the notion of fuzzy prime ideal and highlighted the difference between fuzzy prime ideal and prime fuzzy ideal of a lattice. In 2010, Kumbhojkar [2010] added a condition to the notion of fuzzy ideals defined by Liu [1982] to define in a new way to obtain fuzzy ideal, called fuzzy h-ideal.

Recently, in Mezzomo et al. [2012b], fuzzy ideals and fuzzy filters of a fuzzy lattice (X, A) have been defined in the sense of Chon [2009] as a crisp set $Y \subseteq X$ endowed with the fuzzy order $A|_{Y \times Y}$. In paper Mezzomo et al. [2012a], both definitions of ideal and filter related to a fuzzy lattice (X, A) were proposed, including a discussion on some kinds of ideals and filters. The intersection of families for each class of such ideals and filters together with preserved properties were also studied.

In paper Mezzomo et al. [2013c], using the notion of fuzzy lattice defined by Chon [2009], we introduce a new notion of fuzzy ideal and fuzzy filter for fuzzy lattices and some types of fuzzy ideals and fuzzy filters of fuzzy lattice were considered, such as, fuzzy principal ideals (filters), proper fuzzy ideals (filters),

fuzzy prime ideals (filters) and fuzzy maximal ideals (filters). In addition, we studied some properties analogous to the classical theory of fuzzy ideals (filters). In this study, it is proved that the class of proper fuzzy ideals (filters) is closed under fuzzy intersection operation. We also prove that if a bounded fuzzy lattice admits a maximal fuzzy ideal, then it is prime. Moreover, we define a fuzzy homomorphism h from fuzzy lattices $\mathcal{L}$ and $\mathcal{M}$ and prove some results involving fuzzy homomorphism and fuzzy ideals as if h is a fuzzy monomorphism and the fuzzy image of a fuzzy set $\tilde{h}(I)$ is a fuzzy ideal, then I is a fuzzy ideal. Similarly, we prove for proper, prime and maximal fuzzy ideals. Finally, we prove that h is a fuzzy homomorphism from fuzzy lattices $\mathcal{L}$ into $\mathcal{M}$ if the inverse image of all principal fuzzy ideals of $\mathcal{M}$ is a fuzzy ideal of $\mathcal{L}$.

Additionally, α-ideals and α-filters for fuzzy lattices were defined in paper Mezzomo et al. [2013a], providing a characterization for α-ideals and α-filters of fuzzy lattices by using their support and α-level sets. Moreover, in that paper, we prove some similar properties to the classical theory of α-ideals and α-filters, such as, the class of α-ideals and α-filters were shown to be closed under intersection operation. In sequence, the operations of product and collapsed sum on bounded fuzzy lattice were defined in Mezzomo et al. [2013b] as an extension of the classical theory. Furthermore, the product and collapsed sum on bounded fuzzy lattices were stated as fuzzy posets, and, consequently, as bounded fuzzy lattices.

Extending these studies, in paper Mezzomo et al. [2013e] we focus on the lifting, opposite, interval and intuitionist operations on bounded fuzzy lattices. They are conceived as extensions of their analogous operations on the classical theory, by using the fuzzy partial order relation and the definition of fuzzy lattices, as conceived by Chon. In addition, we prove that lifting, opposite, interval and intuitionist on (complete) bounded fuzzy lattices are (complete) bounded fuzzy lattices by introducing new results from both operators, product and collapsed sum, which had already been defined in our previous paper Mezzomo et al. [2013b].

Finally, in Mezzomo et al. [2013d], we characterize a fuzzy ideal on operation of product between bounded fuzzy lattices $\mathcal{L}$ and $\mathcal{M}$ and define fuzzy α-ideals of fuzzy lattices. Moreover, we characterize a fuzzy α-ideal on operation of product between bounded fuzzy lattices $\mathcal{L}$ and $\mathcal{M}$ and prove that given a fuzzy α-ideal

H_α of the product between $\mathcal{L}$ and $\mathcal{M}$, there exist fuzzy α-ideals I_α of $\mathcal{L}$ and J_α of $\mathcal{M}$ such that $H_\alpha \subseteq I_\alpha \times J_\alpha$.

1.3 Objectives

The main objective of this work is introducing a new theory of fuzzy ideals and fuzzy filters of fuzzy lattices that preserve several properties of classical ideals and filters. In this sense, considering the notion of fuzzy order of Zadeh [1971], we use the fuzzy lattice defined by Chon [2009] and we turn our attention to investigate the following specific goals:

- Define types of fuzzy lattices and prove some properties using the fuzzy lattice defined by Chon;
- Define fuzzy homomorphism on bounded fuzzy lattices and prove some properties analogous to the classical theory;
- Define some characteristics and operations of product, collapsed sum, lifting, opposite, interval and intuitionistic of bounded fuzzy lattices;
- Define ideals and filters of fuzzy lattices and prove some properties analogous to the classical theory;
- Define a new notion of fuzzy ideals and filters of fuzzy lattices, study some types of fuzzy ideals and fuzzy filters of fuzzy lattices and prove some properties, such as, the class of proper fuzzy ideals and fuzzy filters are closed under fuzzy intersection;
- Define α-ideals and α-filters of fuzzy lattices and prove some properties analogous the classical theory;
- Define fuzzy α-ideals of fuzzy lattices, prove some properties analogous the classical theory and study the relation involving fuzzy α-ideals and the product operation on bounded fuzzy lattices.

1.4 Methodology

There are several different forms to define fuzzy partial order relations, as we can see in Belohlávek [2004]; Bodenhofer and Kung [2004]; Fodor and Roubens [1994]; Gerla [2004]; Yao and Lu [2009]; Zadeh [1971]. Similarly, the concept of fuzzy partial order, fuzzy partially ordered set, fuzzy lattice and fuzzy ideal can be found in several forms in the literature. However, our choice for using the fuzzy lattice defined by Chon [2009] is because their notion of fuzzy lattice is very similar to the usual notion of lattice as a partially ordered set. The notion of fuzzy order relation used by Chon was first defined by Zadeh [1971].

The methodology that will be used in this work is to study the fuzzification of mathematical concepts as the new notion of ideals, filters, fuzzy ideals, fuzzy filters and fuzzy α-ideals using the fuzzy lattice notion defined by Chon. We will define some fuzzy operations of this fuzzy lattice analogous to the operations of classical lattice and define some types of fuzzy ideals and fuzzy filters for each one of the new lattices formed among these operations.

1.5 Structure of Thesis

This thesis is composed of eight chapters, which are divided in sections where we discuss about fuzzy order relation, fuzzy lattices, fuzzy homomorphism, operations, ideals and filters, fuzzy ideals and filters, α-ideals and fuzzy α-ideals of fuzzy lattices. We begin describing in Chapter 1 the research problem and what has been researched about the problem. Other chapters are organized as follows:

- Chapter 2 points out basic definitions about classical lattices, ideals and filters of lattices, operations on lattices, fuzzy theory of fuzzy sets and fuzzy relations;

- Chapter 3 focuses on fuzzy lattices, their characteristics and fuzzy homomorphism on bounded fuzzy lattices;

- Chapter 4 defines the operations of product, collapsed sum, lifting, opposite, interval and intuitionistic analogous the operations for classical lattices and prove some results;

- Chapter 5 defines ideals and filters of fuzzy lattice, some types of ideals and prove properties of ideals analogous to the classical theory;
- Chapter 6 defines fuzzy ideals and fuzzy filters of fuzzy lattices, some types of ideals and prove properties of ideals analogous to the classical theory. It also defines fuzzy homomorphism between fuzzy ideals and prove some results;
- Chapter 7 defines α-ideals of fuzzy lattices and prove some results and properties analogous to the classical theory. It also defines fuzzy α-ideals of fuzzy lattices and prove results involving fuzzy α-ideals and product operations on fuzzy lattices;
- Finally, chapter 8 presents our main contributions and publications beyond the present ideas and further works.

Chapter 2

Preliminaries

In this chapter, we will briefly review some basic concepts of lattices, ideals and filters from both the algebraic and partial order points of view as necessary for the development of other sections. This presentation is quite introductory and can be found in many books on lattice theory. Readers who are familiar with the basic concepts may wish to proceed to the next section.

2.1 Classical Lattices

According to Davey and Priestley [2002], many important properties of ordered sets are expressed in terms of the existence of upper and lower bounds on their subsets. In this way, lattices and complete lattices are two of the most important classes of the ordered sets.

2.1.1 Partial Orders

According to Davey and Priestley [2002], order permeates mathematics and everyday life in such a way that it is taken for granted. It appears in many guises: first, second, third, ...; bigger versus smaller; better versus worse. Notions of progression, precedence and preference may all be brought under its umbrella. Our first task is crystallizing these imprecise ideas and formalizing the relationship of "less-than-or-equal-to".

So, the following statements have something to do with order.

(*i*) $0 < 1$ and $1 < 2$;

(*ii*) Two brothers have a common mother;

(*iii*) Given any two distinct real numbers x and y, either x is greater than y or y is greater than x.

Order is not an intrinsic property of a single object. It concerns comparison between pairs of objects: 0 is smaller than 1; Mars is farther from sun than Earth, a seraphim ranks above an angel, etc. In mathematics terms, an ordering is a binary relation on a set of objects. In our example, the relation may be taken to be "less than" on $\mathbb{N}$ in (*i*), "is descendant of" in the set of all human begins in (*ii*).

What distinguishes an order relation from some other kind of relation? First, ordering is transitive. From the fact that $0 < 1$ and $1 < 2$ we can deduce that $0 < 2$. Secondly, order is antisymmetric: 3 is bigger than 1 but 1 is not bigger than 3. Third, an order is reflexive if 1 is bigger than or equal 1 (and vice versa), we have that $1 = 1$. The order theory is based on the properties of reflexivity, transitivity and antisymmetry.

By Davey and Priestley [2002], Definition 1.2, when P is a nonempty set, a **partial order** on P is a binary relation $\leq$ on P such that, for all $x, y, z \in P$, the condition holds

(*i*) $x \leq x$ (reflexivity);

(*ii*) $x \leq y$ and $y \leq x$ imply $x = y$ (antisymmetry);

(*iii*) $x \leq y$ and $y \leq z$ imply $x \leq z$ (transitivity).

These conditions are referred to, respectively, as reflexivity, antisymmetry and transitivity. A set P equipped with an order relation $\leq$ is said to be an ordered set (or partially ordered set). Usually we shall be a little slovenly and say simply "P is an ordered set". Where it is necessary to specify the order relation overtly we write $(P, \leq)$. On any set, $=$ is an order, the discrete order. A relation $\leq$ on a set P, which is reflexive and transitive but not necessarily antisymmetric is called a quasi-order or pre-order.

According to Davey and Priestley [2002], we have the following definitions:

(*i*) Chains and antichains as: Let P be an ordered set. Then $\mathcal{P} = (P, \leq)$ is a chain if, for all $x, y \in P$, either $x \leq y$ or $y \leq x$, that is, any two elements of P are comparable. Alternative names for a chain are linearly ordered set or totally ordered set. At the opposite extreme from a chain is an antichain. The ordered set $\mathcal{P}$ is an antichain $x \leq y$ in P only if $x = y$.

(*ii*) The dual of an ordered set: Given any ordered set $\mathcal{P} = (P, \leq)$ we can form a new ordered set $\mathcal{P}^d = (P^d, \leq^d)$ (the dual of $\mathcal{P}$) by defining $x \leq y$ to hold in P^d if and only if $x \geq y$ holds in P. For $\mathcal{P}$ finite , we obtain a diagram for $\mathcal{P}^d$ simply by "tuning upside down" a diagram for $\mathcal{P}$.

(*iii*) The Duality Principle: Given a statement Φ about ordered sets which is true in all ordered sets, the dual statement Φ^d is also true in all ordered sets.

We next introduce some important special elements.

(*iv*) Bottom and top elements: Let $\mathcal{P} = (P, \leq)$ be an ordered set. We say P has a **bottom element** if there exists $\perp \in P$ (called bottom) with the property that $\perp \leq x$ for all $x \in P$. Dually, P has a **top element** if there exists $\top \in P$ such that $x \leq \top$ for all $x \in P$. A simple instance of the Duality Principle note that the true statement "$\perp$ is unique when it exists" has as its dual version the statement "$\top$ is unique when it exists".

(*v*) Lifting: Given an ordered set $\mathcal{P} = (P, \leq)$ (with or without $\perp$), we form $\mathcal{P}_\perp$, called "lifted", as follows. Take an element $\perp \notin P$ and define $\leq$ on $P_\perp := P \cup \{\perp\}$

$$x \leq y \text{ if and only if } x = \perp \text{ or } x \leq y \text{ in } P.$$

Any set S gives rise to an ordered set with $\perp$, as follows. Order S by making it an antichain, $\overline{S}$, and then form $\overline{S}_\perp$. Ordered sets obtained in this way are called flat. In applications it is likely that $S \subseteq \mathbb{R}$. In this context we shall, for simplicity, write $S_\perp$ instead of the more correct $\overline{S}_\perp$. Since we shall not have occasion to apply lifting to subsets of $\mathbb{R}$ ordered as chains, this should cause no confusion.

(vi) Products: Let $P_1, ..., P_n$ be ordered sets. The Cartesian product $P_1 \times ... \times P_n$ can be made into an ordered set by imposing the coordinatewise order defined by

$$(x_1, ..., x_n) \leq (y_1, ..., y_n) \Leftrightarrow (\forall i)\ x_i \leq y_i \text{ in } P_i.$$

Given an ordered set P, the notation P^n is used as shorthand for the n-fold product $P \times .... \times P$. Informally, a product $P_1 \times P_2$ is drawn by replacing each point of a diagram of P_1 by copy of a diagram for P_2, and connecting corresponding points.

(vii) Down-sets and Up-sets: Let P be an ordered set and $S \subseteq P$.

(i) S is a down-set if, whenever $x \in S, y \in P$ and $y \leq x$, then $y \in S$.

(ii) Dually, S is a up-set if, whenever $x \in S, y \in P$ and $y \geq x$, then $y \in S$.

A down-set can be as one which is "closed under going down". Dually, a up-set can be as one which is "closed under going up". So, given $S \subseteq P$, we define the down-set $\downarrow S$ and the up-set $\uparrow S$, as:

$$\downarrow S = \{y \in P : y \leq x \text{ for some } x \in S\};$$
$$\uparrow S = \{y \in P : y \geq x \text{ for some } x \in S\}.$$

It is easily checked that $\downarrow S$ ($\uparrow S$) is the smallest down-set (up-set) containing S.

Similarly, given $x \in P$, the down-set $\downarrow x$ and the up-set $\uparrow x$ are defined as:

$$\downarrow x = \{y \in P : y \leq x\};$$
$$\uparrow x = \{y \in P : y \geq x\}.$$

Down-sets (up-sets) $\downarrow x$ ($\uparrow x$) are called principal. Clearly, $\downarrow \{x\} = \downarrow x$ and $\uparrow \{x\} = \uparrow x$.

By Davey and Priestley [2002], Lemma 1.30, let P be an ordered set and $x, y \in P$. Then the following are equivalent:

(i) $x \leq y$;

(ii) $\downarrow x \subseteq \downarrow y$;

(*iii*) For all $S \subseteq P$, if $y \in \downarrow S$ implies $x \in \downarrow S$.

Many important properties of an ordered set P are expressed in terms of the existence of certain upper bounds or lower bounds of subsets of P. A set $P \neq \emptyset$ equipped with a partial order relation $\leq$ is said to be a partially ordered set or poset. When we need to specify the order relation we write $\mathcal{P} = (P, \leq)$. According to Davey and Priestley [2002], lattices and complete lattices are two of the most important classes of ordered sets.

Based on Davey and Priestley [2002], Definition 2.1, when $\mathcal{P} = (P, \leq)$ is a poset and let $S \subseteq P$, an element $x \in P$ is an **upper** (**lower**) **bound** of S if $y \leq x$ ($y \geq x$) for all $y \in S$. The set of all upper (lower) bounds of S is denoted by S^u (S^l).

$$S^u = \{x \in P : y \leq x \text{ for all } y \in S\};$$
$$S^l = \{x \in P : y \geq x \text{ for all } y \in S\}.$$

If S^u (S^l) has a least (greatest) element x, then x is called the **least upper bound** or **supremum** (**greatest lower bound** or **infimum**) of S, denoted by $\sup S$ ($\inf S$). Since the least (greatest) elements are unique, if the supremum (infimum) exists, it is unique in S^u (S^l).

When P has no top element, we have $P^u = \emptyset$ and hence $\sup P$ does not exist. By duality, $\inf P = \perp$ whenever P has a bottom element. Now let S be the empty subset of P. Then any element $x \in P$ satisfies (vacuously) $y \leq x$ for all $y \in S$. Thus $\emptyset^u = P$ and hence $\sup \emptyset$ exists if and only if P has a bottom element, and in that case $\sup \emptyset = \perp$. Dually, $\inf \emptyset = \top$ whenever P has a top element.

We use the following notation: we write $x \vee y$ in place of $\sup\{x, y\}$ when it exists and $x \wedge y$ in place of $\inf\{x, y\}$ when it exists. Similarly, we write $\bigvee S$ and $\bigwedge S$ instead of $\sup S$ and $\inf S$ when these exist.

2.1.2 Lattices

In Davey and Priestley [2002], Definition 2.4, if $\mathcal{P} = (P, \leq)$ is a non-empty poset, we have that

(*i*) $(P, \leq)$ is called a **lattice** if $\sup\{x, y\}$ and $\inf\{x, y\}$ exist, for all $x, y \in P$;

(ii) $(P, \leq)$ is called a **complete lattice** if $\sup S$ and $\inf S$ exist, for all $S \subseteq P$.

We introduced lattices as ordered sets of a special type. However, we may adopt an alternative viewpoint. Given a lattice $\mathbf{L}$, we may define binary operations **join** and **meet** on the nonempty set L by

$$x \vee y = \sup\{x, y\} \text{ and } x \wedge y = \inf\{x, y\} \qquad (x, y \in L).$$

Note that the operations $\vee : L^2 \to L$ and $\wedge : L^2 \to L$ are order-preserving (or monotone). A lattice need not have $\top$ or $\perp$, so $\inf \emptyset$ and $\sup \emptyset$ may not exist. This simple proof can be varied to yield a large number of equally trivial statements about lattices and partially ordered sets, in general. Additionally, a lattice $\mathbf{L} = (L, \leq)$ which has top and bottom elements is called as **bounded lattice**. To make the use of the Duality Principle legitimate for lattices, note:

$$\text{If } (L, \leq) \text{ is a lattice, so is its dual } (L, \geq).$$

Thus the Duality Principle applies to lattices. That is, by Duality Principle, the dual of "$(L, \leq)$ has a $\perp$" is "$(L, \geq)$ has a $\top$".

In this section we view a lattice also as an algebraic structure $(L, \vee, \wedge)$. We stress the connection between $\vee, \wedge$ and $\leq$.

By Davey and Priestley [2002], Lemma 2.8, let $\mathbf{L}$ be a lattice and let $x, y \in L$. Then the following are equivalent:

(i) $x \leq y$;

(ii) $x \vee y = y$;

(iii) $x \wedge y = x$.

We are seeing that lattices are defined as posets. Despite of alternative viewpoints, in accordance with Gratzer [2000], an algebraic structure $\mathbf{L} = (L, \wedge, \vee)$ where L is a nonempty set and $\wedge$, $\vee$ are binary operations is a **lattice** if, for each $x, y, z \in L$, the following properties are verified:

(i) $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$ (commutativity);

(ii) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$ (associativity);

(*iii*) $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$ (absorption laws);

(*iv*) $x \wedge x = x$ and $x \vee x = x$ (idempotency).

Definition 2.1.1 *[Davey and Priestley, 2002, Definition 2.12] Let $\boldsymbol{L}$ be a lattice. We say $\boldsymbol{L}$ has a top element if there exists $1 \in \boldsymbol{L}$ such that $a = a \wedge 1$ for all $a \in \boldsymbol{L}$. Dually, we say $\boldsymbol{L}$ has a bottom element if there exists $0 \in \boldsymbol{L}$ such that $a = a \vee 0$ for all $a \in \boldsymbol{L}$. The lattice $(L, \vee, \wedge)$ has a 1 iff $(L, \leq)$ has a top element $\top$ and, in that case, $1 = \top$. A dual statement holds for 0 and $\perp$. A lattice $(L, \vee, \wedge)$ possessing 0 and 1 is called* ***bounded lattice****.*

A finite lattice is automatically bounded, with $1 = \bigvee L$ and $0 = \bigwedge L$. For more detailed study we refer to Davey and Priestley [2002]; Gratzer [2000].

A subset M of L is called a sublattice if, for all $x, y \in M$ implies $x \vee y \in M$ and $x \wedge y \in M$.

Remark 2.1.1 *Let $\mathbf{L} = (L, \wedge, \vee)$ be a lattice.*

(*i*) *$(L, \wedge, \vee, 1, 0)$ is a bounded lattice when $0, 1 \in L$ and for all $x \in L$, $x \wedge 1 = x$ and $x \vee 0 = x$, meaning that 0 and 1 are the bottom and top elements, respectively.*

(*ii*) *$(L, \wedge, \vee)$ is a lattice in the sense of definition of the algebraic lattice $\mathbf{L}$, if L with the binary relation $\leq_{\mathbf{L}} \subseteq L \times L$ defined by $x \leq_{\mathbf{L}} y \Leftrightarrow x \wedge y = x$, is a lattice. In addition, if $\boldsymbol{L}$ is bounded, then 0 is a bottom and 1 is a top, i.e., 0 and 1 correspond to the infimum and supremum of $\mathbf{L}$.*

(*iii*) *$(L, \leq_{\mathbf{L}})$ is a lattice in the sense of Definition 2.4, when L join with the binary operations $\vee, \wedge : L^2 \to L$, called join and meet operations, defined by $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$, for all $x, y \in L$, respectively, is a lattice.*

Therefore, by (ii) and (iii), both notions of lattices (order relation and algebraic) are equivalents.

Thus, lattices can be completely characterized in terms of the join and meet operations. We may henceforth say "let $\mathbf{L}$ be a lattice", replacing $\mathbf{L}$ by $(L, \leq)$ or

by $(L, \vee, \wedge)$ if we want to emphasize that we are thinking of it as a special kind of ordered set or as an algebraic structure. By considering the algebraic approach for lattices, the notion of bounded lattice has an algebraic counterpart.

The top and bottom elements are the identity of $\wedge$ and $\vee$, respectively.

When thinking of $\mathbf{L}$ as $(L, \vee, \wedge)$, it is appropriate to view these elements from a more algebraic standpoint.

Now, we define an ideal in the algebraic viewpoint. In Koguep et al. [2008] is defined the ideal and filter of a lattice $\mathbf{L}$ as follows.

Definition 2.1.2 *[Davey and Priestley, 2002, Definition 2.20] Let L be a nonempty set and $\boldsymbol{L} = (L, \wedge, \vee)$ a lattice. A nonempty subset I of L is called an **ideal** of $\boldsymbol{L}$ if for all $x, y \in L$*

(i) If $y \in I$ and $x \leq y$, then $x \in I$;

(ii) $x, y \in I$ implies $x \vee y \in I$.

Using the Duality Principle we define a filter of a lattice as:

Definition 2.1.3 *[Davey and Priestley, 2002, Definition 2.20] Let L be a nonempty set and $\boldsymbol{L} = (L, \wedge, \vee)$ a lattice. A nonempty subset F of L is called a **filter** of $\boldsymbol{L}$ if for all $x, y \in L$*

(i) If $y \in F$ and $y \leq x$, then $x \in F$;

(ii) $x, y \in F$ implies $x \wedge y \in F$.

An ideal and a filter is called **proper** if it does not coincide with L. It is very easy to show that an ideal I of a lattice with $\top$ is proper if and only if $\top \notin I$, and dually, a filter F of a lattice with $\perp$ is proper if and only if $\perp \notin F$. A proper ideal I of L is **prime** if and only if $x, y \in L$ and $x \wedge y \in I$ imply that $x \in I$ or $y \in I$. Dually, a proper filter F of L is prime if and only if $x, y \in L$ and $x \vee y \in I$ imply that $x \in I$ or $y \in I$. For each $x \in L$, the set $\downarrow x = \{y \in L : y \leq x\}$ is an ideal, it is known as the **principal ideal** generated by x. Dually, $\uparrow x = \{y \in L : y \geq x\}$ is a **principal filter**.

For more detailed study, we refer to Birkhoff [1967]; Davey and Priestley [2002]; Gierz et al. [2003]; Gratzer [2000].

2.1.3 Lattice Homomorphisms

In this section, homomorphisms on lattices are considered by preserving their main properties.

Let $\mathbf{L} = (L, \wedge_{\mathbf{L}}, \vee_{\mathbf{L}}, 0_{\mathbf{L}}, 1_{\mathbf{L}})$ and $\mathbf{M} = (M, \wedge_{\mathbf{M}}, \vee_{\mathbf{M}}, 0_{\mathbf{M}}, 1_{\mathbf{M}})$ be bounded lattices. A mapping $h : \mathbf{L} \to \mathbf{M}$ is said to be a **lattice alg-homomorphism**[1] if, for all $x, y \in L$, it follows that

(i) $h(x \wedge_{\mathbf{L}} y) = h(x) \wedge_{\mathbf{M}} h(y)$;

(ii) $h(x \vee_{\mathbf{L}} y) = h(x) \vee_{\mathbf{M}} h(y)$;

(iii) $h(0_{\mathbf{L}}) = 0_{\mathbf{M}}$;

(iv) $h(1_{\mathbf{L}}) = 1_{\mathbf{M}}$.

Let $\mathbf{L} = (L, \leq_{\mathbf{L}}, \perp_{\mathbf{L}}, \top_{\mathbf{L}})$ and $\mathbf{M} = (M, \leq_{\mathbf{M}}, \perp_{\mathbf{M}}, \top_{\mathbf{M}})$ be bounded lattices. A mapping $h : \mathbf{L} \to \mathbf{M}$ is said to be a **lattice ord-homomorphism** if, for all $x, y \in L$, it follows that

(i) If $x \leq y$ then $h(x) \leq h(y)$;

(ii) $h(\perp_{\mathbf{L}}) = \perp_{\mathbf{M}}$;

(iii) $h(\top_{\mathbf{L}}) = \top_{\mathbf{M}}$.

In classical logic, every alg-homomorphism is an ord-homomorphism however, in some cases, the reciprocal does not hold. In the following, we consider only alg-homomorphism since alg-homomorphism implies ord-homomorphism as we can see in the following example.

Example 2.1.1 *Consider the following lattices L and M as the following diagrams in Figure 2.1.*

Notice that the map $h : L \to M$ defined by $h(x) = x', h(y) = y', h(z) = z'$ and $h(w) = w'$, is an order preserving and preserves the infimum and supremum elements, hence is an ord-homomorphism. However, it is not an alg-homomorphism

[1] Usually it is used the term homomorphism and $\{0, 1\}$-homomorphism if the lattices are bounded (see Davey and Priestley [2002]; Gratzer [2000]). For emphasis, we will call by lattice alg-homomorphism to make clear that is a algebraic homomorphism between lattices.

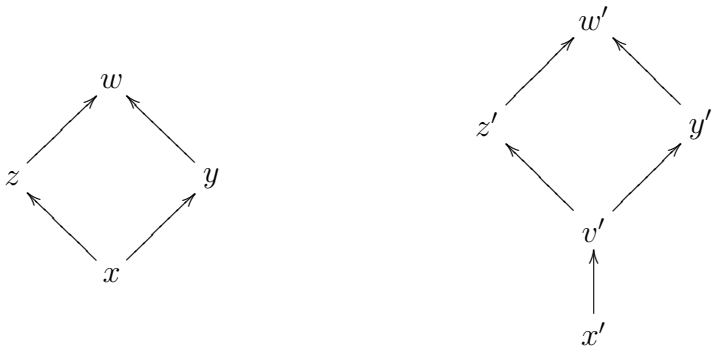

Figure 2.1: Representations of lattices L and M, respectively.

because the $\wedge$ operator is not preserved, that is, $x' = h(y \wedge_L z) \neq h(y) \wedge_M h(z) = v'$.
■

For the sake of simplicity, alg-homomorphism are referred as homomorphisms, unless in the cases where distinctions should be necessary.

2.1.4 Operations on lattices

Operators on lattices are reported in this section. In order to simplify denotation, $\mathbf{L_1}$ and $\mathbf{L_2}$ will denote the lattices $\mathbf{L_1} = (L_1, \leq_1)$ and $\mathbf{L_2} = (L_2, \leq_2)$, respectively.

The **product** of $\mathbf{L_1}$ and $\mathbf{L_2}$ is a lattice denoted by $\mathbf{L_1} \times \mathbf{L_2} = (L_1 \times L_2, \leq_\times)$, where for each $x_1, x_2 \in L_1$ and $y_1, y_2 \in L_2$

$$(x_1, y_1) \leq_\times (x_2, y_2) \text{ iff } x_1 \leq_1 x_2 \text{ and } y_1 \leq_2 y_2.$$

Clearly, the binary operations $\vee_\times$ and $\wedge_\times$ on $\mathbf{L_1} \times \mathbf{L_2}$ are characterized as follow

$$\begin{aligned} (x_1, y_1) \wedge_\times (x_2, y_2) &= (x_1 \wedge_1 x_2, y_1 \wedge_2 y_2), \\ (x_1, y_1) \vee_\times (x_2, y_2) &= (x_1 \vee_1 x_2, y_1 \vee_2 y_2). \end{aligned}$$

The **collapsed sum** of $\mathbf{L_1}$ and $\mathbf{L_2}$ is a lattice denoted by $\mathbf{L_1} \oplus \mathbf{L_2} = (L_1 \oplus L_2, \leq_\oplus)$, where

$$L_1 \oplus L_2 = (L_1 - \{\bot_1, \top_1\} \times \{1\}) \cup (L_2 - \{\bot_2, \top_2\} \times \{2\}) \cup \{\bot, \top\}$$

and the partial order $\leq_{\oplus}$ satisfies the next three conditions

(i) $(x,i) \leq_{\oplus} (y,j)$ iff $i = j$ and $x \leq_i y$;

(ii) $\perp \leq_{\oplus} x$, for all $x \in L_1 \oplus L_2$;

(iii) $x \leq_{\oplus} \top$, for all $x \in L_1 \oplus L_2$.

A characterization of $x \wedge_{\oplus} y$ and $x \vee_{\oplus} y$ is in the following

$$x \wedge_{\oplus} y = \begin{cases} (\hat{x} \wedge_i \hat{y}, i), & \text{if } x = (\hat{x}, i) \text{ and } y = (\hat{y}, i) \\ x, & \text{if } y = \top \\ y, & \text{if } x = \top \\ 0, & \text{otherwise;} \end{cases}$$

$$x \vee_{\oplus} y = \begin{cases} (\hat{x} \vee_i \hat{y}, i), & \text{if } x = (\hat{x}, i) \text{ and } y = (\hat{y}, i) \\ x, & \text{if } y = \perp \\ y, & \text{if } x = \perp \\ 1, & \text{otherwise.} \end{cases}$$

Let $\mathbf{L} = (L, \leq)$ be a bounded lattice and $\perp_{\uparrow} \notin L$. The **lifting** of $\mathbf{L}$ is a lattice denoted by $\mathbf{L}_{\uparrow} = (L_{\uparrow}, \leq_{\uparrow})$, where $L_{\uparrow} = L \cup \{\perp_{\uparrow}\}$ and, for each $x, y \in L$, the following conditions are held

(i) $x \leq_{\uparrow} y$ whenever $x \neq \perp_{\uparrow}, y \neq \perp_{\uparrow}$ and $x \leq y$

(ii) $\perp_{\uparrow} \leq_{\uparrow} x$, for all $x \in L$.

Additionally, characterizations of $x \wedge_{\uparrow} y$ and $x \vee_{\uparrow} y$ are given as

$$\begin{aligned} x \wedge_{\uparrow} y &= x \wedge y, \\ x \vee_{\uparrow} y &= x \vee y, \\ x \wedge_{\uparrow} 0 &= 0 \wedge_{\uparrow} x = 0, \\ x \vee_{\uparrow} 0 &= 0 \vee_{\uparrow} x = x. \end{aligned}$$

The **opposite (or dual)** of a lattice $\mathbf{L} = (L, \leq)$ is a lattice denoted by $\mathbf{L^{op}} = (L^{op}, \leq^{op})$, where $L^{op} = L$. Thus, the following equivalence holds

$$x \leq^{op} y \text{ iff } y \leq x.$$

Moreover, based on the algebraic structure point of view, the opposite (or dual) of a lattice $(L, \vee, \wedge)$ is $(L, \vee^{op}, \wedge^{op})$ and, for each $x, y \in L$, we have that

$$\begin{aligned} (x \vee^{op} y) &= (x \wedge y), \\ (x \wedge^{op} y) &= (x \vee y). \end{aligned}$$

The **interval operator** on a lattice $\mathbf{L}$ is also a lattice denoted by $\mathbb{I}\mathbf{L} = (IL, \leq_{\mathbb{I}})$, where $IL = \{[\underline{l}, \bar{l}] : \underline{l}, \bar{l} \in L \text{ and } \underline{l} \leq \bar{l}\}$. The associated partial order $\leq_{\mathbb{I}}$ over this lattice agrees with the product order, meaning that

$$[\underline{x}, \bar{x}] \leq_{\mathbb{I}} [\underline{y}, \bar{y}] \text{ iff } \underline{x} \leq \underline{y} \text{ and } \bar{x} \leq \bar{y}.$$

A characterization of $[\underline{x}, \bar{x}] \wedge_{\mathbb{I}} [\underline{y}, \bar{y}]$ and $[\underline{x}, \bar{x}] \vee_{\mathbb{I}} [\underline{y}, \bar{y}]$ are given as

$$\begin{aligned} [\underline{x}, \bar{x}] \wedge_{\mathbb{I}} [\underline{y}, \bar{y}] &= [\underline{x} \wedge \underline{y}, \bar{x} \wedge \bar{y}]; \\ [\underline{x}, \bar{x}] \vee_{\mathbb{I}} [\underline{y}, \bar{y}] &= [\underline{x} \vee \underline{y}, \bar{x} \vee \bar{y}]. \end{aligned}$$

For more detailed study, we refer to Bedregal and Santos [2006]; Davey and Priestley [2002]; Gratzer [2000].

These concepts could be presented on fuzzy sets as defined by Zadeh [1965] has been generalized in several forms. In Atanassov [1983], Atanassov introduced the notion of an intuitionistic fuzzy set A in a non-empty universal set X, which is expressed as $A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$, where $\mu_A, \nu_A : X \to [0, 1]$ are respectively called the membership and non-membership functions verifying, for all $x \in X$, the following condition

$$\mu_A(x) + \nu_A(x) \leq 1.$$

In Deschrijver and Kerre [2003], Deschrijver and Kerre gave an alternative approach for intuitionistic fuzzy set and proved that intuitionistic fuzzy set can also be seen as an L-fuzzy set in the sense of Goguen [1967] by considering the complete lattice $(L^*, \leq_{L^*})$, where

$$L^* = \{(x, y) \in [0, 1] \times [0, 1] : x + y \leq 1\}$$

and $(x_1, x_2) \leq_{L^*} (y_1, y_2)$ iff $x_1 \leq y_1$ and $y_2 \leq x_2$.

Let $\mathbf{L}$ be a complete lattice and $N : \mathbf{L} \to \mathbf{L}$ be an involutive and order reversing operation. In Atanassov and Stoeva [1984], Atanassov and Stoeva defined an intuitionistic fuzzy set $A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$, where the functions $\mu_A, \nu_A : X \to L$ verified, for all $x \in X$, the condition

$$\mu_A(x) \leq N(\nu_A(x)).$$

In the following, we will consider the lattice operator L^*_N analogous to L^*, according to the Atanassov and Stoeva's approach.

Let $\mathbf{L}^{op}$ be the opposite lattice of $\mathbf{L}$. The **classical intuitionistic** lattice is denoted by $\mathbf{L}^*_N = (L^*_N, \leq^*)$, where N is an ord-homomorphism from $\mathbf{L}$ into $\mathbf{L}^{op}$ and $L^*_N = \{(x, y) \in L \times L^{op}\colon\ N(x) \leq^{op} y\}$ such that, for all $(x_1, x_2), (y_1, y_2) \in L \times L^{op}$, the following inequality holds

$$(x_1, x_2) \leq^* (y_1, y_2) \text{ iff } x_1 \leq y_1 \text{ and } x_2 \leq^{op} y_2.$$

Clearly, we have that

$$(x_1, x_2) \wedge^* (y_1, y_2) = (x_1 \wedge y_1, x_2 \wedge^{op} y_2)$$
$$(x_1, x_2) \vee^* (y_1, y_2) = (x_1 \vee y_1, x_2 \vee^{op} y_2).$$

The next theorem guarantees that the above lattice operations are preserved by bounded and complete lattices.

Theorem 2.1.1 *[Mezzomo et al., 2013e, Theorem 2.1] Let* $\mathbf{L}$ *and* $\mathbf{M}$ *lattices. Then (i)* $\mathbf{L} \times \mathbf{M}$*, (ii)* $\mathbf{L} \oplus \mathbf{M}$*, (iii)* $\mathbf{L}_\uparrow$*, (iv)* $\mathbf{L^{op}}$*, (v)* $\mathbb{I}\mathbf{L}$ *and (vi)* $\mathbf{L}^*_N$ *are lattices.*

Proof: The proofs from (i) to (iv) are straightforward.

(v) Let $\mathbf{L}$ be a lattice. Then, for all $\underline{x}, \bar{x}, \underline{y}, \bar{y} \in L$, there exist $\underline{x} \wedge_L \underline{y}$ and $\bar{x} \wedge_L \bar{y}$. Therefore, for all $[\underline{x}, \bar{x}], [\underline{y}, \bar{y}] \in IL$, both elements $[\underline{x}, \bar{x}] \wedge_{\mathbb{I}} [\underline{y}, \bar{y}]$ and $[\underline{x}, \bar{x}] \vee_{\mathbb{I}} [\underline{y}, \bar{y}]$ are well defined. Clearly, the former is the infimum and the later is the maximum, with respect to $\leq_{\mathbb{I}}$, for all $[\underline{x}, \bar{x}], [\underline{y}, \bar{y}] \in IL$. Therefore, $\mathbb{I}\mathbf{L}$ is a lattice.

(*vi*) Since $\mathbf{L}$ and $\mathbf{L^{op}}$ are lattices, then there exists $x \wedge y$, for all $x, y \in L$. Therefore, for all $(x_1, x_2), (y_1, y_2) \in L_N^*$, both elements $(x_1, x_2) \wedge^* (y_1, y_2) = (x_1 \wedge y_1, x_2 \wedge^{op} y_2)$ and $(x_1, x_2) \vee^* (y_1, y_2) = (x_1 \vee y_1, x_2 \vee^{op} y_2)$ are well defined. Clearly, the former is the infimum and the later is the maximum with respect to $\leq^*$. Therefore, L_N^* is a lattice.

■

Theorem 2.1.2 *Let* $\mathbf{L}$ *and* $\mathbf{M}$ *bounded lattices. Then (i)* $\mathbf{L} \times \mathbf{M}$*, (ii)* $\mathbf{L} \oplus \mathbf{M}$*, (iii)* $\mathbf{L}_{\uparrow}$*, (iv)* $\mathbf{L^{op}}$*, (v)* $\mathbb{I}\mathbf{L}$ *and (vi)* $\mathbf{L}_N^*$ *are bounded lattices.*

Proof: The proofs from (*i*) to (*iv*) are straightforward.

(*v*) Let $\mathbf{L}$ be a bounded lattice. Then, $\mathbf{L}$ has $\perp$ and $\top$. So, for all $\underline{l}, \bar{l} \in L$, $\perp \leq_L \underline{l}$ and $\bar{l} \leq_L \top$. Thus, $[\perp, \bar{l}], [\underline{l}, \top] \in IL$ and $[\perp, \bar{l}] \leq_{\mathbb{I}} [\underline{l}, \top]$. By Theorem 2.1.1 (*v*), we have that $\mathbb{I}\mathbf{L}$ is a lattice. Therefore, $\mathbb{I}\mathbf{L}$ is a bounded lattice.

(*vi*) Since $\mathbf{L}$ and $\mathbf{L^{op}}$ are bounded lattices, then $\mathbf{L}$ and $\mathbf{L^{op}}$ has $\perp$ and $\top$. So, for all $x_1, x_2 \in L$, $\perp \leq_L x_1$ and $\top \leq_{L^{op}}^{op} x_2$. Thus, $(\perp, \top), (x_1, x_2) \in L_N^*$ and $(\perp, \top) \leq^* (x_1, x_2)$. By Theorem 2.1.1 (*vi*), we have that $\mathbf{L}_N^*$ is a lattice. Therefore, $\mathbf{L}_N^*$ is a bounded lattice.

■

Theorem 2.1.3 *Let* $\mathbf{L}$ *and* $\mathbf{M}$ *be complete lattices. Then (i)* $\mathbf{L} \times \mathbf{M}$*, (ii)* $\mathbf{L} \oplus \mathbf{M}$*, (iii)* $\mathbf{L}_{\uparrow}$*, (iv)* $\mathbf{L^{op}}$*, (v)* $\mathbb{I}\mathbf{L}$ *and (vi)* $\mathbf{L}_N^*$ *are also complete lattices.*

Proof: The proofs from (*i*) to (*iv*) are straightforward. Then all $S \subseteq \mathbf{L}$

(*v*) Let $\mathbf{L}$ be a complete lattice. Then all subsets on $\mathbf{L}$ has supremum and infimum. So, analogous Theorem 2.1.1 (*v*), we have that $\mathbb{I}\mathbf{L}$ is a complete lattice.

(*vi*) Since $\mathbf{L}$ and $\mathbf{L^{op}}$ are bounded lattices, then all subsets on $\mathbf{L}$ and $\mathbf{L^{op}}$ has supremum and infimum. So, analogous the Theorem 2.1.1 (*vi*), we have that $\mathbf{L}_N^*$ is a complete lattice.

■

2.2 Fuzzy Sets

A set χ is considered as universe of discourse and it includes all possible elements related with the given problem. Consider a subset X of χ. According to Pedrycz and Gomide [1998], if we denote the accept decision by 1 and the reject decision by 0, for short, then we may express the classification decision though the characteristic membership function $f_X : \chi \to \{0, 1\}$, of the form: for all $x \in \chi$, we have that

$$f_X(x) = \begin{cases} 1, & \text{if} \quad x \in X \\ 0, & \text{if} \quad x \notin X \end{cases}$$

Clearly, the empty set $\emptyset$ has a null characteristic function and the universe set χ a unity characteristic function.

According to Ross [2004], in classical sets the transition for an element in the universe between membership and nonmembership in a given set is abrupt and well defined. For an element in a universe that contains fuzzy sets, this transition can be gradual. This transition among various degrees of membership can be thought of as conforming to the fact that the boundaries are vague and ambiguous. A fuzzy set is a set containing elements that have varying degrees of membership in the set. According to Zadeh [1965], fuzzy sets are formally defined as follows:

Definition 2.2.1 *A fuzzy set $\widetilde{X}$ in χ is characterized by a membership function $\mu_{\widetilde{X}}$ mapping the elements of the domain, space or universe of discourse χ to the unit interval $[0, 1]$, that is,*

$$\mu_{\widetilde{X}} : \chi \to [0, 1]$$

where $[0, 1]$ means real numbers between 0 and 1 (including 0 and 1).

The fuzzy set $\widetilde{X}$ in χ may be represented as a set of ordered pairs of a generic element $x \in \chi$ and its grade of membership, i.e., $\widetilde{X} = \{(x, \mu_{\widetilde{X}}(x)) : x \in \chi\}$.

Example 2.2.1 *Let $\chi = \{a, b, c\}$ be a universal set. $\widetilde{X}_1 = \{(a, 0.3), (b, 1.0), (c, 0.7)\}$ and $\widetilde{X}_2 = \{(a, 0.0), (b, 0.9), (c, 0.7)\}$ would be fuzzy sets on X denoted by $\widetilde{X}_1 \subseteq \chi$ and $\widetilde{X}_2 \subseteq \chi$.*

We can characterize fuzzy sets in more detail by referring to the features used in characterizing the membership functions.

Definition 2.2.2 *[Pedrycz and Gomide, 1998, Definition 1.3] By the* ***support*** *of a fuzzy set $\widetilde{X}$, denoted by $S(\widetilde{X})$, we mean all elements of χ that belong to $\widetilde{X}$ to a nonzero degree. That is, $S(\widetilde{X})$ is a classical set defined by*

$$S(\widetilde{X}) = \{x \in \chi : \mu_{\widetilde{X}}(x) > 0\}.$$

According to Pedrycz and Gomide [1998], the **α-level set** of $\widetilde{X}$, denoted by $\widetilde{X}_\alpha$, is a set consisting of those elements of a universe χ whose membership values are equal or exceed the value of α. That is,

$$\widetilde{X}_\alpha = \{x \in \chi : \mu_{\widetilde{X}}(x) \geq \alpha\}.$$

Note that $\alpha \in (0, 1]$ is arbitrary and the α-level set is a crisp set. In other words, $\widetilde{X}_\alpha$ consists of elements of χ identified with $\widetilde{X}$ to a degree of at least α. Clearly, the lower level of α, the more elements are admitted to the corresponding α-level. That is, if $\alpha_1 > \alpha_2$, then $\widetilde{X}_{\alpha_1} \subset \widetilde{X}_{\alpha_2}$.

2.2.1 Fuzzy Connectives

2.2.1.1 Negation

The notion of the complement of $\widetilde{X}$ can be generalized by studying the negation operation and it is defined as:

Definition 2.2.3 *A function $N : [0, 1] \to [0, 1]$ is called* ***negation*** *if*

(i) Monotonicity: N is nonincreasing;

(ii) Boundary condition: $N(0) = 1$ and $N(1) = 0$.

The negation $N : [0, 1] \to [0, 1]$ is called ***strict negation*** *if, additionally,*

(iii) Continuity: N is a continuous function;

(iv) Strictly decreasing: N is a strictly decreasing function.

The strict negation $N : [0,1] \to [0,1]$ is called ***strong negation*** *if is a involution, that is, if*

(*v*) *Involution:* $N(N(x)) = x$, *for* $x \in [0,1]$.

It is obvious that $N : [0,1] \to [0,1]$ is a strict negation if and only if, it is a strictly decreasing bijection.

The more important and more used strong negation is the **standard negation** given by $N_S(x) = 1 - x$. According to Trillas [1979], $N : [0,1] \to [0,1]$ is a strong negation if and only if, there exists a monotone bijection $M : [0,1] \to [0,1]$ such that, for all $x \in [0,1]$

$$N(x) = M^{-1}(N_S(M(x))).$$

That is, all strong negation is a monotone transformation of standard negation. Note that all strong negation is a strict negation.

2.2.1.2 Triangular Norms

According to Pedrycz and Gomide [1998], the concept of triangular norm comes from the ideas of the called probabilistic metric space first proposed in 1942 by Menger. In fuzzy set theory, triangular norms play a key role by providing generic models for intersection and union operations on fuzzy sets, which must possess the properties of commutativity, associativity and monotonicity. Boundary condition must also be satisfied, to assure that they have the same behavior that conjunctions for bounded values, i.e., 0 and 1. Therefore, triangular norms form general classes of intersection and union operators.

Definition 2.2.4 *[Pedrycz and Gomide, 1998, Definition 2.1] A* ***triangular norm (t-norm)*** *is a binary operation* $t : [0,1] \times [0,1] \to [0,1]$, *satisfying the following conditions:*

(*i*) *Commutativity:* $x \ t \ y = y \ t \ x$;

(*ii*) *Associativity:* $x \ t \ (y \ t \ z) = (x \ t \ y) \ t \ z$;

(*iii*) *Monotonicity: If* $y \leq z$, *then* $x \ t \ y \leq x \ t \ z$;

(*iv*) *Identity:* $1\ t\ x = x$.

Notice that for all t-norm t, $0\ t\ x = 0$.

2.2.1.3 Triangular Co-Norms

As a formal construction, triangular co-norms are dual to triangular norms.

Definition 2.2.5 *[Pedrycz and Gomide, 1998, Definition 2.2] A* ***triangular co-norm (s-norm)*** *is a binary operation* $s : [0,1] \times [0,1] \to [0,1]$*, satisfying the following conditions:*

(*i*) *Commutativity:* $x\ s\ y = y\ s\ x$*;*

(*ii*) *Associativity:* $x\ s\ (y\ s\ z) = (x\ s\ y)\ s\ z$*;*

(*iii*) *Monotonicity: If* $y \leq z$*, then* $x\ s\ y \leq y\ s\ z$*;*

(*iv*) *Identity:* $1\ s\ x = 1$.

Notice that for all t-Conorm s, $0\ s\ x = x$.

Clearly, the min operator is a t-norm and the max operator is a s-norm. They are used to define set intersection and union operators, respectively.

2.2.2 Operations and Relations

According to Zadeh [1965], we say that $\widetilde{X}$ is included in $\widetilde{Y}$ if and only if the membership function of $\widetilde{X}$ is less than that of $\widetilde{Y}$, for all $x \in \chi$. That is,

$$\widetilde{X} \subseteq \widetilde{Y} \text{ iff } \mu_{\widetilde{X}}(x) \leq \mu_{\widetilde{Y}}(x).$$

According to Bojadziev and Bojadziev [1998], when a fuzzy set $\widetilde{X}$ is included in $\widetilde{Y}$, then $\widetilde{X}$ is called a fuzzy subset of $\widetilde{Y}$.

Union, intersection and complement represent the standard operations of fuzzy theory. According to Zadeh [1965], we define union, intersection and complement as:

Definition 2.2.6 *Let $\widetilde{X}$ and $\widetilde{Y}$ be fuzzy sets on the universe χ. For a given element $x \in \chi$, the operations of union, intersection and complement are defined as:*

$$\begin{aligned}
\textit{Union:} \qquad \mu_{\widetilde{X}\cup\widetilde{Y}}(x) &= \max\{\mu_{\widetilde{X}}(x), \mu_{\widetilde{Y}}(x)\} \\
&= \mu_{\widetilde{X}}(x) \vee \mu_{\widetilde{Y}}(x); \\
\textit{Intersection:} \qquad \mu_{\widetilde{X}\cap\widetilde{Y}}(x) &= \min\{\mu_{\widetilde{X}}(x), \mu_{\widetilde{Y}}(x)\} \\
&= \mu_{\widetilde{X}}(x) \wedge \mu_{\widetilde{Y}}(x); \\
\textit{Complement:} \qquad \mu_{\overline{\widetilde{X}}}(x) &= 1 - \mu_{\widetilde{X}}(x).
\end{aligned}$$

We generally use the sign $\vee$ for max operation and $\wedge$ for min operation. The operations of union, intersection and complement can be generalized by consider t-conorms, t-norms and negations, respectively, as follows:

(i) $\mu_{\widetilde{X}\cup_s\widetilde{Y}}(x) = s\{\mu_{\widetilde{X}}(x), \mu_{\widetilde{Y}}(x)\}$;

(ii) $\mu_{\widetilde{X}\cap_t\widetilde{Y}}(x) = t\{\mu_{\widetilde{X}}(x), \mu_{\widetilde{Y}}(x)\}$;

(iii) $\mu_{\overline{\widetilde{X}}_N}(x) = N(\mu_{\widetilde{X}}(x))$.

2.2.3 Cartesian Product on Fuzzy Set

According to Lee [2005], the Cartesian product applied to n fuzzy sets can be defined as follows: Let $\mu_{\widetilde{X}_1}, \ldots, \mu_{\widetilde{X}_n}$ be membership functions of $\widetilde{X}_1, \ldots, \widetilde{X}_n$. Then, the membership degree of $(x_1, \ldots, x_n) \in X_1 \times \ldots \times X_n$ on the fuzzy set $\widetilde{X}_1 \times \ldots \times \widetilde{X}_n$ is,

$$\mu_{\widetilde{X}_1\times\ldots\times\widetilde{X}_n}(x_1, \ldots, x_n) = \min\{\mu_{\widetilde{X}_1}(x_1), \ldots, \mu_{\widetilde{X}_n}(x_n)\}.$$

Clearly, this notion can be generalized by consider an arbitrary (n-ary) t-norm instead of the minimum t norm.

2.2.4 Operations on Fuzzy Set

According to Ross [2004], fuzzy sets follow the some properties as crisp sets. Because of this fact and because the membership values of a crisp set are values

(1) Involution	$\overline{\overline{\widetilde{X}}} = \widetilde{X}$
(2) Commutativity	$\widetilde{X} \cup \widetilde{Y} = \widetilde{Y} \cup \widetilde{X}$ $\widetilde{X} \cap \widetilde{Y} = \widetilde{Y} \cap \widetilde{X}$
(3) Associativity	$(\widetilde{X} \cup \widetilde{Y}) \cup \widetilde{Z} = \widetilde{X} \cup (\widetilde{Y} \cup \widetilde{Z})$ $(\widetilde{X} \cap \widetilde{Y}) \cap \widetilde{Z} = \widetilde{X} \cap (\widetilde{Y} \cap \widetilde{Z})$
(4) Distributivity	$\widetilde{X} \cup (\widetilde{Y} \cap \widetilde{Z}) = (\widetilde{X} \cup \widetilde{Y}) \cap (\widetilde{X} \cup \widetilde{Z})$ $\widetilde{X} \cap (\widetilde{Y} \cup \widetilde{Z}) = (\widetilde{X} \cap \widetilde{Y}) \cup (\widetilde{X} \cap \widetilde{Z})$
(5) Idempotency	$\widetilde{X} \cup \widetilde{X} = \widetilde{X}$ $\widetilde{X} \cap \widetilde{X} = \widetilde{X}$
(6) Identity	$\widetilde{X} \cup (\emptyset) = \widetilde{X}$ $\widetilde{X} \cap \emptyset = \emptyset$ $\widetilde{X} \cup \chi = \chi$ $\widetilde{X} \cap \chi = \widetilde{X}$
(7) Transitivity	If $\widetilde{X} \subseteq \widetilde{Y}$ and $\widetilde{Y} \subseteq \widetilde{Z}$, then $\widetilde{X} \subseteq \widetilde{Z}$
(8) De Morgan's law	$\overline{\widetilde{X} \cap \widetilde{Y}} = \overline{\widetilde{X}} \cup \overline{\widetilde{Y}}$ $\overline{\widetilde{X} \cup \widetilde{Y}} = \overline{\widetilde{X}} \cap \overline{\widetilde{Y}}$

Table 2.1: Characteristics of standard fuzzy set operations.

of the interval $[0, 1]$, classical sets can be thought of as a special case of fuzzy sets. The standard characteristics are represented in the Table 2.1.

Note that there are some characteristics of crisp operators do not hold here. For example, the excluded middle law and contradiction law are, in general, $\widetilde{X} \cup \overline{\widetilde{X}} \neq \chi$ and $\widetilde{X} \cap \overline{\widetilde{X}} \neq \emptyset$, respectively. The reason for this occurrence is that the boundary of complement of $\widetilde{X}$ is ambiguous.

2.2.5 Fuzzy Relations and Composition

A binary relation is defined as a subset of cartesian product. In fuzzy sets, a fuzzy relation in χ is a fuzzy set in the product space $X \times X$. Relations between elements of crisp sets can be extend to fuzzy relations, and the relations will be considered as fuzzy sets. A relation R between the crisp sets X and Y is the function $R : X \times Y \to \{0, 1\}$ meaning that:

(*i*) If the value of the relation is $R(x, y) = 1$ for some $x \in X$ and $y \in Y$, then x and y are related;

(*ii*) Otherwise, if $R(x, y) = 0$, we say that x and y are unrelated.

According to Ross [2004], fuzzy relations also map elements of two universes through the Cartesian product of them. However, the strength of the relation between ordered pairs of the two universes is not measured with the characteristic function, but rather with a membership function expressing various degrees of strength of the relation on the unit interval $[0, 1]$. Hence, a fuzzy relation A is a mapping from Cartesian space $X \times Y$ to the interval $[0, 1]$. If $X = Y$ the we say that A is a binary fuzzy relation in X.

According to Bojadziev and Bojadziev [1998], a fuzzy relation on $X \times Y$, denoted by A, is defined as the fuzzy set

$$A = \{((x, y), \mu_A(x, y)) : (x, y) \in X \times Y\},$$

where the function $\mu_A : X \times Y \to [0, 1]$ is called membership function. It gives the degree of membership of the ordered pair (x, y) in A associating with each pair (x, y) in $X \times Y$ a real number in interval $[0, 1]$.

Two fuzzy relations A and B are equal if and only if, for every pair $(x, y) \in A \times B$,

$$A(x, y) = B(x, y).$$

Here, $A(x, y)$ is interpreted as strength of relations between x and y. When $A(x_1, y_1) \geq A(x_2, y_2)$, then (x_1, y_1) is more strongly related than (x_2, y_2).

Remark 2.2.1 *Because a fuzzy relations is a fuzzy set, then the α-level sets and support of fuzzy relations is defined as in fuzzy sets, i.e., the* ***α-level set*** *of a fuzzy relation $A : X \times Y \to [0, 1]$ is defined as, for all $x \in X$ and $y \in Y$,*

$$A_\alpha = \{(x, y) \in X \times Y : A(x, y) \geq \alpha\}.$$

In the same way, we define the ***support*** *of a fuzzy relation $S(A)$ as*

$$S(A) = \{(x, y) \in X \times Y : A(x, y) > 0\}.$$

2.2.5.1 Max-Min Composition

According to Bojadziev and Bojadziev [1998], the operation composition combines fuzzy relations in the variables $x \in X, y \in Y$ and $z \in Z$ where X, Y, Z are classical sets. Consider the relations

$$\begin{aligned} A &= \{((x,y), \mu_A(x,y)) : (x,y) \in X \times Y\} \\ B &= \{((y,z), \mu_A(y,z)) : (y,z) \in Y \times Z\} \end{aligned}$$

The **max-min composition**, denoted by $A \circ B$ with membership functions $\mu_{A \circ B}$, is defined by

$$A \circ B = \{((x,z), \sup_{y \in B}(\min(\mu_A(x,y), \mu_B(y,z))) : (x,z) \in X \times Z\}.$$

Note that the composition of fuzzy relations has the associative property, that is,

$$A \circ (B \circ C) = (A \circ B) \circ C.$$

Hence, $A \circ B$ is a relation of $X \times Z$. The relations A and B are considered as matrices and the operation composition resembles the multiplication in the matrix, linking row and columns, after which cell is occupied by max-min value (the product is replaced by min and the sum by max). Then the max-min operation can be interpreted as strength of relations between rows in A and columns in B.

2.2.6 Types of Fuzzy Relations

Now, we will developed some characteristics of fuzzy relations and we will assume that A is a binary fuzzy relation defined on $X \times X$.

Let X be a nonempty set and $x, y, z \in X$. Some main properties of binary fuzzy relation A in X is defined in the following:

(i) A is a **fuzzy reflexive** relation if $A(x,x) = 1$, for all $x \in X$;

(ii) A is a **fuzzy symmetric** relation if $A(x,y) = A(y,x)$, for all $x, y \in X$;

(*iii*) A is a **fuzzy antisymmetric** relation if $A(x,y) > 0$ and $A(y,x) > 0$ implies $x = y$, for all $x, y \in X$;

(*iv*) A is a **fuzzy transitive** relation if $A(x,y) \geq \sup_{z \in X} \min\{A(x,z), A(z,y)\}$.

The fuzzy reflexivity, symmetry, antisymmetry and transitivity notion were first defined by Zadeh [1971].

A binary fuzzy relation $A : X \times X \longrightarrow [0,1]$ is called a **fuzzy equivalence relation** in X if A is reflexive, transitive and symmetric.

2.2.7 Fuzzy Order Relation

Orders are everywhere in mathematics and related fields as computer science. The first order often discussed is the standard order on the natural numbers. This intuitive concept can be extended to orders on other sets of numbers, such as the integers and the reals. The idea of being greater than or less than another number is one of the basic intuitions of number systems, in general, although one usually is also interested in the actual difference of two numbers, which is not given by the order. There are two types of relations which are widely used in mathematics: equivalence relation and order relation.

According to Chon [2009], a fuzzy relation A in X is called **fuzzy pre-order relation**, if it is a fuzzy reflexive and transitive relation. In the same way, a fuzzy relation A in X is a **fuzzy partial order relation** if A is fuzzy reflexive, antisymmetric and transitive. A fuzzy partial order relation A is a **fuzzy total order relation** if either $A(x,y) > 0$ or $A(y,x) > 0$, for all $x, y \in X$. If A is a fuzzy partial order relation on a set X, then (X, A) is called a **fuzzy partially ordered set** or **fuzzy poset**. If A is a fuzzy total order relation on a set X, then (X, A) is called **fuzzy totally ordered set** or a **fuzzy chain**.

Remark 2.2.2 *From now on, for simplicity of notation, we will use A and B for fuzzy order relations and instead of $\mu_A(x,y)$ we will use $A(x,y)$.*

In the literature there are several other ways to define a fuzzy reflexive, symmetric and transitive relation, see e.g Fodor and Roubens [1994]; Fodor and Yager

[2000]. Additionally, we can find several other forms to define fuzzy partial order relations, see Beg [2012]; Belohlávek [2004]; Chon [2009]; Yao and Lu [2009] for additional information.

Remark 2.2.3 *In Mezzomo et al. [2012b], Remark 3.1, when A is fuzzy reflexive, then the fuzzy transitivity can be rewritten by replacing the "$\geq$" by "$=$". In other words, A is fuzzy transitive iff $A(x,y) = \sup_{z\in X} \min\{A(x,z), A(z,y)\}$, for all $x, y, z \in X$.*

The statement that is claimed in the last remark can be easily proved. First, we know that $A(x,y) \geq \sup_{z\in X} \min\{A(x,z), A(z,y)\}$ and second, trivially, $\sup_{z\in X} \min\{A(x,z), A(z,y)\} \geq \min\{A(x,x), A(x,y)\} = \min\{1, A(x,y)\} = A(x,y)$. Therefore, we have that $A(x,y) = \sup_{z\in X} \min\{A(x,z), A(z,y)\}$.

Proposition 2.2.1 *[Mezzomo et al., 2013c, Proposition 2.1] Let (X, A) be a fuzzy poset and $x, y, z \in X$. If $A(x,y) > 0$ and $A(y,z) > 0$, then $A(x,z) > 0$.*

Proof: Straightforward by definition of fuzzy transitivity. ■

Proposition 2.2.2 *Let (X, A) be a fuzzy poset, $\alpha \in (0,1]$ and $x, y, z \in X$. If $A(x,y) \geq \alpha$ and $A(y,z) \geq \alpha$, then $A(x,z) \geq \alpha$.*

Proof: Suppose $\alpha \in (0,1]$ such that $A(x,y) \geq \alpha$ and $A(y,z) \geq \alpha$. Then, $\min\{A(x,y), A(y,z)\} \geq \alpha$. So, $\sup_{y\in X} \min\{A(x,y), A(y,z)\} \geq \min\{A(x,u), A(u,z)\} \geq \alpha$. Therefore, by definition of fuzzy transitivity, $A(x,z) \geq \alpha$. ■

For more detailed study about fuzzy sets, we refer to Bojadziev and Bojadziev [1998]; Klir and Yuan [1995]; Pedrycz and Gomide [1998]; Ross [2004]; Zadeh [1965, 1971]; Zimmermann [1991].

Chapter 3

Fuzzy Lattices

In this chapter, we use the notion of fuzzy lattice given by Chon [2009] defined as a fuzzy partial order relation and develop some properties of fuzzy lattices. In addition, we characterize a fuzzy lattice (X, A) as a classical set X under a fuzzy partial order relation A and characterize the fuzzy lattice (X, A) via its α-level set and its support.

The results in this chapter were used as preliminary results in Mezzomo et al. [2012a,b, 2013a,b,c,d,e].

3.1 Definitions and Properties

Our choice for using the fuzzy lattice defined by Chon [2009] is because their notion of fuzzy lattice is very similar to the usual notion of lattice as a fuzzy partial order relation. The notion of fuzzy order relation used by Chon was first defined by Zadeh [1971].

Before defining a fuzzy lattice as a fuzzy partial order relation, we need to introduce the idea of supremum and infimum of a fuzzy poset.

Definition 3.1.1 *[Chon, 2009, Definition 3.1] Let (X, A) be a fuzzy poset and let $Y \subseteq X$. An element $u \in X$ is said to be an* ***upper bound*** *for Y if $A(y, u) > 0$ for all $y \in Y$. An upper bound u_0 for Y is a* ***least upper bound or supremum*** *of Y if $A(u_0, u) > 0$ for every upper bound u for Y. An element $v \in X$ is said to be a* ***lower bound*** *for a subset Y if $A(v, y) > 0$ for all $y \in Y$. A lower bound v_0*

for Y is a ***greatest lower bound or infimum*** *of Y if $A(v, v_0) > 0$ for every lower bound v for Y.*

A least upper bound of Y will be denoted by $\sup Y$ and a greatest lower bound by $\inf Y$. We denote a least upper bound of the set $\{x, y\}$ by $x \vee y$ and denote a greatest lower bound of the set $\{x, y\}$ by $x \wedge y$.

Remark 3.1.1 *Since A is fuzzy antisymmetric, then the least upper (greatest lower) bound of $Y \subseteq X$, if it exists, is unique.*

The statement claimed in the above remark is easily proved. Suppose that u_0 and u_1 are two least upper bounds of a subset $Y \subseteq X$. Then, by definition, u_0 and u_1 are upper bounds of Y. Thus as u_0 is a $\sup Y$, then $A(u_0, u_1) > 0$ and as u_1 is a $\sup Y$, then $A(u_1, u_0) > 0$. Therefore, by the antisymmetry of A, $u_0 = u_1$. Similarly, we prove that $\inf Y$ is unique.

Example 3.1.1 *Let $X = \{x_1, y_1, z_1, w_1\}$ and let $A : X \times X \longrightarrow [0, 1]$ be a fuzzy relation such that $A(x_1, x_1) = A(y_1, y_1) = A(z_1, z_1) = A(w_1, w_1) = 1.0, A(y_1, x_1) = A(z_1, x_1) = A(w_1, x_1) = A(z_1, y_1) = A(w_1, y_1) = A(w_1, z_1) = 0.0, A(z_1, w_1) = 0.3, A(y_1, w_1) = 0.5, A(x_1, w_1) = 0.8, A(y_1, z_1) = 0.2, A(x_1, z_1) = 0.4$, and $A(x_1, y_1) = 0.1$. In the following, both graphical representations related of the table and oriented graph of the fuzzy partial order relation A are presented in Figure 3.1.*

It is easily checked that A is a fuzzy total order relation and therefore, (X, A) is a fuzzy lattice. Let $Y = \{x_1, y_1\}$, then w_1, z_1 and y_1 are the upper bounds of Y and since $A(y_1, x_1) = 0$ and $A(x_1, y_1) > 0$, it follows that the supremum of Y is y_1 and the infimum is x_1.

Analogously, $x_1 \vee z_1 = z_1,\ x_1 \vee w_1 = w_1,\ y_1 \vee z_1 = z_1,\ y_1 \vee w_1 = w_1,\ z_1 \vee w_1 = w_1,\ x_1 \wedge z_1 = x_1,\ x_1 \wedge w_1 = x_1,\ y_1 \wedge z_1 = y_1,\ y_1 \wedge w_1 = y_1$, and $z_1 \wedge w_1 = z_1$. ■

But, not every set of elements of a fuzzy poset has a least upper (greatest lower) bound as can be seen in the following example.

Example 3.1.2 *Let $X = \{x, y, z, w\}$ and let $A : X \times X \longrightarrow [0, 1]$ be a fuzzy relation such that $A(x, x) = A(y, y) = A(z, z) = A(w, w) = 1.0, A(x, y) = A(y, x) =$*

A	x_1	y_1	z_1	w_1
x_1	1.0	0.1	0.4	0.8
y_1	0.0	1.0	0.2	0.5
z_1	0.0	0.0	1.0	0.3
w_1	0.0	0.0	0.0	1.0

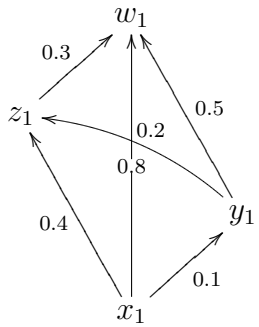

Figure 3.1: Representations of the fuzzy order relation A on X.

$A(z,x) = A(w,x) = A(z,y) = A(w,y) = A(w,z) = 0.0, A(x,y) = 0.1, A(x,z) = 0.8, A(x,w) = 0.7, A(y,z) = 0.4,$ *and* $A(y,w) = 0.3$.

Then it is easily checked that A is a fuzzy partial order relation. Also, $x \vee y = y,\ x \vee z = z,\ x \vee w = w,\ y \vee z = z,\ y \vee w = w,\ x \wedge y = x,\ x \wedge z = x,\ x \wedge w = x,\ y \wedge z = y, y \wedge w = y$ *and* $z \wedge w = y$. *Notice that $z \vee w$ does not exist.*

In the following, both graphical representations related of the table and oriented graph of the fuzzy partial order relation A are presented in Figure 3.2. ■

A	x	y	z	w
x	1.0	0.1	0.8	0.7
y	0.0	1.0	0.4	0.3
z	0.0	0.0	1.0	0.0
w	0.0	0.0	0.0	1.0

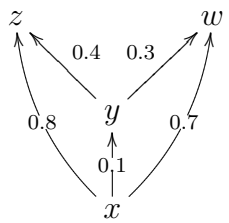

Figure 3.2: Representations of the fuzzy order relation.

Definition 3.1.2 *[Chon, 2009, Definition 3.2] A fuzzy poset (X, A) is called a* ***fuzzy lattice*** *if $x \vee y$ and $x \wedge y$ exist, for all $x, y \in X$.*

Remark 3.1.2 *The Example 3.1.1 is an example of fuzzy lattice whereas the Example 3.1.2, is not a fuzzy lattice.*

In Chon [2009], Proposition 2.2, by taking (X, A) as a fuzzy poset (or chain) and $Y \subseteq X$, if $B = A|_{Y \times Y}$, that is, B is the fuzzy relation on Y such that for all $x, y \in Y$, $B(x, y) = A(x, y)$, then (Y, B) is a fuzzy poset (or chain).

Definition 3.1.3 *[Mezzomo et al., 2012b, Definition 3.3] Let (X, A) be a fuzzy lattice. (Y, B) is a **fuzzy sublattice** of (X, A) if $Y \subseteq X$, $B = A|_{Y \times Y}$ and (Y, B) is a fuzzy lattice.*

Definition 3.1.4 *[Mezzomo et al., 2013e, Definition 3.1] $\mathcal{L} = (X, A)$ is a **bounded fuzzy lattice** if there exists $\perp$ and $\top$ in X such that for any $x \in X$ we have that $A(\perp, x) > 0$ and $A(x, \top) > 0$.*

Proposition 3.1.1 *Let (X, A) be a fuzzy lattice and let $x, y, z \in X$. Then*

(i) $A(x, x \vee y) > 0$, $A(y, x \vee y) > 0$, $A(x \wedge y, x) > 0$, $A(x \wedge y, y) > 0$;

(ii) $A(x, z) > 0$ and $A(y, z) > 0$ implies $A(x \vee y, z) > 0$;

(iii) $A(z, x) > 0$ and $A(z, y) > 0$ implies $A(z, x \wedge y) > 0$;

(iv) $A(x, y) > 0$ iff $x \vee y = y$;

(v) $A(x, y) > 0$ iff $x \wedge y = x$;

(vi) If $A(y, z) > 0$, then $A(x \wedge y, x \wedge z) > 0$ and $A(x \vee y, x \vee z) > 0$;

(vii) If $A(x \vee y, z) > 0$, then $A(x, z) > 0$ and $A(y, z) > 0$;

(viii) If $A(x, y \wedge z) > 0$, then $A(x, y) > 0$ and $A(x, z) > 0$.

Proof: The proofs from (i) to (vi), see in Chon [2009], Proposition 3.3.

(vii) By (i) we have that $A(y, x \vee y) > 0$ and by hypothesis $A(x \vee y, z) > 0$. So, by transitivity, $A(y, z) > 0$. Similarly, we prove that $A(x, z) > 0$.

$(viii)$ Analogous to (vii).

■

In the previous chapter, we defined the α-level set A_α and the support $S(A)$ of a fuzzy relation A on X as $A_\alpha = \{(x,y) \in X \times X : A(x,y) \geq \alpha\}$ and $S(A) = \{(x,y) \in X \times X : A(x,y) > 0\}$, respectively.

The following Proposition was transcribed from paper Chon [2009] Proposition 2.4, but the transitivity proof is not correct. The following proposition corrects the proof.

Proposition 3.1.2 *[Mezzomo et al., 2012b, Proposition 3.2] Let $A : X \times X \longrightarrow [0,1]$ be a fuzzy relation. Then, A is a fuzzy partial order relation on X iff for each $\alpha \in (0,1]$, the α-level set A_α is a partial order relation in X.*

Proof:

($\Rightarrow$) Let A be a fuzzy partial order relation on X and $\alpha \in (0,1]$. Since $A(x,x) = 1$ for all $x \in X$, $(x,x) \in A_\alpha$ for all α such that $\alpha \in (0,1]$. Suppose $(x,y) \in A_\alpha$ and $(y,x) \in A_\alpha$. Then, $A(x,y) \geq \alpha > 0$ and $A(y,x) \geq \alpha > 0$, and hence, because A is fuzzy antisymmetric, $x = y$. Suppose $(x,y) \in A_\alpha$ and $(y,z) \in A_\alpha$. Then, $A(x,y) \geq \alpha$ and $A(y,z) \geq \alpha$. Since $A(x,z) \geq \sup_{y \in X} \min\{A(x,y), A(y,z)\}$, then $A(x,z) \geq \min\{A(x,y), A(y,z)\} \geq \alpha$, that is, $(x,z) \in A_\alpha$.

($\Leftarrow$) Let A_α be a partial order relation for all α such that $\alpha \in (0,1]$. Then, $(x,x) \in A_\alpha$ for all α such that $\alpha \in (0,1]$. Thus, $(x,x) \in A_1$, that is, $A(x,x) = 1$. Suppose $A(x,y) > 0$ and $A(y,x) > 0$. Then, $A(x,y) > v > 0$ for some $v \in [0,1]$ and $A(y,x) > w > 0$ for some $w \in [0,1]$. Let $u = \min(v,w)$. Then, $A(x,y) > u > 0$ and $A(y,x) > u > 0$. Thus, $(x,y),(y,x) \in A_u$. Since A_u is antisymmetric, $x = y$. Let $x,y,z \in X$ and $\alpha_y = \min(A(x,y), A(y,z))$. So, $(x,y),(y,z) \in A_{\alpha_y}$ and because A_{α_y} is by hypothesis a partial order, then $(x,z) \in A_{\alpha_y}$. Therefore, $A(x,z) \geq \alpha_y = \min\{A(x,y), A(y,z)\}$, for all $y \in X$ and therefore, $A(x,z) \geq \sup_{y \in X} \min\{A(x,y), A(y,z)\}$, that is, A is fuzzy transitive.

■

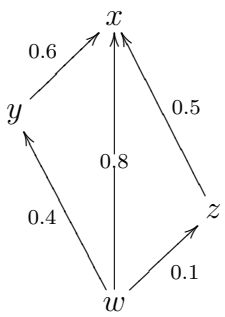

Figure 3.3: Representations of the fuzzy lattice.

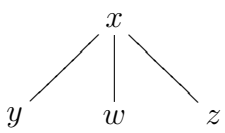

Figure 3.4: Representations of the α-cut equal to 0.5.

Moreover, in work Chon [2009], Proposition 3.5, let $A : X \times X \to [0,1]$ be a fuzzy relation. If (X, A_α) is a lattice for every α with $\alpha \in (0,1]$, then (X, A) is a fuzzy lattice.

At first view, we might think that the converse is also true. That is, if (X, A) is a fuzzy lattice, then (X, A_α) is a lattice for every α with $\alpha \in (0,1]$. But, we can see that depending on the α-level set, (X, A_α) may not be a lattice as seen in the following example.

Example 3.1.3 *Let $X = \{x, y, z, w\}$ and let $A : X \times X \to [0,1]$ be a fuzzy relation such that $A(x,x) = A(y,y) = A(z,z) = A(w,w) = 1, A(x,y) = A(x,z) = A(x,w) = A(y,z) = A(z,y) = A(y,w) = A(z,w) = 0, A(y,x) = 0.6, A(z,x) = 0.5, A(w,x) = 0.8, A(w,y) = 0.4$ and $A(w,z) = 0.1$. Then it is easily checked that A is a fuzzy partial order relation and that (X, A) is a fuzzy lattice, c.f. Figure 3.3. But, if we choose the α-cut equal to 0.5, (X, A_α) is not a lattice because the $y \wedge z$ does not exist, c.f. Figure 3.4.* ■

Moreover, we can not claim that given a fuzzy lattice (X, A) always there is an α-level set that is a lattice (X, A_α). The following example shows this situation.

Example 3.1.4 *Let $L = ((0,1] \times \{a,b\}) \cup \{\bot, \top\}$ and the following fuzzy partial order on L:*

$$A(x,y) = \begin{cases} 1, & \text{if } x = y \\ \frac{n-m}{2}, & \text{if } x = (m,c), y = (n,c), c \in \{a,b\} \text{ and } m < n \\ \frac{n}{2}, & \text{if } x = \bot \text{ and } y = (n,c) \text{ for } c \in \{a,b\} \text{ and } n \in (0,1] \\ 0.5, & \text{if } y = \top \text{ and } x \neq \top \\ 0, & \text{otherwise} \end{cases} \tag{3.1}$$

First we will prove that (L, A) is a fuzzy lattice.

(i) Reflexibility: Straightforward because $A(x,y) = 1$ if $x = y$;

(ii) Antisymmetry: Straightforward.

(iii) Transitivity: Let $x, y \in L$.

Case 1: *If $x = y$, then $A(x,y) = 1 \geq \min\{A(x,z), A(z,y)\}$, for all $z \in L$.*

Case 2: *If $x = (m,c)$, $y = (n,c)$, $c \in \{a,b\}$ and $m < n$, we have that $A(x,y) = \frac{n-m}{2} > 0$. Let $z \in L$.*

(a) If either $z \in \{\bot, \top\}$ or $z = (p,d)$ with $d \neq c$, then $A(x,y) > 0 = \min\{A(x,z), A(z,y)\}$.

(b) Let $z = (p,c)$ with $p \in (0,1]$.

(b.1) If $p < m < n$, then

$$\begin{aligned} \min\{A(x,z), A(z,y)\} &= \min\left\{0, \frac{n-p}{2}\right\} \\ &= 0 \\ &< A(x,y). \end{aligned}$$

(b.2) If $m < p < n$, then $\frac{p-m}{2} < \frac{n-m}{2}$ and $\frac{n-p}{2} < \frac{n-m}{2}$. So,

$$\min\{A(x,z), A(z,y)\} = \min\left\{\frac{p-m}{2}, \frac{n-p}{2}\right\}$$

$$
\begin{aligned}
&< \frac{n-m}{2} \\
&= A(x,y).
\end{aligned}
$$

(b.3) If $m < n < p$, then

$$
\begin{aligned}
\min\{A(x,z), A(z,y)\} &= \min\left\{\frac{p-m}{2}, 0\right\} \\
&= 0 \\
&< A(x,y).
\end{aligned}
$$

(b.4) If either $p = m$ or $p = n$. Then

$$
\begin{aligned}
\min\{A(x,z), A(z,y)\} &= \min\left\{1, \frac{n-m}{2}\right\} \\
&= \frac{n-m}{2} \\
&= A(x,y).
\end{aligned}
$$

Hence, for all $z \in L$, $A(x,y) \geq \min\{A(x,z), A(z,y)\}$. Therefore, $A(x,y) \geq \sup_{z \in L} \min\{A(x,z), A(z,y)\}$.

Case 3: *Let $x = \perp$, $y = (n,c)$ such that $c \in \{a,b\}$ and $n \in (0,1]$. Then, we have that $A(x,y) = \frac{n}{2} > 0$. Let $z \in L$. So*

(a) If $z = (p,d)$ with $d \neq c$, then $A(x,y) > 0 = \min\{\frac{p}{2}, 0\} = \min\{A(x,z), A(z,y)\}$.

(b) If $z = \perp$, then

$$
\begin{aligned}
\min\{A(x,z), A(z,y)\} &= \min\left\{1, \frac{n}{2}\right\} \\
&= \frac{n}{2} \\
&= A(x,y).
\end{aligned}
$$

(c) If $z = \top$, then

$$
\begin{aligned}
\min\{A(x,z), A(z,y)\} &= \min\{0.5, 0\} \\
&= 0
\end{aligned}
$$

$$< \quad A(x,y).$$

(d) Let $z = (p,c)$ with $p \in (0,1]$.

(d.1) If $p < n$, then

$$\begin{aligned} \min\{A(x,z), A(z,y)\} &= \min\left\{\frac{p}{2}, \frac{n-p}{2}\right\} \\ &< \frac{n}{2} \\ &= A(x,y). \end{aligned}$$

(d.2) If $n < p$, then

$$\begin{aligned} \min\{A(x,z), A(z,y)\} &= \min\left\{\frac{p}{2}, 0\right\} \\ &= 0 \\ &< A(x,y). \end{aligned}$$

Hence, for all $z \in L$, $A(x,y) \geq \min\{A(x,z), A(z,y)\}$. Therefore, $A(x,y) \geq \sup_{z \in L} \min\{A(x,z), A(z,y)\}$.

Case 4: *If $y = \top$ and $x \neq \top$. Then, we have $A(x,y) = 0.5 > 0$. Let $z \in L$. So*

(a) If $z \neq \top$, then $A(x,y) = 0.5 \geq \min\{A(x,z), 0.5\} = \min\{A(x,z), A(z,y)\}$.

(b) If $z = \top$, then

$$\begin{aligned} \min\{A(x,z), A(z,y)\} &= \min\{0.5, 1\} \\ &= 0.5 \\ &= A(x,y). \end{aligned}$$

Hence, for all $z \in L$, $A(x,y) \geq \min\{A(x,z), A(z,y)\}$. Therefore, $A(x,y) \geq \sup_{z \in L} \min\{A(x,z), A(z,y)\}$.

Case 5: *We have three situations that are not considered in the previous cases.*

(a) If $x = \top$ and $y = \bot$, then $A(x, y) = 0$. Let $z \in L$. So

(a.1) If either $z = \bot$ or $z = \top$, then

$$\begin{aligned} \min\{A(x,z), A(z,y)\} &= \min\{0, 1\} \\ &= 0 \\ &= A(x,y). \end{aligned}$$

(a.2) If $z = (m, c)$, then

$$\begin{aligned} \min\{A(x,z), A(z,y)\} &= \min\{0, 0\} \\ &= 0 \\ &= A(x,y). \end{aligned}$$

(b) If $x = (m, c)$ and $y = (n, c)$ with $n < m$, then $A(x, y) = 0$. Let $z \in L$. So

(b.1) If $z = \bot$, then

$$\begin{aligned} \min\{A(x,z), A(z,y)\} &= \min\left\{0, \frac{n}{2}\right\} \\ &= 0 \\ &= A(x,y). \end{aligned}$$

(b.2) If $z = \top$, then

$$\begin{aligned} \min\{A(x,z), A(z,y)\} &= \min\{0.5, 0\} \\ &= 0 \\ &= A(x,y). \end{aligned}$$

(b.3) Let $z = (p, c)$ and $p < n < m$, then

$$\begin{aligned} \min\{A(x,z), A(z,y)\} &= \min\left\{0, \frac{n-p}{2}\right\} \\ &= 0 \\ &= A(x,y). \end{aligned}$$

(b.4) Let $z = (p, c)$ and $n < p < m$, then

$$\begin{aligned} \min\{A(x,z), A(z,y)\} &= \min\{0, 0\} \\ &= 0 \\ &= A(x,y). \end{aligned}$$

(b.5) Let $z = (p, c)$ and $n < m < p$, then

$$\begin{aligned} \min\{A(x,z), A(z,y)\} &= \min\left\{\frac{p-m}{2}, 0\right\} \\ &= 0 \\ &= A(x,y). \end{aligned}$$

(b.6) Let $z = (p, c)$ and either $p = m$ or $p = n$. Then

$$\begin{aligned} \min\{A(x,z), A(z,y)\} &= \min\{1, 0\} \\ &= 0 \\ &= A(x,y). \end{aligned}$$

(c) If $x = (m, c)$ and $y = (n, d)$ with $c, d \in \{a, b\}$ and $c \neq d$.

(c.1) If $z = (p, c)$ and $m < p$ and $c \in \{a, b\}$. Then

$$\begin{aligned} \min\{A(x,z), A(z,y)\} &= \min\left\{\frac{p-m}{2}, 0\right\} \\ &= 0 \\ &= A(x,y). \end{aligned}$$

(c.2) If $z = (p, c)$ and $p < m$ and $c \in \{a, b\}$. Then

$$\begin{aligned} \min\{A(x,z), A(z,y)\} &= \min\{0, 0\} \\ &= 0 \\ &= A(x,y). \end{aligned}$$

(c.3) If $z = (p, d)$ and $p < n$ and $d \in \{a, b\}$. Then

$$\begin{aligned} \min\{A(x,z), A(z,y)\} &= \min\left\{0, \frac{n-p}{2}\right\} \\ &= 0 \\ &= A(x,y). \end{aligned}$$

(c.4) If $z = (p, d)$ and $n < p$ and $d \in \{a, b\}$. Then

$$\begin{aligned} \min\{A(x,z), A(z,y)\} &= \min\{0, 0\} \\ &= 0 \\ &= A(x,y). \end{aligned}$$

(c.5) If either $z = (p, c)$ such that $p = m$ or $z = (p, d)$ such that $p = n$. Then

$$\begin{aligned} \min\{A(x,z), A(z,y)\} &= \min\{1, 0\} \\ &= 0 \\ &= A(x,y). \end{aligned}$$

(c.6) If $z = \perp$. Then

$$\begin{aligned} \min\{A(x,z), A(z,y)\} &= \min\left\{0, \frac{n}{2}\right\} \\ &= 0 \\ &= A(x,y). \end{aligned}$$

(c.7) If $z = \top$. Then

$$\begin{aligned} \min\{A(x,z), A(z,y)\} &= \min\{0.5, 0\} \\ &= 0 \\ &= A(x,y). \end{aligned}$$

Hence, for all $z \in L$, $A(x,y) \geq \min\{A(x,z), A(z,y)\}$. Therefore,

$$A(x,y) \geq \sup_{z \in L} \min \{A(x,z), A(z,y)\}.$$

Therefore, (L, A) is a fuzzy partial order relation. To prove that (L, A) is a fuzzy lattice, we need to show that there exist infimum and supremum, for all $x, y \in L$.

Case 1: *If $A(x,y) > 0$, then by Proposition 3.1.1 (iv) and (v), we have that $x \wedge y = x$ and $x \vee y = y$.*

Case 2: *If $A(x,y) = 0$, then we have two situations:*

(a) If $x = \top$ and $y = \perp$;

(b) If $x = (m,c), y = (n,d), c, d \in \{a,b\}$ and $c \neq d$.

In both situations $x \wedge y = \perp$ and $x \vee y = \top$.

Case 3: *If $A(y,x) \geq 0$, then is analogous to the previous cases.*

Therefore, (L, A) is a fuzzy lattice.

Nevertheless, for all $\alpha > 0$, A_α is not a lattice since if $x = (\alpha, a)$ and $y = (\alpha, b)$, then $\{x, y\}$ has no lower bound in A_α. Suppose $l \in L$ is a lower bound in (L, A_α) for $\{x, y\}$, then $A(l,x) \geq \alpha$ and $A(l,y) \geq \alpha$, thus, by definition of A:

(i) If $l = x$, then $A(x,y) \geq \alpha$, but by definition, $A(l,y) = A((\alpha,a),(\alpha,b)) = 0$. Analogously, if $l = y$.

(ii) If $l = (z,a)$, then by definition, $A(l,y) = A((z,a),(\alpha,b)) = 0$. Analogously, if $l = (z,b)$.

(iii) If $l = \perp$, then $A(l,x) = A(\perp,(\alpha,a)) = \alpha/2 < \alpha$. Analogously, $A(l,y) = \alpha/2$.

Hence, $\{x, y\}$ has no lower bound in (L, A_α). Notice that the other two conditions are not applicable. ■

But we can build a lattice from a fuzzy lattice by consider their support as follows:

Proposition 3.1.3 *[Mezzomo et al., 2012b, Proposition 3.4] Let $A : X \times X \longrightarrow [0,1]$ be a fuzzy relation. If A is a fuzzy partial order relation on X, then $S(A)$ is a partial order relation on X.*

Proof: Let A be a fuzzy partial order relation on X. Since $A(x,x) = 1$ for all $x \in X$, $(x,x) \in S(A)$. Suppose $(x,y) \in S(A)$ and $(y,x) \in S(A)$. Then, $A(x,y) > 0$ and $A(y,x) > 0$, and hence, because A is fuzzy antisymmetric, $x = y$. Suppose $(x,y) \in S(A)$ and $(y,z) \in S(A)$. Then, $A(x,y) > 0$ and $A(y,z) > 0$. Since $A(x,z) \geq \sup_{y \in X} \min\{A(x,y), A(y,z)\}$, then $A(x,z) \geq \min\{A(x,y), A(y,z)\} > 0$, that is, $(x,z) \in S(A)$. ■

The following example show us that the converse does not hold.

Example 3.1.5 *Let $X = \{x,y,z\}$ be a set and let $A : X \times X \to [0,1]$ be the fuzzy order defined as $A(x,x) = A(y,y) = A(z,z) = 1.0$, $A(y,x) = A(z,x) = A(z,y) = 0.0$, $A(x,y) = 0.3, A(x,z) = 0.1$ and $A(y,z) = 0.8$. Clearly, $S(A)$ is a partial order relation whereas A is not a fuzzy partial order relation because it is not transitive, i.e., $0.1 = A(x,z) \ngeq \sup_{y \in X} \min\{A(x,y), A(y,z)\} = \min\{0.3, 0.8\} = 0.3$.* ■

Proposition 3.1.4 *[Mezzomo et al., 2013c, Proposition 2.3] Let (X,A) be a fuzzy poset, then $x \vee y$ and $x \wedge y$ coincides with $x \vee y$ and $x \wedge y$ in $(X, S(A))$.*

Proof: Consider $x, y \in X$. Suppose $A(x,y) > 0$, then $x \vee y = y$ in (X,A). Since $A(x,y) > 0$, then $(x,y) \in S(A)$ and $x \vee y = y$ in $(X, S(A))$. Now, suppose that $(x,y) \in S(A)$, then $A(x,y) > 0$ and $x \vee y = y$. Therefore, $x \vee y = y$ in terms of (X,A). If $A(y,x) > 0$, is analogously the previous. If the conditions $A(x,y) = 0$ and $A(y,x) = 0$ occur simultaneously, then we suppose $x \vee y = z$ in terms of (X,A). So, since $A(x,y) = 0$ and $A(y,x) = 0$, then $(x,y) \notin S(A)$ and $(y,x) \notin S(A)$. Hence, by hypothesis, $x \vee y = z$ in terms of $(X, S(A))$. On the other hands, if $(x,y) \notin S(A)$ and $(y,x) \notin S(A)$, then $A(x,y) = 0$ and $A(y,x) = 0$. Hence, by hypothesis, $x \vee y = z$ in terms of (X,A). The proof for $x \wedge y$ is analogous. ■

Corollary 3.1.1 *[Mezzomo et al., 2013c, Proposition 3.1] Let (X,A) be a fuzzy poset. (X,A) is a fuzzy lattice iff $(X, S(A))$ is a crisp lattice.*

PROOF:

($\Rightarrow$) Straightforward from Propositions 3.1.4 and 3.1.1.

($\Leftarrow$) Straightforward from Proposition 3.1.4.

■

Proposition 3.1.5 *Let (X, A) be a fuzzy poset. (X, A) is a bounded fuzzy lattice iff $(X, S(A))$ is a bounded crisp lattice.*

PROOF:

($\Rightarrow$) Let (X, A) be a bounded fuzzy lattice. Then, for all $x \in X$, $A(\bot, x) > 0$ and $A(x, \top) > 0$. So, $(\bot, x) \in S(A)$ and $(x, \top) \in S(A)$ and by Corollary 3.1.1, we have that $(X, S(A))$ is a lattice. Therefore, $(X, S(A))$ is a bounded lattice.

($\Leftarrow$) Let $(X, S(A))$ be a bounded lattice. Then, for all $x \in X$, $(\bot, x) \in S(A)$ and $(x, \top) \in S(A)$. So, $A(\bot, x) > 0$ and $A(x, \top) > 0$ and by Corollary 3.1.1, we have that (X, A) is a fuzzy lattice.

■

Definition 3.1.5 *[Mezzomo et al., 2013c, Definition 4.4] A fuzzy poset (X, A) is called **fuzzy sup-lattice** if each pair of elements has supremum on X. Duality, a fuzzy poset (X, A) is called **fuzzy inf-lattice** if each pair of elements has infimum on X.*

Remark 3.1.3 *Notice that a fuzzy poset is a fuzzy semi-lattice iff it is either fuzzy sup-lattice or fuzzy inf-lattice.*

We define supremum and infimum of a fuzzy set I on X as follows:

Definition 3.1.6 *[Mezzomo et al., 2013c, Definition 4.5] Let (X, A) be a fuzzy poset and I be a fuzzy set on X. $\sup I$ is an element of X such that if $x \in X$ and $\mu_I(x) > 0$, then $A(x, \sup I) > 0$ and if $u \in X$ is such that $A(x, u) > 0$ when $\mu_I(x) > 0$, then $A(\sup I, u) > 0$. Similarly, $\inf I$ is an element of X such that if $x \in X$ and $\mu_I(x) > 0$, then $A(\inf I, x) > 0$ and if $v \in X$ is such that $A(v, x) > 0$ when $\mu_I(x) > 0$, then $A(v, \inf I) > 0$.*

Definition 3.1.7 *[Mezzomo et al., 2013c, Definition 4.6] A fuzzy inf-lattice is called* ***fuzzy inf-complete*** *if its all nonempty fuzzy set admits infimum. Similarly, a fuzzy sup-lattice is called* ***fuzzy sup-complete*** *if its all nonempty fuzzy set admits supremum. A fuzzy lattice is called* ***complete fuzzy lattice*** *if it is simultaneously fuzzy inf-complete and fuzzy sup-complete.*

Proposition 3.1.6 *[Mezzomo et al., 2013c, Proposition 4.15 and 4.16] Let (X, A) be a sup-complete (inf-complete) fuzzy lattice and I be a fuzzy set on X. Then, $\sup I$ ($\inf I$) exists and it is unique.*

PROOF: The existence of $\sup I$ is guaranteed by Definition 3.1.7. Just let us prove the uniqueness of $\sup I$. Suppose u and v are $\sup I$. Then by Definition 3.1.6, $A(v, u) > 0$ and $A(u, v) > 0$. So, by antisymmetry, $u = v$.

Analogous we prove that if (X, A) be a complete fuzzy inf-lattice and I be a fuzzy set on X, then $\inf I$ exists and it is unique. ■

Proposition 3.1.7 *If (X, A) is a complete fuzzy lattice, then $(X, S(A))$ is a complete crisp lattice.*

PROOF: Let (X, A) be a complete fuzzy lattice and $Y \subseteq X$. Since, for each $x, y \in Y$, if $A(x, y) > 0$ then we have that $(x, y) \in S(A)$. So, by Proposition 3.1.1 (iv) and (v), all $Y \subseteq X$ has supremum and infimum. Therefore, $(X, S(A))$ is a complete lattice. ■

3.2 Fuzzy Homomorphism on Bounded Fuzzy Lattices

In this section, we define the fuzzy homomorphism between fuzzy lattices and prove some results involving fuzzy homomorphism on fuzzy lattices. The bounded fuzzy lattices denoted by $\mathcal{L} = (X, A)$ and $\mathcal{M} = (Y, B)$ are related to nonempty universal sets X and Y and fuzzy relations $A : X \times X \to [0, 1]$ and $B : Y \times Y \to [0, 1]$, respectively.

Definition 3.2.1 *[Mezzomo et al., 2013c, Definition 5.1] Let $\mathcal{L} = (X, A)$ and $\mathcal{M} = (Y, B)$ be bounded fuzzy lattices. A mapping $h : X \to Y$ is a* ***fuzzy homomorphism*** *from $\mathcal{L}$ into $\mathcal{M}$ if, for all $x, y \in X$, it satisfies the following conditions:*

(i) $h(x \wedge_{\mathcal{L}} y) = h(x) \wedge_{\mathcal{M}} h(y)$;

(ii) $h(x \vee_{\mathcal{L}} y) = h(x) \vee_{\mathcal{M}} h(y)$;

(iii) $h(0_{\mathcal{L}}) = 0_{\mathcal{M}}$;

(iv) $h(1_{\mathcal{L}}) = 1_{\mathcal{M}}$.

Example 3.2.1 *Let $\mathcal{L} = (X, A)$ be the fuzzy lattice defined in Example 3.1.1 and let $Y = \{x', y', z', v', w'\}$ and let $B : Y \times Y \longrightarrow [0, 1]$ be the fuzzy order relation over the set Y defined in the Figure 3.5.*
It is easy to prove that the fuzzy poset $\mathcal{M} = (Y, B)$ is a fuzzy lattice. Then, a fuzzy homomorphism h from $\mathcal{L}$ into $\mathcal{M}$ can be defined by $h(x) = x', h(y) = y', h(z) = z'$ and $h(w) = w'$. ■

B	x'	y'	z'	v'	w'
x'	1.0	0.1	0.4	0.7	0.9
y'	0.0	1.0	0.0	0.5	0.7
z'	0.0	0.0	1.0	0.5	0.6
v'	0.0	0.0	0.0	1.0	0.3
w'	0.0	0.0	0.0	0.0	1.0

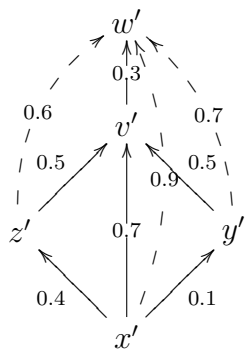

Figure 3.5: Representations of the fuzzy order relation B on Y.

Like in crisp algebra, fuzzy homomorphisms can be classified as:

(i) A **fuzzy monomorphism** is an injective fuzzy homomorphism;

(ii) A **fuzzy epimorphism** is a surjective fuzzy homomorphism;

(*iii*) A **fuzzy isomorphism** is a bijective fuzzy homomorphism.

Proposition 3.2.1 *[Mezzomo et al., 2013c, Proposition 5.1] Let $\mathcal{L} = (X, A)$ and $\mathcal{M} = (Y, B)$ be bounded fuzzy lattices and let a mapping $h : X \to Y$ be a fuzzy homomorphism. For all $x, y \in X$, if $A(x, y) > 0$, then $B(h(x), h(y)) > 0$.*

PROOF: Since $A(x, y) > 0$, then $x \wedge_{\mathcal{L}} y = x$. So, $h(x) = h(x \wedge_{\mathcal{L}} y) = h(x) \wedge_{\mathcal{M}} h(y)$ and therefore, $B(h(x), h(y)) > 0$. ■

Proposition 3.2.2 *[Mezzomo et al., 2013c, Proposition 5.2] Let $\mathcal{L} = (X, A)$ and $\mathcal{M} = (Y, B)$ be bounded fuzzy lattices and $h : X \to Y$ a map. Then, h is fuzzy order-preserving (i.e., if $A(x, y) > 0$ then $B(h(x), h(y)) > 0$) iff $B(h(x) \vee_{\mathcal{M}} h(y), h(x \vee_{\mathcal{L}} y)) > 0$, for all $x, y \in X$.*

PROOF:

($\Rightarrow$) For all $x, y \in X$, we have by Proposition 3.1.1 (*i*) that $A(x \wedge_{\mathcal{L}} y, x) > 0$ and $A(x \wedge_{\mathcal{L}} y, y) > 0$. Because h is a fuzzy order-preserving, then $B(h(x \wedge_{\mathcal{L}} y), h(x)) > 0$ and $B(h(x \wedge_{\mathcal{L}} y), h(y)) > 0$. And, by Proposition 3.1.1 (*iii*), we have that $B(h(x \wedge_{\mathcal{L}} y), h(x) \wedge_{\mathcal{M}} h(y)) > 0$.

($\Leftarrow$) For all $x, y \in X$, if $A(x, y) > 0$, then $x \vee_{\mathcal{L}} y = y$ and therefore $h(x \vee_{\mathcal{L}} y) = h(y)$. By hypothesis $B(h(x) \vee_{\mathcal{M}} h(y), h(x \vee_{\mathcal{L}} y)) > 0$. So, by Proposition 3.1.1 (*vii*), $B(h(x), h(x \vee_{\mathcal{L}} y)) > 0$. Hence $h(x \vee_{\mathcal{L}} y) = h(y)$, we have that $B(h(x), h(y)) > 0$.

■

Proposition 3.2.3 *[Mezzomo et al., 2013c, Proposition 5.3] Let $\mathcal{L} = (X, A)$ and $\mathcal{M} = (Y, B)$ be bounded fuzzy lattices and $h : X \to Y$ a map. Then, h is fuzzy order-preserving iff $B(h(x \wedge_{\mathcal{L}} y), h(x) \wedge_{\mathcal{M}} h(y)) > 0$, for all $x, y \in X$.*

PROOF: Analogous the Proposition 3.2.2. ■

Definition 3.2.2 *[Mezzomo et al., 2013c, Definition 5.2] Let $\mathcal{L} = (X, A)$ and $\mathcal{M} = (Y, B)$ be bounded fuzzy lattices. A mapping $h : X \to Y$ is a **fuzzy order-homomorphism** from $\mathcal{L}$ into $\mathcal{M}$ if, for all $x, y \in X$, satisfies the following conditions:*

(i) If $A(x, y) > 0$ then $B(h(x), h(y)) > 0$;

(ii) $h(\perp_{\mathcal{L}}) = \perp_{\mathcal{M}}$;

(iii) $h(\top_{\mathcal{L}}) = \top_{\mathcal{M}}$.

*If h is bijective then it is called a **fuzzy order-isomorphism**.*

Remark 3.2.1 *In particular, if h is a fuzzy homomorphism, then h is fuzzy order-preserving.*

Proposition 3.2.4 *[Mezzomo et al., 2013c, Proposition 5.4] Let $\mathcal{L} = (X, A)$ and $\mathcal{M} = (Y, B)$ be bounded fuzzy lattices. A map $h : X \to Y$ is a fuzzy order-isomorphism iff h is bijective and fuzzy order-preserving.*

PROOF:

($\Rightarrow$) Straightforward by Definition 3.2.2.

($\Leftarrow$) Suppose h is bijective and fuzzy order-preserving. The condition (i) of the Definition 3.2.2 is straightforward. So, we need to prove the conditions (ii) and (iii).
(ii) Suppose $h(\perp_{\mathcal{L}}) = a \neq \perp_{\mathcal{M}}$. By hypothesis h is bijective, then there exists $x \in X$ such that $h(x) = \perp_{\mathcal{M}}$. Because $A(\perp_{\mathcal{L}}, X) > 0$ then, by Definition 3.2.2 (i), $B(h(\perp_{\mathcal{L}}), h(x)) > 0$, i.e., $B(a, \perp_{\mathcal{M}}) > 0$. Therefore, $a = \perp_{\mathcal{M}}$ that is a contradiction.
(iii) Analogous to the previous case.

■

Note that if h is a fuzzy order-isomorphism, then its inverse is also a fuzzy order-isomorphism. Therefore, $A(x, y) > 0 \Leftrightarrow B(h(x), h(y)) > 0$.

Proposition 3.2.5 *[Mezzomo et al., 2013c, Proposition 5.5] Let $\mathcal{L} = (X, A)$ and $\mathcal{M} = (Y, B)$ be bounded fuzzy lattices and $h : X \to Y$ be a map. Then, h is a fuzzy isomorphism iff h is a fuzzy order-isomorphism.*

PROOF:

($\Rightarrow$) Let h be a fuzzy isomorphism and $x, y \in X$. Then by Proposition 3.1.1 (v) and Definition 3.2.1 (i),

$$\begin{aligned} A(x,y) > 0 &\Leftrightarrow x \wedge_{\mathcal{L}} y = x \\ &\Leftrightarrow h(x \wedge_{\mathcal{L}} y) = h(x) \\ &\Leftrightarrow h(x) \wedge_{\mathcal{M}} h(y) = h(x) \\ &\Leftrightarrow B(h(x), h(y)) > 0. \end{aligned}$$

Hence, since fuzzy isomorphism is bijective, then h is a fuzzy order-isomorphism.

($\Leftarrow$) Considering the Proposition 3.2.2, to prove that h is a fuzzy isomorphism, we need show that $B(h(x) \wedge_{\mathcal{M}} h(y), h(x \wedge_{\mathcal{L}} y)) > 0$ and $B(h(x \vee_{\mathcal{L}} y), h(x) \vee_{\mathcal{M}} h(y)) > 0$, for all $x, y \in X$. By hypothesis h is surjective, then there exists $z \in X$ such that $h(z) = h(x) \wedge_{\mathcal{M}} h(y)$. Then, $B(h(z), h(x)) > 0$ and $B(h(z), h(y)) > 0$. So, because h is fuzzy order-isomorphism, $A(z, x) > 0, A(z, y) > 0$. Thus, $A(z, x \wedge_{\mathcal{L}} y) > 0$ and, because h is fuzzy order-isomorphism, $B(h(z), h(x \wedge_{\mathcal{L}} y) > 0$. Hence, $B(h(x) \wedge_{\mathcal{M}} h(y), h(x \wedge_{\mathcal{L}} y)) > 0$. Therefore, by antisymmetry, we have that $h(x \wedge_{\mathcal{L}} y) = h(x) \wedge_{\mathcal{M}} h(y)$. By duality, we prove that $B(h(x \vee_{\mathcal{L}} y), h(x) \vee_{\mathcal{M}} h(y)) > 0$ and so, $h(x \vee_{\mathcal{L}} y) = h(x) \vee_{\mathcal{M}} h(y)$. Finally, because h is injective and surjective, then $h(\perp_{\mathcal{L}}) = \perp_{\mathcal{M}}$ and $h(\top_{\mathcal{L}}) = \top_{\mathcal{M}}$. Therefore, h is a bijective fuzzy homomorphism.

■

Definition 3.2.3 *[Mezzomo et al., 2013c, Definition 5.3] Let $\mathcal{L} = (X, A)$ and $\mathcal{M} = (Y, B)$ be bounded fuzzy lattices and $h : X \to Y$ be a map. Let $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ be the set of all fuzzy sets of X and Y, respectively. The function $\tilde{h} : \mathcal{F}(X) \to \mathcal{F}(Y)$ is defined by $\mu_{\tilde{h}(Z)}(x') = \sup\{\mu_Z(x) : h(x) = x' \text{ and } x \in X\}$ for each $Z \in \mathcal{F}(X)$. In addition, $\tilde{h}(Z)$ is called the **fuzzy image** of $Z \in \mathcal{F}(X)$ induced by h. Similarly, for each $W \in \mathcal{F}(Y)$, $\mu_{\overleftarrow{\tilde{h}}(W)}(x) = \mu_W(h(x))$. In addition, $\overleftarrow{\tilde{h}}(W)$ is called **fuzzy inverse image** from $W \in \mathcal{F}(Y)$ induced by h.*

Proposition 3.2.6 *[Mezzomo et al., 2013c, Proposition 5.6] Let $\mathcal{L} = (X, A)$ and $\mathcal{M} = (Y, B)$ be bounded fuzzy lattices, $h : X \to Y$ be a map, $Z \in \mathcal{F}(X)$ and*

$W \in \mathcal{F}(Y)$ be fuzzy sets. Then, for $x \in X$ and $y' \in Y$ we have that $\mu_{\overleftarrow{\tilde{h}}(\tilde{h}(Z))}(x) \geq \mu_Z(x)$ and $\mu_{\tilde{h}(\overleftarrow{\tilde{h}}(W))}(y') \leq \mu_W(y')$.

PROOF: Let $x \in X$. Then,

$$\begin{aligned} \mu_{\overleftarrow{\tilde{h}}(\tilde{h}(Z))}(x) &= \mu_{\tilde{h}(Z)}(h(x)) \\ &= \sup\{\mu_Z(z) : h(z) = h(x)\} \\ &\geq \mu_Z(x). \end{aligned}$$

Let $y' \in Y$. Then,

$$\begin{aligned} \mu_{\tilde{h}(\overleftarrow{\tilde{h}}(W))}(y') &= \sup\{\mu_{\overleftarrow{\tilde{h}}(W)}(x) : h(x) = y'\} \\ &= \sup\{\mu_W(h(x)) : h(x) = y'\}. \end{aligned}$$

If there exists at least a $x \in X$ such that $h(x) = y'$, i.e., $h^{-1}(y') \neq \emptyset$, then $\mu_{\tilde{h}(\overleftarrow{\tilde{h}}(W))}(y') = \mu_W(y')$. On the other hand, if $h^{-1}(y') = \emptyset$ then $\mu_{\tilde{h}(\overleftarrow{\tilde{h}}(W))}(y') = 0 \leq \mu_W(y')$. ■

Corollary 3.2.1 *[Mezzomo et al., 2013c, Corollary 5.1] Let $\mathcal{L} = (X, A)$ and $\mathcal{M} = (Y, B)$ be bounded fuzzy lattices. If $h : X \to Y$ is bijective, then $\mu_{\overleftarrow{\tilde{h}}(\tilde{h}(Z))}(x) = \mu_Z(x)$ and $\mu_{\tilde{h}(\overleftarrow{\tilde{h}}(W))}(y') = \mu_W(y')$.*

PROOF: Straightforward from Proposition 3.2.6. ■

Corollary 3.2.2 *[Mezzomo et al., 2013c, Corrolary 5.2] Let $\mathcal{L} = (X, A)$ and $\mathcal{M} = (Y, B)$ be bounded fuzzy lattices. If $h : X \to Y$ is surjective, then $\mu_{\tilde{h}(\overleftarrow{\tilde{h}}(W))}(y') = \mu_W(y')$.*

PROOF: Straightforward from Proposition 3.2.6. ■

Chapter 4

Operations on Bounded Fuzzy Lattices

In this chapter we will define the operations of product, collapsed sum, lifting, opposite, interval and intuitionist on the bounded fuzzy lattices as defined in the previous section for classical lattices, also providing a characterization based on their main properties.

The operations of product and collapsed sum on bounded fuzzy lattice were defined in Mezzomo et al. [2013b] as an extension of the classical theory. Furthermore, the product and collapsed sum on bounded fuzzy lattices were stated as fuzzy posets, and, consequently, as bounded fuzzy lattices. Extending these previous studies, in paper Mezzomo et al. [2013e], we focus on the lifting, opposite, interval and intuitionist operations on bounded fuzzy lattices. They are conceived as extensions of their analogous operations on the classical theory, by using the fuzzy partial order relation and the definition of fuzzy lattices, as conceived by Chon. In addition, we prove that lifting, opposite, interval and intuitionist on (complete) bounded fuzzy lattices are (complete) bounded fuzzy lattices introducing new results from both operators, product and collapsed sum, which already were defined in our previous paper Mezzomo et al. [2013b].

The bounded fuzzy lattices denoted by $\mathcal{L} = (X, A)$ and $\mathcal{M} = (Y, B)$ are related to nonempty universal sets X and Y and fuzzy relations $A : X \times X \to [0, 1]$ and $B : Y \times Y \to [0, 1]$, respectively.

4.1 Product Operator

Let $\mathcal{L}$ and $\mathcal{M}$ be fuzzy posets. The product of $\mathcal{L}$ and $\mathcal{M}$ is denoted by $\mathcal{L} \times \mathcal{M} = (X \times Y, C)$ and, for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, its fuzzy partial order defined as

$$C((x_1, y_1), (x_2, y_2)) = \min\{A(x_1, x_2), B(y_1, y_2)\}.$$

Lemma 4.1.1 *[Mezzomo et al., 2013b, Lemma 4.1] Let $\mathcal{L} = (X, A)$ and $\mathcal{M} = (Y, B)$ be fuzzy posets. Then $\mathcal{L} \times \mathcal{M}$ is a fuzzy poset.*

Proof: The reflexivity and antisymmetry are straightforward. In order to prove the transitivity, consider $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.

$$\begin{aligned}
& C((x_1, y_1), (x_2, y_2)) = \min\{A(x_1, x_2), B(y_1, y_2)\} \\
\geq\ & \min\{\sup_{x_3 \in X} \min\{A(x_1, x_3), A(x_3, x_2)\}, \sup_{y_3 \in Y} \min\{B(y_1, y_3), B(y_3, y_2)\}\} \\
=\ & \sup_{(x_3, y_3) \in X \times Y} \{\min\{\min\{A(x_1, x_3), A(x_3, x_2)\}, \min\{B(y_1, y_3), B(y_3, y_2)\}\}\} \\
=\ & \sup_{(x_3, y_3) \in X \times Y} \min\{A(x_1, x_3), A(x_3, x_2), B(y_1, y_3), B(y_3, y_2)\} \\
=\ & \sup_{(x_3, y_3) \in X \times Y} \min\{\min\{A(x_1, x_3), B(y_1, y_3)\}, \min\{A(x_3, x_2), B(y_3, y_2)\}\} \\
=\ & \sup_{(x_3, y_3) \in X \times Y} \min\{C((x_1, y_1), (x_3, y_3)), C((x_3, y_3), (x_2, y_2))\}.
\end{aligned}$$

Therefore, we conclude that $\mathcal{L} \times \mathcal{M}$ is a fuzzy poset. ■

Proposition 4.1.1 *[Mezzomo et al., 2013b, Proposition 4.1] Let $\mathcal{L} = (X, A)$ and $\mathcal{M} = (Y, B)$ be bounded fuzzy lattices. Then $\mathcal{L} \times \mathcal{M}$ is a bounded fuzzy lattice.*

Proof: Consider $x_1, x_2, x_3 \in X$ and $y_1, y_2, y_3 \in Y$. Clearly, $(x_1, y_1) \wedge_\times (x_2, y_2) = (x_1 \wedge_\mathcal{L} x_2, y_1 \wedge_\mathcal{M} y_2)$ is well defined and $(x_1, y_1) \wedge_\times (x_2, y_2)$ is a lower bound of $\{(x_1, y_1), (x_2, y_2)\}$. Suppose that $(x_3, y_3) \in X \times Y$ is also a lower bound of $\{(x_1, y_1), (x_2, y_2)\}$. Then it holds that $C((x_3, y_3), (x_1, y_1)) > 0$ and $C((x_3, y_3), (x_2, y_2)) > 0$. So, $A(x_3, x_1) > 0$, $A(x_3, x_2) > 0$, $B(y_3, y_1) > 0$ and $B(y_3, y_2) > 0$. Hence, $A(x_3, x_1 \wedge_\mathcal{L} x_2) > 0$ and $B(y_3, y_1 \wedge_\mathcal{M} y_2) > 0$. So, we obtain that $C((x_3, y_3), (x_1, y_1) \wedge_\times (x_2, y_2)) = C((x_3, y_3), (x_1 \wedge_\mathcal{L} x_2, y_1 \wedge_\mathcal{M} y_2)) > 0$ and therefore

$(x_1, y_1) \wedge_\times (x_2, y_2)$ is the infimum of $\{(x_1, y_1), (x_2, y_2)\}$ in $\mathcal{L} \times \mathcal{M}$. Analogously, we prove that $(x_1, y_1) \vee_\times (x_2, y_2)$ is the supremum of $\{(x_1, y_1), (x_2, y_2)\}$ in $\mathcal{L} \times \mathcal{M}$. As $\mathcal{L}$ and $\mathcal{M}$ are bounded fuzzy lattices, then trivially, $(\bot_\mathcal{L}, \bot_\mathcal{M})$ and $(\top_\mathcal{L}, \top_\mathcal{M})$ are the bottom and top elements, respectively. Therefore, by Lemma 4.1.1 and the last results we can conclude that $\mathcal{L} \times \mathcal{M}$ is a bounded fuzzy lattice. ■

Proposition 4.1.2 *[Mezzomo et al., 2013e, Proposition 4.2] Let $\mathcal{L} = (X, A)$ and $\mathcal{M} = (Y, B)$ be complete fuzzy lattices. Then, $\mathcal{L} \times \mathcal{M} = (X \times Y, C)$ is a complete fuzzy lattice.*

Proof: First we will to prove that $\mathcal{L} \times \mathcal{M}$ is a sup-complete fuzzy lattice. For that, let $\mathcal{L} = (X, A)$ and $\mathcal{M} = (Y, B)$ be complete fuzzy lattices and let I be a nonempty set on $X \times Y$. Let $I_x = \{x \in X : (x, y) \in I \text{ for some } y \in Y\}$ and $I_y = \{y \in Y : (x, y) \in I \text{ for some } x \in X\}$. By hypothesis $\mathcal{L}$ and $\mathcal{M}$ are complete fuzzy lattices, then there exist $\sup I_x$ and $\sup I_y$. We will prove that $(\sup I_x, \sup I_y)$ is the supremum of I. Clearly, $(\sup I_x, \sup I_y)$ is an upper bound of I. Suppose $(x_2, y_2) \in X \times Y$ is also an upper bound of I. Then, $A(\sup I_x, x_2) > 0$ and $B(\sup I_y, y_2) > 0$ and so, $C((\sup I_x, \sup I_y), (x_2, y_2)) > 0$. Therefore, $(\sup I_x, \sup I_y)$ is the supremum of I and $\mathcal{L} \times \mathcal{M}$ is a sup-complete fuzzy lattice. In the same manner, we prove that $(\inf I_x, \inf I_y)$ is the infimum of I and $\mathcal{L} \times \mathcal{M}$ is a inf-complete fuzzy lattice. Therefore, together with Proposition 4.1.1 and the above results we can conclude that $\mathcal{L} \times \mathcal{M}$ is a complete fuzzy lattice. ■

Example 4.1.1 *Let $\mathcal{L} = (X, A)$ be the fuzzy lattice defined in Example 3.1.1 and let $Y = \{x_2, y_2, z_2, w_2\}$ and $B : Y \times Y \to [0, 1]$ be the fuzzy lattice of the Figure (4.1). The related complete fuzzy lattice $\mathcal{L} \times \mathcal{M} = (X \times Y, C)$ is described in Table 4.1.* ■

B	x_2	y_2	z_2	w_2
x_2	1.0	0.1	0.3	0.9
y_2	0.0	1.0	0.0	0.6
z_2	0.0	0.0	1.0	0.4
w_2	0.0	0.0	0.0	1.0

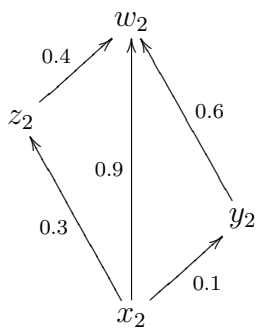

Figure 4.1: Representations of the fuzzy lattice $\mathcal{M} = (Y, B)$.

C	(x_1,x_2)	(x_1,y_2)	(x_1,z_2)	(x_1,w_2)	(y_1,x_2)	(y_1,y_2)	(y_1,z_2)	(y_1,w_2)	(z_1,x_2)	(z_1,y_2)	(z_1,z_2)	(z_1,w_2)	(w_1,x_2)	(w_1,y_2)	(w_1,z_2)	(w_1,w_2)
(x_1,x_2)	1.0	0.1	0.3	0.9	0.1	0.1	0.1	0.1	0.4	0.1	0.3	0.4	0.8	0.1	0.3	0.8
(x_1,y_2)	0.0	1.0	0.0	0.6	0.0	0.1	0.0	0.1	0.0	0.4	0.0	0.4	0.0	0.8	0.0	0.8
(x_1,z_2)	0.0	0.0	1.0	0.4	0.0	0.0	0.1	0.1	0.0	0.0	0.4	0.4	0.0	0.0	0.8	0.8
(x_1,w_2)	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.4	0.0	0.0	0.0	0.8
(y_1,x_2)	0.0	0.0	0.0	0.0	1.0	0.1	0.3	0.9	0.2	0.1	0.2	0.2	0.5	0.1	0.3	0.5
(y_1,y_2)	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.6	0.0	0.2	0.0	0.2	0.0	0.5	0.0	0.5
(y_1,z_2)	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.4	0.0	0.0	0.2	0.2	0.0	0.0	0.5	0.4
(y_1,w_2)	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.2	0.0	0.0	0.0	0.3
(z_1,x_2)	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.1	0.3	0.9	0.3	0.1	0.3	0.3
(z_1,y_2)	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.6	0.0	0.3	0.0	0.3
(z_1,z_2)	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.4	0.0	0.0	1.0	0.4
(z_1,w_2)	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.3
(w_1,x_2)	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.1	0.3	0.9
(w_1,y_2)	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.6
(w_1,z_2)	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.4
(w_1,w_2)	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0

Table 4.1: Product Operator $\mathcal{L} \times \mathcal{M}$.

4.2 Collapsed Sum Operator

The collapsed sum between fuzzy posets $\mathcal{L}$ and $\mathcal{M}$, denoted by $\mathcal{L}\oplus\mathcal{M} = (X\oplus Y, C)$, is defined by the set

$$X \oplus Y = (X - \{\bot_{\mathcal{L}}, \top_{\mathcal{L}}\} \times \{l\}) \cup (Y - \{\bot_{\mathcal{M}}, \top_{\mathcal{M}}\} \times \{m\}) \cup (\{\bot, \top\}),$$

and the corresponding fuzzy relation on $X \oplus Y$, given as

$$C(x,y) = \begin{cases} A(\hat{x},\hat{y}), & \text{if } x = (\hat{x}, l) \text{ and } y = (\hat{y}, l); \\ B(\hat{x},\hat{y}), & \text{if } x = (\hat{x}, m) \text{ and } y = (\hat{y}, m); \\ 1, & \text{if } x = \bot \text{ or } y = \top; \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

In the next lemma and two propositions, the fuzzy complete (bounded) poset constructions of $\mathcal{L} = (X, A)$ and $\mathcal{M} = (Y, B)$ are preserved by the collapsed sum $\mathcal{L} \oplus \mathcal{M}$.

Lemma 4.2.1 *[Mezzomo et al., 2013b, Lemma 4.2] Let $\mathcal{L} = (X, A)$ and $\mathcal{M} = (Y, B)$ be fuzzy posets. Then $\mathcal{L} \oplus \mathcal{M}$ is a fuzzy poset.*

Proof: The reflexivity and antisymmetry properties are straightforward. In order to prove the transitivity, consider the following cases: Let $x, z \in X \oplus Y$.

(i) If $x = (\hat{x}, l)$, $y = (\hat{y}, l)$, then $C((\hat{x}, l), (\hat{z}, l)) = A(\hat{x}, \hat{z})$. Since we have that $A(\hat{x}, \hat{z}) \geq \sup_{\hat{y}\in X} \min \{A(\hat{x}, \hat{y}), A(\hat{y}, \hat{z})\}$, then it holds that

$$\begin{aligned} C(x,z) &= C((\hat{x}, l), (\hat{z}, l)) \\ &= A(\hat{x}, \hat{z}) \\ &\geq \sup_{\hat{y}\in X} \min\{A(\hat{x}, \hat{y}), A(\hat{y}, \hat{z})\} \\ &\geq \sup_{\hat{y}\in X} \min\{C((\hat{x}, l), (\hat{y}, l)), C((\hat{y}, l), (\hat{z}, l))\} \\ &= \sup_{y\in X\oplus Y} \min\{C((\hat{x}, l), y), C(y, (\hat{z}, l))\} \\ &= \sup_{y\in X\oplus Y} \min\{C(x, y), C(y, z)\}. \end{aligned}$$

(ii) The case $\hat{x}, \hat{y} \in Y - \{\perp_{\mathcal{M}}, \top_{\mathcal{M}}\}$ is analogous to the previous case.

(iii) If $\hat{x} \in X - \{\perp_{\mathcal{L}}, \top_{\mathcal{L}})$ and $\hat{z} \in Y - \{\perp_{\mathcal{M}}, \top_{\mathcal{M}})$, then $C((\hat{x}, l), (\hat{z}, m)) = 0$. Moreover, for all $y \in X \oplus Y$,

(a) if $y = (\hat{y}, l)$ then $C((\hat{y}, l), (\hat{z}, m)) = 0$ and we can conclude that $\min\{C((\hat{x}, l), (\hat{y}, l)), C((\hat{y}, l), (\hat{z}, m))\} = 0$.

(b) if $y = (\hat{y}, m)$ then $C((\hat{x}, l), (\hat{y}, m)) = 0$ and we can also conclude that $\min\{C((\hat{x}, l), (\hat{y}, m)), C((\hat{y}, m), (\hat{z}, m))\} = 0$.

(c) if $y = \perp$, $C((\hat{x}, l), \perp) = 0$ and then $\min\{C((\hat{x}, l), \perp), C(\perp, (\hat{z}, m))\} = 0$.

(d) $y = \top$, $C(\top, (\hat{y}, m)) = 0$ and $\min\{C((\hat{x}, l), \top), C(\top, (\hat{z}, m))\} = 0$. So, $C((\hat{x}, l), (\hat{z}, m)) = 0 = \sup\{\perp\} = \sup\limits_{y \in X \oplus Y} \min\{C((\hat{x}, l), y), C(y, (\hat{z}, m))\}$.

(iv) If $x = \top$ and $z \neq \top$, then $C(x, z) = 0$ and for all $y \in X \oplus Y$, either $C(x, y) = 0$ or $C(y, z) = 0$ and therefore, $\min\{C(x, y), C(y, z)\} = 0$. So, $C(x, z) = 0 = \sup\limits_{y \in X \oplus Y} \min\{C(x, y), C(y, z)\}$.

(v) If $x \neq \perp$ and $z = \perp$, then for all $y \in X \oplus Y$, either $C(x, y) = 0$ or $C(y, z) = 0$ and therefore, $\min\{C(x, y), C(y, z)\} = 0$. Concluding, $C(x, z) = 0 = \sup\limits_{y \in X \oplus Y} \min\{C(x, y), C(y, z)\}$.

(vi) If $x = \perp$ or $z = \top$, then for all $y \in X \oplus Y$, $C(x, y) > 0$ or $C(y, z) > 0$. So $\min\{C(x, y), C(y, z)\} > 0$ and $C(x, z) \geq \sup\limits_{y \in X \oplus Y} \min\{C(x, y), C(y, z)\}$. Therefore, for all $x, z \in X \oplus Y$, $C(x, z) \geq \sup\limits_{y \in X \oplus Y} \min\{C(x, y), C(y, z)\}$.

Concluding, $\mathcal{L} \oplus \mathcal{M}$ is a fuzzy poset. ■

Proposition 4.2.1 *[Mezzomo et al., 2013b, Proposition 4.2] Let $\mathcal{L} = (X, A)$ and $\mathcal{M} = (Y, B)$ be bounded fuzzy lattices. Then $\mathcal{L} \oplus \mathcal{M}$ is a bounded fuzzy lattice.*

PROOF: Let $x, y, z \in X \oplus Y$. The binary operator $\wedge_\oplus$ on $\mathcal{L} \oplus \mathcal{M}$ is given as

$$x \wedge_\oplus y = \begin{cases} (\hat{x} \wedge_\mathcal{L} \hat{y}, l), & \text{if } x = (\hat{x}, l) \text{ and } y = (\hat{y}, l); \\ (\hat{x} \wedge_\mathcal{M} \hat{y}, m), & \text{if } x = (\hat{x}, m) \text{ and } y = (\hat{y}, m); \\ x, & \text{if } x \neq \top \text{ and } y = \top; \\ y, & \text{if } x = \top \text{ and } y \neq \top; \\ \perp, & \text{otherwise.} \end{cases}$$

Clearly, $x \wedge_\oplus y$ is well defined, i.e., $x \wedge_\oplus y \in X \oplus Y$, and $x \wedge_\oplus y$ is a lower bound of x and y. Suppose that $z \in X \oplus Y$ is also a lower bound of x and y, i.e., $C(z, x) > 0$ and $C(z, y) > 0$. Then, the following cases hold.

(i) If $x = (\hat{x}, l)$ and $y = (\hat{y}, l)$, then $z = \perp$ or $z = (\hat{z}, l)$. If $z = \perp$, then trivially, $C(z, x \wedge_\oplus y) > 0$. If $z = (\hat{x}, l)$, then $A(\hat{z}, \hat{x}) > 0$ and $A(\hat{z}, \hat{y}) > 0$, meaning that $A(\hat{z}, \hat{x} \wedge_\mathcal{L} \hat{y}) > 0$. Therefore, in both inequalities, $C(z, x \wedge_\oplus y) > 0$.

(ii) If $x = (\hat{x}, m)$ and $y = (\hat{y}, m)$ the proof is analogous to the previous case.

(iii) If $x = (\hat{x}, i)$, $y = (\hat{y}, j)$ and $i \neq j$ is straightforward.

(iv) In the case of $x \in \{0, 1\}$ or $y \in \{0, 1\}$, it is straightforward.

Therefore, $x \wedge_\oplus y$ is the infimum in $\mathcal{L} \oplus \mathcal{M}$. Similarly, we define $\vee_\oplus$ by

$$x \vee_\oplus y = \begin{cases} (\hat{x} \vee_\mathcal{L} \hat{y}, l), & \text{if } x = (\hat{x}, l) \text{ and } y = (\hat{y}, l); \\ (\hat{x} \vee_\mathcal{M} \hat{y}, m), & \text{if } x = (\hat{x}, m) \text{ and } y = (\hat{y}, m); \\ x, & \text{if } x \neq \perp \text{ and } y = \perp; \\ y, & \text{if } x = \perp \text{ and } y \neq \perp; \\ \top, & \text{otherwise.} \end{cases}$$

and the proof that $x \vee_\oplus y$ is the supremum in $\mathcal{L} \oplus \mathcal{M}$ is analogously obtained. From equation (4.1), it holds that $\perp$ and $\top$ are the bottom and top, respectively. ■

Proposition 4.2.2 *[Mezzomo et al., 2013e, Proposition 4.4] Let $\mathcal{L} = (X, A)$ and $\mathcal{M} = (Y, B)$ be complete fuzzy lattices. Then, $\mathcal{L} \oplus \mathcal{M} = (X \oplus Y, C)$ is a complete fuzzy lattice.*

C	$\perp$	(y_1, l)	(z_1, l)	(y_2, m)	(z_2, m)	$\top$
$\perp$	1.0	1.0	1.0	1.0	1.0	1.0
(y_1, l)	0.0	1.0	0.2	0.0	0.0	1.0
(z_1, l)	0.0	0.0	1.0	0.0	0.0	1.0
(y_2, m)	0.0	0.0	0.0	1.0	0.0	1.0
(z_2, m)	0.0	0.0	0.0	0.0	1.0	1.0
$\top$	0.0	0.0	0.0	0.0	0.0	1.0

Table 4.2: Collapsed Sum Operator $\mathcal{L} \oplus \mathcal{M}$

PROOF: First we will to prove that $\mathcal{L} \oplus \mathcal{M}$ is a fuzzy lattice inf-complete. Let $\mathcal{L} = (X, A)$ and $\mathcal{M} = (Y, B)$ be complete fuzzy lattices and let I be a nonempty fuzzy set on $X \oplus Y$, that is, $\mu_I : X \oplus Y \to [0, 1]$. Additionally, let I_X be a fuzzy set on X and let I_Y be a fuzzy set on Y, such that, $\mu_I(x, y) = \min(\mu_{I_X}(x), \mu_{I_Y}(y))$. Then, the next conditions are hold:

(i) If $\inf_{\mathcal{L}}(I_X) \neq \perp_{\mathcal{L}}$ and $\inf_{\mathcal{M}}(I_Y) \neq \perp_{\mathcal{M}}$, $\inf_{\mathcal{L}\oplus\mathcal{M}}(I) = (\inf_{\mathcal{L}}(I_X), \inf_{\mathcal{M}}(I_Y))$;

(ii) If $\inf_{\mathcal{L}}(I_X) = \perp_{\mathcal{L}}$ and $\inf_{\mathcal{M}}(I_Y) \neq \perp_{\mathcal{M}}$ then $\inf_{\mathcal{L}\oplus\mathcal{M}}(I) = (\perp, \inf_{\mathcal{M}}(I_Y))$;

(iii) If $\inf_{\mathcal{L}}(I_X) \neq \perp_{\mathcal{L}}$ and $\inf_{\mathcal{M}}(I_Y) = \perp_{\mathcal{M}}$ then $\inf_{\mathcal{L}\oplus\mathcal{M}}(I) = (\inf_{\mathcal{L}}(I_X), \perp)$;

(iv) If $\inf_{\mathcal{L}}(I_X) = \perp_{\mathcal{L}}$ and $\inf_{\mathcal{M}}(I_Y) = \perp_{\mathcal{M}}$ then clearly, $\inf_{\mathcal{L}\oplus\mathcal{M}}(I) = (\perp, \perp)$.

Similarly we define $\mu_I(x, y) = \max(\mu_{I_X}(x), \mu_{I_Y}(y))$ and find $\sup_{\mathcal{L}\oplus\mathcal{M}}(I)$. Therefore $\mathcal{L} \oplus \mathcal{M}$ is a complete fuzzy lattice. ■

Example 4.2.1 *Let $\mathcal{L} = (X, A)$ and $\mathcal{M} = (Y, B)$ be fuzzy lattices defined in Example 3.1.1 and Example 4.1.1, respectively. The collapsed sum $\mathcal{L} \oplus \mathcal{M}$ is represented in the Table 4.2.* ■

4.3 Lifting Operator

The lifting of fuzzy posets $\mathcal{L} = (X, A)$ is denoted by $\mathcal{L}_\uparrow = (X_{\perp_\uparrow}, A_\uparrow)$, where $\perp_\uparrow \notin X$, $X_{\perp_\uparrow} = X \cup \{\perp_\uparrow\}$ and, for all $x, y \in X_{\perp_\uparrow}$, $A_\uparrow$ is defined by

$$A_\uparrow(x,y) = \begin{cases} A(x,y), & \text{if } x \neq \perp_\uparrow \text{ and } y \neq \perp_\uparrow; \\ 1, & \text{if } x = \perp_\uparrow; \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 4.3.1 *[Mezzomo et al., 2013e, Lemma 4.3] Let $\mathcal{L} = (X, A)$ be a fuzzy poset. Then $\mathcal{L}_\uparrow$ is a fuzzy poset.*

Proof: The reflexivity and antisymmetry is straightforward. In order to prove he transitivity, let $x, y \in X_{\perp_\uparrow}$. So, the next cases are held.

(i) If $x \neq \perp_\uparrow$ and $y \neq \perp_\uparrow$, by definition $A_\uparrow(x,y) = A(x,y)$. Because $\mathcal{L}$ is a fuzzy poset then, for all $x, z \in X$, $A(x,z) \geq \sup_{y \in X} \min\{A(x,y), A(y,z)\}$. Moreover, since $A_\uparrow(x,z) = A(x,z)$, it implies that

$$A_\uparrow(x,z) \geq \sup_{y \in X} \min\{A(x,y), A(y,z)\} = \sup_{y \in X_{\perp_\uparrow}} \min\{A_\uparrow(x,y), A_\uparrow(y,z)\}$$

(ii) If $x = \perp_\uparrow$, then $A_\uparrow(x,z) = 1$, for all $z \in X_{\perp_\uparrow}$. So, we can conclude that $A_\uparrow(x,z) = 1 \geq \sup_{y \in X_{\perp_\uparrow}} \min\{A_\uparrow(x,y),\ A_\uparrow(y,z)\}$.

(iii) If $z = \perp_\uparrow$, then $A_\uparrow(\perp_\uparrow, \perp_\uparrow) = 1$ and $A_\uparrow(x, \perp_\uparrow) = 0$, for all $x \in X$. So, trivially $A_\uparrow(\perp_\uparrow, \perp_\uparrow) = 1 \geq \sup_{y \in X_{\perp_\uparrow}} \min\{A_\uparrow(\perp_\uparrow, y), A_\uparrow(y, \perp_\uparrow)\}$ and, for all $x \in X$,

$$\begin{aligned} 0 &= \min\{A_\uparrow(x,y), A_\uparrow(y, \perp_\uparrow)\} \\ &\leq \sup_{y \in X_{\perp_\uparrow}} \min\{A_\uparrow(x,y), A_\uparrow(y, \perp_\uparrow)\} \\ &\leq A_\uparrow(x, \perp_\uparrow) \\ &= 0 \end{aligned}$$

Therefore, $\mathcal{L}_\uparrow$ is a fuzzy poset. ■

Proposition 4.3.1 *[Mezzomo et al., 2013e, Proposition 4.5] Let $\mathcal{L} = (X, A)$ be bounded fuzzy lattice. Then $\mathcal{L}_\uparrow$ is a bounded fuzzy lattice.*

PROOF: Let $x, y \in X_{\perp_\uparrow}$. Consider the operator $\wedge_\uparrow$ on $\mathcal{L}_\uparrow = (X, A)$ defined by

$$x \wedge_\uparrow y = \begin{cases} x \wedge_{\mathcal{L}} y, & \text{if } x \neq 0 \text{ and } y \neq 0; \\ \perp_{\mathcal{L}}, & \text{otherwise.} \end{cases}$$

Clearly, $x \wedge_\uparrow y$ is well defined and it is a lower bound of $\{x, y\}$. Suppose that $z \in X_{\perp_\uparrow}$ is also a lower bound of $\{x, y\}$, i.e., $A_\uparrow(z, x) > 0$ and $A_\uparrow(z, y) > 0$. The following three cases hold:

(*i*) If $z = \perp_\uparrow$ then trivially, $A_\uparrow(z, x \wedge_\uparrow y) > 0$;

(*ii*) If $z \neq \perp_\uparrow$, $z \neq \perp_\uparrow$ and $y \neq \perp_\uparrow$ then $A(z, x) > 0$ and $A(z, y) > 0$. So, $A(z, x \wedge y) > 0$ and hence we obtain that $A_\uparrow(z, x \wedge_\uparrow y) > 0$, for all $x, y \in X$;

(*iii*) If $x = \perp_\uparrow$ or $y = \perp_\uparrow$, then $z = \perp_\uparrow$ and therefore, $A_\uparrow(z, x \wedge_\uparrow y) > 0$.

Thus, $x \wedge_\uparrow y$ is the infimum in $\mathcal{L}_\uparrow$. Similarly, we are able to provide the analogous proof for the supremum in $\mathcal{L}_\uparrow$ by taking the binary operator $\vee_\uparrow$ on $\mathcal{L}_\uparrow = (X, A)$ which is, for all $x, y \in X_{\perp_\uparrow}$, expressed as

$$x \vee_\uparrow y = \begin{cases} x \vee_{\mathcal{L}} y, & \text{if } x \neq 0 \text{ and } y \neq 0; \\ \perp_{\mathcal{L}}, & \text{if } x = 0 \text{ and } y = 0; \\ \max\{x, y\}, & \text{otherwise.} \end{cases}$$

Therefore, we can conclude that $\mathcal{L}_\uparrow$ is a bounded fuzzy lattice. ■

Proposition 4.3.2 *[Mezzomo et al., 2013e, Proposition 4.6] Let $\mathcal{L} = (X, A)$ be complete fuzzy lattice. Then, $\mathcal{L}_\uparrow = (X_{\perp_\uparrow}, A_\uparrow)$ is a complete fuzzy lattice.*

PROOF: In order to prove that $\mathcal{L}_\uparrow$ is a fuzzy lattice inf-complete, let $\mathcal{L} = (X, A)$ be a complete fuzzy lattice, I be a non-empty fuzzy set on $X_{\perp_\uparrow}$, meaning that

$\mu_I : X_{\perp_\uparrow} \to [0,1]$ and I_X be a fuzzy set on X such that $\mu_I(x) = \min(\mu_{I_X}(x))$. By hypothesis, $\mathcal{L}$ is a complete fuzzy lattice, then clearly, $\inf_{\mathcal{L}_\uparrow} = (\inf_{\mathcal{L}}(I_X))$ if $\inf_{\mathcal{L}}(I_X) \neq \perp_\uparrow$ and $\inf_{\mathcal{L}_\uparrow} = \perp_\uparrow$ if $\inf_{\mathcal{L}}(I_X) = \perp_\uparrow$.
Similarly, we define $\mu_I(x) = \max(\mu_{I_X}(x))$ and clearly $\sup_{\mathcal{L}_\uparrow} = (\sup_{\mathcal{L}}(I_X))$, for all I on X. Therefore $\mathcal{L}_\uparrow$ is a complete fuzzy lattice. ■

Example 4.3.1 *Let $\mathcal{L} = (X, A)$ be fuzzy lattice defined in Example 3.1.1. The lifting $\mathcal{L}_\uparrow$ is represented in the Figure 4.2.* ■

$A_\uparrow$	$\perp_\uparrow$	x_1	y_1	z_1	w_1
$\perp_\uparrow$	1.0	1.0	1.0	1.0	1.0
x_1	0.0	1.0	0.1	0.4	0.8
y_1	0.0	0.0	1.0	0.2	0.5
z_1	0.0	0.0	0.0	1.0	0.3
w_1	0.0	0.0	0.0	0.0	1.0

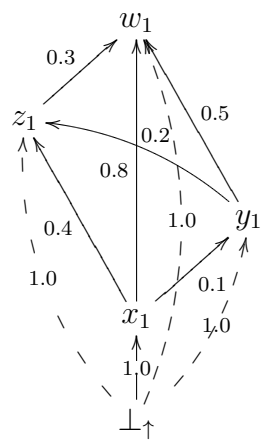

Figure 4.2: Lifting Operator $\mathcal{L}_\uparrow$.

4.4 Opposite Operator

The opposite (or dual) of a fuzzy poset $\mathcal{L} = (X, A)$ is denoted by $\mathcal{L}^{op} = (X^{op}, A^{op})$ and, for all $x, y \in X^{op}$ [1], the operator A^{op} is defined by

$$A^{op}(x, y) = A(y, x).$$

Thus, for all $x, y \in X$ such that $A(x, y) = 0$ and $A(y, x) = \alpha$, $\alpha \in (0, 1]$, then $A^{op}(x, y) = \alpha$ and $A^{op}(y, x) = 0$, respectively. So, the corresponding definition of the operations $\wedge_{op}$ and $\vee_{op}$ are given as

[1] Notice that $X^{op} = X$ because what differs $\mathcal{L}$ from $\mathcal{L}^{op}$ is the fuzzy order relation and not the elements. We denote the set of the fuzzy lattice $\mathcal{L}^{op}$ by X^{op} for do not confusing with a set X of the fuzzy lattice $\mathcal{L}$.

$$x \wedge^{op} y = x \vee y \quad \text{and} \quad x \vee^{op} y = x \wedge y.$$

Lemma 4.4.1 *[Mezzomo et al., 2013e, Lemma 4.4] Let $\mathcal{L} = (X, A)$ be a fuzzy poset. Then $\mathcal{L}^{op}$ is a fuzzy poset.*

PROOF: Straightforward. ■

Proposition 4.4.1 *[Mezzomo et al., 2013e, Proposition 4.7] Let $\mathcal{L} = (X, A)$ be a (complete) bounded fuzzy lattice. Then $\mathcal{L}^{op}$ is a (complete) bounded fuzzy lattice.*

PROOF: Straightforward. ■

Example 4.4.1 *Let $\mathcal{L}$ =(X, A) be fuzzy lattice defined in Example 3.1.1. The opposite $\mathcal{L}^{op}$ is represented in the Figure 4.3.* ■

A^{op}	x_1	y_1	z_1	w_1
x_1	1.0	0.0	0.0	0.0
y_1	0.1	1.0	0.0	0.0
z_1	0.4	0.2	1.0	0.0
w_1	0.8	0.5	0.3	1.0

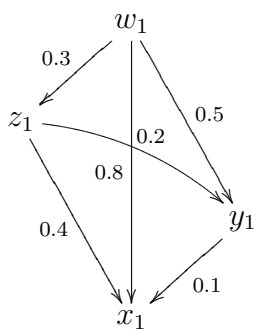

Figure 4.3: Opposite Operator $\mathcal{L}^{op}$.

4.5 Interval Operator

The interval of a fuzzy poset $\mathcal{L} = (X, A)$ is $\mathbb{L} = (\mathbb{X}, \mathbb{A})$ where $\mathbb{X} = \{[\underline{x}, \overline{x}] : \underline{x}, \overline{x} \in X \text{ and } A(\underline{x}, \overline{x}) > 0\}$, and $\mathbb{A}$ is defined by

$$\mathbb{A}\left([\underline{x}, \overline{x}], [\underline{y}, \overline{y}]\right) = \min\{A(\underline{x}, \underline{y}), A(\overline{x}, \overline{y})\}.$$

Lemma 4.5.1 *[Mezzomo et al., 2013e, Lemma 4.5] Let $\mathcal{L} = (X, A)$ be a fuzzy poset. Then $\mathbb{L}$ is a fuzzy poset.*

PROOF: The reflexivity and antisymmetry is straightforward. In order to prove the transitivity, consider $\underline{x}, \overline{x}, \underline{z}, \overline{z} \in X$. Then, it holds that

$$\begin{aligned}
\mathbb{A}\left([\underline{x}, \overline{x}], [\underline{z}, \overline{z}]\right) &= \min\{A\left(\underline{x}, \underline{z}\right), A\left(\overline{x}, \overline{z}\right)\} \\
&\geq \min\{\sup_{\underline{y} \in X} \min\{A(\underline{x}, \underline{y}), A(\underline{y}, \underline{z})\}, \sup_{\overline{y} \in X} \min\{A(\overline{x}, \overline{y}), A(\overline{y}, \overline{z})\}\} \\
&= \sup_{(\underline{y}, \overline{y}) \in X \times X} \{\min\{\min\{A(\underline{x}, \underline{y}), A(\underline{y}, \underline{z})\}, \min\{A(\overline{x}, \overline{y}), A(\overline{y}, \overline{z})\}\}\} \\
&\geq \sup_{[\underline{y}, \overline{y}] \in \mathbb{X}} \{\min\{\min\{A(\underline{x}, \underline{y}), A(\underline{y}, \underline{z})\}, \min\{A(\overline{x}, \overline{y}), A(\overline{y}, \overline{z})\}\}\} \\
&= \sup_{[\underline{y}, \overline{y}] \in \mathbb{X}} \min\{A(\underline{x}, \underline{y}), A(\underline{y}, \underline{z}), A(\overline{x}, \overline{y}), A(\overline{y}, \overline{z})\} \\
&= \sup_{[\underline{y}, \overline{y}] \in \mathbb{X}} \min\{\min\{A(\underline{x}, \underline{y}), A(\overline{x}, \overline{y})\}, \min\{A(\underline{y}, \underline{z}), A(\overline{y}, \overline{z})\}\} \\
&= \sup_{[\underline{y}, \overline{y}] \in \mathbb{X}} \min\{\mathbb{A}([\underline{x}, \overline{x}], [\underline{y}, \overline{y}]), \mathbb{A}([\underline{y}, \overline{y}], [\underline{z}, \overline{z}])\}.
\end{aligned}$$

Therefore, $\mathbb{L}$ is a fuzzy poset. ■

Proposition 4.5.1 *[Mezzomo et al., 2013e, Proposition 4.8] Let $\mathcal{L} = (X, A)$ be a bounded fuzzy lattice. Then $\mathbb{L}$ is a bounded fuzzy lattice.*

PROOF: Let $[\underline{x}, \overline{x}], [\underline{y}, \overline{y}] \in \mathbb{X}$. We define $\wedge_{\mathbb{L}}$ by

$$[\underline{x}, \overline{x}] \wedge_{\mathbb{L}} [\underline{y}, \overline{y}] = [\underline{x} \wedge_{\mathcal{L}} \underline{y}, \overline{x} \wedge_{\mathcal{L}} \overline{y}].$$

Clearly $[\underline{x}, \overline{x}] \wedge_{\mathbb{L}} [\underline{y}, \overline{y}]$ is well defined and it is a lower bound of $[\underline{x}, \overline{x}]$ and $[\underline{y}, \overline{y}]$. Suppose that $[\underline{z}, \overline{z}] \in \mathbb{L}$ is also a lower bound of $[\underline{x}, \overline{x}]$ and $[\underline{y}, \overline{y}]$, then $\mathbb{A}([\underline{z}, \overline{z}], [\underline{x}, \overline{x}]) > 0$ and $\mathbb{A}([\underline{z}, \overline{z}], [\underline{y}, \overline{y}]) > 0$. So, $A(\underline{z}, \underline{x}) > 0$, $A(\underline{z}, \underline{y}) > 0$, $A(\overline{z}, \overline{x}) > 0$ and $A(\overline{z}, \overline{y}) > 0$. Thus, $A(\underline{z}, \underline{x} \wedge_{\mathcal{L}} \underline{y}) > 0$ and $A(\overline{z}, \overline{x} \wedge_{\mathcal{L}} \overline{y}) > 0$. Concluding, $\mathbb{A}([\underline{z}, \overline{z}], [\underline{x} \wedge_{\mathcal{L}} \underline{y}, \overline{x} \wedge_{\mathcal{L}} \overline{y}]) > 0$ and hence $[\underline{x}, \overline{x}] \wedge_{\mathbb{L}} [\underline{y}, \overline{y}]$ is the infimum of x and y. Analogously, we define $\vee_{\mathbb{L}}$ by

$$[\underline{x}, \overline{x}] \vee_{\mathbb{L}} [\underline{y}, \overline{y}] = [\underline{x} \vee_{\mathcal{L}} \underline{y}, \overline{x} \vee_{\mathcal{L}} \overline{y}]$$

and we prove that $[\underline{x}, \overline{x}] \vee_{\mathbb{L}} [\underline{y}, \overline{y}]$ is the supremum of x and y. Since $\mathcal{L}$ is a bounded fuzzy lattice, $[\bot, \bot]$ and $[\top, \top]$ are bottom and top in $\mathbb{L}$, respectively. Therefore, $\mathbb{L}$ is a bounded fuzzy lattice. ■

Proposition 4.5.2 *[Mezzomo et al., 2013e, Proposition 4.9] Let $\mathcal{L} = (X, A)$ be complete fuzzy lattice. Then, $\mathbb{L} = (\mathbb{X}, \mathbb{A})$ is a complete fuzzy lattice.*

PROOF: First we will to prove that $\mathbb{L}$ is an inf-complete fuzzy lattice. For that, let $\mathcal{L} = (X, A)$ be a complete fuzzy lattice and let I be a nonempty set on $\mathbb{L}$. Let $I_{\underline{x}} = \{\underline{x} \in X : [\underline{x}, \overline{x}] \in I$ for some $\overline{x} \in X\}$ and $I_{\overline{x}} = \{\overline{x} \in X : [\underline{x}, \overline{x}] \in I$ for some $\underline{x} \in X\}$. By hypothesis $\mathcal{L}$ is a complete fuzzy lattice, then there exist $\inf I_{\underline{x}}$ and $\inf I_{\overline{x}}$. We will prove that $(\inf I_{\underline{x}}, \inf I_{\overline{x}})$ is the infimum of I. Clearly, $(\inf I_{\underline{x}}, \inf I_{\overline{x}})$ is a lower bound of I. Suppose $[\underline{y}, \overline{y}] \in \mathbb{X}$ is also a lower bound of I. Then, $A(\underline{y}, \inf I_{\underline{x}}) > 0$ and $A(\overline{y}, \inf I_{\overline{x}}) > 0$ and so, $\mathbb{A}([\underline{y}, \overline{y}], [\inf I_{\underline{x}}, \inf I_{\overline{x}}]) > 0$. Therefore, $(\inf I_{\underline{x}}, \inf I_{\overline{x}})$ is the infimum of I and $\mathbb{L}$ is an inf-complete fuzzy lattice. In the same manner, we prove that $(\sup I_{\underline{x}}, \sup I_{\overline{x}})$ is the supremum of I and $\mathbb{L}$ is a sup-complete fuzzy lattice. Therefore $\mathbb{L}$ is a complete fuzzy lattice. ■

Example 4.5.1 *Let $\mathcal{L} = (X, A)$ be the fuzzy lattice defined in Example 3.1.1. The fuzzy lattice $\mathbb{L}$ is represented in the Table 4.3.* ■

$\mathbb{A}$	$[x_1,x_1]$	$[x_1,y_1]$	$[x_1,z_1]$	$[x_1,w_1]$	$[y_1,x_1]$	$[y_1,y_1]$	$[y_1,z_1]$	$[y_1,w_1]$	$[z_1,x_1]$	$[z_1,y_1]$	$[z_1,z_1]$	$[z_1,w_1]$	$[w_1,x_1]$	$[w_1,y_1]$	$[w_1,z_1]$	$[w_1,w_1]$
$[x_1, x_1]$	1.0	0.1	0.4	0.8	0.1	0.1	0.1	0.1	0.4	0.1	0.4	0.3	0.8	0.1	0.4	0.8
$[x_1, y_1]$	0.0	1.0	0.2	0.5	0.0	0.1	0.1	0.1	0.0	0.4	0.2	0.3	0.0	0.8	0.2	0.8
$[x_1, z_1]$	0.0	0.0	1.0	0.3	0.0	0.0	0.1	0.1	0.0	0.0	0.4	0.3	0.0	0.0	0.8	0.3
$[x_1, w_1]$	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.4	0.0	0.0	0.0	0.8
$[y_1, x_1]$	0.0	0.0	0.0	0.0	1.0	0.1	0.4	0.8	0.2	0.1	0.2	0.2	0.5	0.1	0.4	0.5
$[y_1, y_1]$	0.0	0.0	0.0	0.0	0.0	1.0	0.2	0.5	0.0	0.2	0.2	0.2	0.0	0.5	0.2	0.5
$[y_1, z_1]$	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.3	0.0	0.0	0.2	0.2	0.0	0.0	0.5	0.3
$[y_1, w_1]$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.2	0.0	0.0	0.0	0.3
$[z_1, x_1]$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.1	0.4	0.8	0.3	0.1	0.3	0.3
$[z_1, y_1]$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.2	0.5	0.0	0.3	0.2	0.3
$[z_1, z_1]$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.3	0.0	0.0	1.0	0.3
$[z_1, w_1]$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.3
$[w_1, x_1]$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.1	0.4	0.8
$[w_1, y_1]$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.2	0.5
$[w_1, z_1]$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.3
$[w_1, w_1]$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0

Table 4.3: Interval Operator $\mathbb{L}$.

4.6 Intuitionistic Operator

In this section we will define the intuitionistic operator between fuzzy lattice. See Atanassov [1983, 1986]; Atanassov and Gargov [1989]; Bustince et al. [2003, 2000]; Cornelis and Deschrijver [2001]; Cornelis et al. [2004]; Costa et al. [2011], for a brief introduction about basic concepts on fuzzy intuitionistic logic.

Consider $\mathcal{L} = (X, A)$ as a fuzzy poset and a related homomorphism $\neg : \mathcal{L} \to \mathcal{L}^{op}$. The intuitionistic operator is denoted by $\mathcal{L}^*_\neg = (X^*_\neg, A^*)$ where $X^*_\neg = \{(x, y) \in X \times X^{op} : A^{op}(\neg(x), y) > 0\}$ and, for all $(x_1, y_1), (x_2, y_2) \in X \times X^{op}$, the fuzzy relation A^* is defined by

$$A^*\left((x_1, x_2), (y_1, y_2)\right) = \min\{A(x_1, y_1), A^{op}(x_2, y_2)\}.$$

Lemma 4.6.1 *[Mezzomo et al., 2013e, Lemma 4.6] When $\mathcal{L} = (X, A)$ is a fuzzy poset, $\mathcal{L}^*_\neg$ is a fuzzy poset.*

Proof: The reflexivity and antisymmetry is straightforward. In order to prove the transitivity, for all $(x_1, x_2), (z_1, z_2) \in X^*_\neg$ the following holds.

$$\begin{aligned}
& A^*((x_1, x_2), (z_1, z_2)) = \min\{A(x_1, z_1), A^{op}(x_2, z_2)\} \\
\geq\ & \min\{\sup_{y_1 \in X} \min\{A(x_1, y_1), A(y_1, z_1)\}, \sup_{y_2 \in X} \min\{A^{op}(y_2, z_2), A^{op}(x_2, y_2)\}\} \\
=\ & \sup_{(y_1, y_2) \in X \times X} \min\{\min\{A(x_1, y_1), A(y_1, z_1)\}, \min\{A(z_2, y_2), A(y_2, x_2)\}\} \\
\geq\ & \sup_{(y_1, y_2) \in X^*_\neg} \min\{\min\{A(x_1, y_1), A(y_1, z_1)\}, \min\{A^{op}(y_2, z_2), A^{op}(x_2, y_2)\}\} \\
=\ & \sup_{(y_1, y_2) \in X^*_\neg} \min\{A(x_1, y_1), A(y_1, z_1), A^{op}(y_2, z_2), A^{op}(x_2, y_2)\} \\
=\ & \sup_{(y_1, y_2) \in X^*_\neg} \min\{\min\{A(x_1, y_1), A^{op}(x_2, y_2)\}, \min\{A(y_1, z_1), A^{op}(y_2, z_2)\}\} \\
=\ & \sup_{(y_1, y_2) \in X^*_\neg} \min\{A^*((x_1, x_2), (y_1, y_2)), A^*((y_1, y_2), (z_1, z_2))\}.
\end{aligned}$$

Therefore, $\mathcal{L}^*_\neg$ is a fuzzy poset. ■

Proposition 4.6.1 *[Mezzomo et al., 2013e, Proposition 4.10] Let $\mathcal{L} = (X, A)$ be bounded fuzzy lattice. Then $\mathcal{L}^*_\neg$ is a bounded fuzzy lattice.*

PROOF: Let $x_1, x_2, y_1, y_2 \in X$. We define $\wedge^*$ by

$$(x_1, x_2) \wedge^* (y_1, y_2) = (x_1 \wedge_{\mathcal{L}} y_1, x_2 \wedge_{\mathcal{L}}^{op} y_2).$$

Clearly $(x_1, x_2) \wedge^* (y_1, y_2)$ is well defined and it is a lower bound of (x_1, x_2) and (y_1, y_2). Suppose that $(z_1, z_2) \in X_{\neg}^*$ is also a lower bound of (x_1, x_2) and (y_1, y_2), then $A^*((z_1, z_2), (x_1, x_2)) > 0$ and $A^*((z_1, z_2), (y_1, y_2)) > 0$. So, $A(z_1, x_1) > 0$, $A^{op}(z_2, x_2) > 0$, $A(z_1, y_1) > 0$ and $A^{op}(z_2, y_2) > 0$. Hence, $A(z_1, x_1 \wedge y_1) > 0$ and $A^{op}(z_2, x_2 \wedge^{op} y_2) > 0$. Therefore, $A^*((z_1, z_2), (x_1, x_2) \wedge^* (y_1, y_2)) > 0$ and $(x_1, x_2) \wedge^* (y_1, y_2)$ is the infimum of x and y. Analogously, $\vee^*$ is defined by

$$(x_1, x_2) \vee^* (y_1, y_2) = (x_1 \vee_{\mathcal{L}} y_1, x_2 \vee_{\mathcal{L}}^{op} y_2)$$

and we prove that $(x_1, x_2) \vee^* (y_1, y_2)$ is the supremum of x and y. Because $\mathcal{L}$ is a bounded fuzzy lattice, then $\perp$ and $\top$ are bottom and top, respectively. ■

Proposition 4.6.2 *[Mezzomo et al., 2013e, Proposition 4.11] Let $\mathcal{L} = (X, A)$ be complete fuzzy lattice. Then, $\mathcal{L}_{\neg}^* = (X_{\neg}^*, A^*)$ is a complete fuzzy lattice.*

PROOF: First we will to prove that $\mathcal{L}_{\neg}^*$ is an inf-complete fuzzy lattice. For that, let $\mathcal{L} = (X, A)$ be a complete fuzzy lattices and let I be a nonempty set on $\mathcal{L}_{\neg}^*$. Let $I_x = \{x \in X : (x, y) \in I \text{ for some } y \in X^{op}\}$ and $I_y = \{y \in X^{op} : (x, y) \in I \text{ for some } x \in X\}$. By hypothesis, $\mathcal{L}$ and $\mathcal{L}^{op}$ are complete fuzzy lattices, then there exist $\inf I_x$ and $\inf I_y$. We will prove that $(\inf I_x, \inf I_y)$ is the infimum of I. Clearly, $(\inf I_x, \inf I_y)$ is a lower bound of I. Suppose $(x_2, y_2) \in X \times X^{op}$ is also a lower bound of I. Then, $A(x_2, \inf I_x) > 0$ and $A^{op}(\inf I_y, y_2) > 0$ and so, $A^*((x_2, y_2), (\inf I_x, \inf I_y)) > 0$. Therefore, $(\inf I_x, \inf I_y)$ is the infimum of I and $\mathcal{L}_{\neg}^*$ is an inf-complete fuzzy lattice. In the same manner, we prove that $(\sup I_x, \sup I_y)$ is the supremum of I and $\mathcal{L}_{\neg}^*$ is a sup-complete fuzzy lattice. Therefore $\mathcal{L}_{\neg}^*$ is a complete fuzzy lattice. ■

Example 4.6.1 *Let $\mathcal{L} = (X, A)$ be fuzzy lattice defined in Example 3.1.1. The intuitionist fuzzy lattice $\mathcal{L}^*$ is represented in the Table 4.4.* ■

A^*	(x_1,x_1)	(x_1,y_1)	(x_1,z_1)	(x_1,w_1)	(y_1,x_1)	(y_1,y_1)	(y_1,z_1)	(y_1,w_1)	(z_1,x_1)	(z_1,y_1)	(z_1,z_1)	(z_1,w_1)	(w_1,x_1)	(w_1,y_1)	(w_1,z_1)	(w_1,w_1)
(x_1, x_1)	1.0	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.4	0.0	0.0	0.0	0.8	0.0	0.0	0.0
(x_1, y_1)	0.1	1.0	0.0	0.0	0.1	0.1	0.0	0.0	0.1	0.4	0.0	0.0	0.1	0.8	0.0	0.0
(x_1, z_1)	0.4	0.2	1.0	0.0	0.1	0.1	0.1	0.0	0.4	0.4	0.4	0.0	0.4	0.2	0.8	0.0
(x_1, w_1)	0.8	0.5	0.3	1.0	0.1	0.1	0.1	0.1	0.4	0.4	0.3	0.3	0.8	0.5	0.3	0.8
(y_1, x_1)	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.2	0.0	0.0	0.0	0.5	0.0	0.0	0.0
(y_1, y_1)	0.0	0.0	0.0	0.0	0.1	1.0	0.0	0.0	0.1	0.2	0.0	0.0	0.1	0.5	0.0	0.0
(y_1, z_1)	0.0	0.0	0.0	0.0	0.4	0.2	1.0	0.0	0.2	0.2	0.2	0.0	0.1	0.2	0.5	0.0
(y_1, w_1)	0.0	0.0	0.0	0.0	0.8	0.5	0.3	1.0	0.2	0.2	0.2	0.2	0.5	0.5	0.3	0.5
(z_1, x_1)	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.3	0.0	0.0	0.0
(z_1, y_1)	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	1.0	0.0	0.0	0.1	0.3	0.0	0.0
(z_1, z_1)	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.4	0.2	1.0	0.0	0.4	0.2	1.0	0.0
(z_1, w_1)	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.8	0.5	0.3	1.0	0.3	0.3	0.3	0.3
(w_1, x_1)	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0
(w_1, y_1)	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	1.0	0.0	0.0
(w_1, z_1)	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.4	0.2	1.0	0.0
(w_1, w_1)	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.8	0.5	0.3	1.0

Table 4.4: Fuzzy Intuitionist Operator $\mathcal{L}^*_{\neg}$

Chapter 5

Ideals and Filters on Fuzzy Lattices

In this chapter we will show the results done in the papers Mezzomo et al. [2012a] and Mezzomo et al. [2012b] using the notion of fuzzy lattices defined by Chon [2009], considering the notion of fuzzy order of Zadeh [1971], for to study classical ideals and filters of fuzzy lattices and prove some properties.

As a first result, in the paper Mezzomo et al. [2012b], we define ideals and filters of a fuzzy lattice (X, A) as a fuzzy sublattice (Y, B), as a crisp set $Y \subseteq X$ join with the fuzzy order $A|_{Y \times Y}$, that is, not as a fuzzy set. Therefore, although we called fuzzy ideal, in the true it is a classical ideal. The paper Mezzomo et al. [2012a] is a continuation of paper Mezzomo et al. [2012b] where we define a ideal and filter as a classical set $Y \subseteq X$ and add new results and defined some kinds of ideals and filters of fuzzy. Lastly, we prove some properties analogous the classical theory of ideals (filters) of classical lattice, such as, the class of proper ideals (filters) of fuzzy lattice is closed under union and intersection.

5.1 Definitions of Ideals and Filters

In this section we will show the main results what was done in the paper Mezzomo et al. [2012a], since we use the same notion of fuzzy lattice (X, A), defined by Chon [2009], where X is a nonempty classical set and A is a fuzzy order relation

as shown in Chapter 3.

In the paper Mezzomo et al. [2012a], we use the same notion of fuzzy lattice to define ideals and filters of a fuzzy lattice (X, A) as a classical set $Y \subseteq X$ and we defined as follows:

Definition 5.1.1 *[Mezzomo et al., 2012a, Definitions 3.1 and 3.2] Let $\mathcal{L} = (X, A)$ be a fuzzy lattice and $Y \subseteq X$. Y is an **ideal** of $\mathcal{L}$ if it satisfies the following conditions:*

(i) If $x \in X$, $y \in Y$ and $A(x, y) > 0$, then $x \in Y$;

(ii) If $x, y \in Y$, then $x \vee y \in Y$.

*On the other hand, Y is a **filter** of $\mathcal{L}$, if*

(iii) If $x \in X$, $y \in Y$ and $A(y, x) > 0$, then $x \in Y$;

(iv) If $x, y \in Y$, then $x \wedge y \in Y$.

In both cases, if $Y \neq X$, then Y is also called **proper**, i.e., **proper ideal** or **proper filter**. The class of all proper ideals of $\mathcal{L}$ will be denoted by $I_p(X)$ and the class of all proper filters will be denoted by $F_p(X)$.

Proposition 5.1.1 *[Mezzomo et al., 2012a, Proposition 3.1] If $Y \subseteq X$ is an ideal (filter) of $\mathcal{L} = (X, A)$, then (Y, B), where $B = A|_{Y \times Y}$, is a fuzzy sublattice of (X, A).*

Proof: Straightforward from Definition 5.1.1. ■

Proposition 5.1.2 *[Mezzomo et al., 2012b, Proposition 3.2] Y is an ideal (filter) of fuzzy lattice $\mathcal{L} = (X, A)$ iff Y is an ideal (filter) of $(X, S(A))$.*

Proof:

($\Rightarrow$) Let Y be an ideal of $\mathcal{L}$ and let $y \in Y$. Then,

(i) If $(x, y) \in S(A)$, then $A(x, y) > 0$. So, by Definition 5.1.1 (i), $x \in Y$.

(ii) Straightforward from Proposition 3.1.4.

($\Leftarrow$) Let Y be an ideal on $(X, S(A))$.

(i) Given $x \in X$ and $y \in Y$, if $A(x, y) > 0$, then $(x, y) \in S(A)$ and $x \in Y$.

(ii) Straightforward from Proposition 3.1.4.

Similarly, we prove that Y is a fuzzy filter of $\mathcal{L} = (X, A)$ iff Y is a filter of $(X, S(A))$.

■

Let (X, A) be a fuzzy lattice and $\alpha \in [0, 1]$. The α-level of A, denoted by A_α, is defined as $A_\alpha = \{(x, y) \in X \times X : A(x, y) \geq \alpha\}$.

Remark 5.1.1 *Let $\mathcal{L} = (X, A)$ be a fuzzy lattice, $\alpha \in (0, 1]$ such that (X, A_α) is a lattice. $Y \subseteq X$ is an ideal (filter) of $\mathcal{L}$ iff Y is an ideal (filter) of (X, A_α).*

5.2 Properties of Ideals and Filters

In Mezzomo et al. [2012a], we also define some kinds of ideals and filters of fuzzy lattice and we prove some notions analogous the classical theory of ideals and filters of classical lattice, such as, the class of proper ideals and filters of fuzzy lattice is closed under union and intersection. In the following propositions we prove some properties analogous the classical theory of ideals and filters of classical lattice.

Proposition 5.2.1 *[Mezzomo et al., 2012a, Proposition 4.1] Let (X, A) be a fuzzy lattice and $Y \subseteq X$ such that (Y, B) is a fuzzy sup-lattice where $B = A|_{Y \times Y}$. The set $\downarrow Y = \{x \in X : A(x, y) > 0$ for some $y \in Y\}$ is an ideal of (X, A).*

Proof:

(i) Let $z \in \downarrow Y$ and $w \in X$ such that $A(w, z) > 0$. Because $z \in \downarrow Y$, then exists $x \in Y$ such that $A(z, x) > 0$, and by transitivity,

$$\begin{aligned} A(w, x) &= \sup_{z \in X} \min\{A(w, z), A(z, x)\} \\ &> \min\{A(w, z), A(z, x)\} \\ &> 0. \end{aligned}$$

Therefore, $w \in \downarrow Y$.

(*ii*) Suppose $x, y \in \downarrow Y$, then exist $z_1, z_2 \in Y$ such that $A(x, z_1) > 0$ and $A(y, z_2) > 0$. So, $A(x, z_1 \vee z_2) > 0$ and $A(y, z_1 \vee z_2) > 0$. By hypothesis (Y, A) is a fuzzy sup-lattice, then $z_1 \vee z_2 \in Y$ and $A(x \vee y, z_1 \vee z_2) > 0$. Therefore, $x \vee y \in \downarrow Y$.

■

Proposition 5.2.2 *[Mezzomo et al., 2012a, Proposition 4.2] Let (X, A) be a fuzzy lattice and $Y \subseteq X$ such that (Y, B) is a fuzzy inf-lattice where $B = A|_{Y \times Y}$. The set $\uparrow Y = \{x \in X : A(y, x) > 0$ for any $y \in Y\}$ is a filter of (X, A).*

PROOF: Analogously the Proposition 5.2.1. ■

Proposition 5.2.3 *[Mezzomo et al., 2012a, Proposition 4.3] Let (X, A) be a fuzzy lattice and $Y \subseteq X$ such that (Y, B) is a complete fuzzy lattice where $B = A|_{Y \times Y}$. Then $\downarrow Y$ and $\uparrow Y$ satisfy the following properties:*

(i) $Y \subseteq \downarrow Y$;

(ii) $Y \subseteq \uparrow Y$;

(iii) $Y \subseteq W \Rightarrow \downarrow Y \subseteq \downarrow W$;

(iv) $Y \subseteq W \Rightarrow \uparrow Y \subseteq \uparrow W$;

(v) $\downarrow\downarrow Y = \downarrow Y$;

(vi) $\uparrow\uparrow Y = \uparrow Y$.

PROOF:

(*i*) If $y \in Y$ and because $A(y, y) = 1$, then $y \in \downarrow Y$.

(*iii*) Suppose $Y \subseteq W$ and $y \in \downarrow Y$, then by definition, exists $z \in Y$ such that $A(z, y) > 0$. Because $Y \subseteq W$, then $z \in W$ and $A(z, y) > 0$. So $y \in \downarrow W$.

(v) ($\Rightarrow$) $\downarrow\downarrow Y \subseteq \downarrow Y$. Suppose $y \in \downarrow\downarrow Y$, then exists $x \in \downarrow Y$ such that $A(y,x) > 0$. Since $x \in \downarrow Y$, then exists $z \in Y$ such that $A(x,z) > 0$. So, $A(y,z) > 0$. Therefore, $y \in \downarrow Y$.

($\Leftarrow$) Straightforward from (i).

In the analogous manner, we prove the properties $(ii), (iv)$ and (vi) for $\uparrow Y$. ■

Corollary 5.2.1 *[Mezzomo et al., 2012a, Corollary 4.1] Let (X, A) be a fuzzy lattice and $Y \subseteq X$ such that (Y, B) is a fuzzy sup(inf)-lattice where $B = A|_{Y \times Y}$. $\downarrow Y$ ($\uparrow Y$) is the least ideal (filter) containing Y.*

Proof: Suppose that there exists an ideal Z such that $Y \subseteq Z \subseteq \downarrow Y$ and suppose $x \in \downarrow Y$ and $x \notin Z$. If $x \in \downarrow Y$, then $A(x,y) > 0$ for some $y \in Y$ and so, $y \in Z$. Thus, because Z is an ideal, then $x \in Z$, that is a contradiction. Similarly we prove for filters. ■

Proposition 5.2.4 *Let $\mathcal{L} = (X, A)$ be a fuzzy lattice and $x \in X$. The set $\downarrow x = \{y \in X : A(y,x) > 0\}$ is an ideal called **principal ideal** generated by x. Dually, the set $\uparrow x = \{y \in X : A(x,y) > 0\}$ is a filter called **principal filter** generated by x.*[1]

Proof: Straightforward from Definition 5.1.1. ■

The family of all ideals of a fuzzy lattice $\mathcal{L} = (X, A)$ will be denoted by $I(\mathcal{L})$ whereas $F(\mathcal{L})$ will denote the family of all filters of a fuzzy lattice $\mathcal{L}$. This families are subsets of parts of X, denoted by $\mathcal{P}(X)$, that is, $I(\mathcal{L}) \subseteq \mathcal{P}(X)$ and $F(\mathcal{L}) \subseteq \mathcal{P}(X)$.

Proposition 5.2.5 *Let $\mathcal{L} = (X, A)$ be a fuzzy lattice. Then,*

(i) $X \in I(\mathcal{L})$ and $X \in F(\mathcal{L})$;

(ii) $\bigcap Z \subseteq I(\mathcal{L})$, for all $Z \subseteq I(\mathcal{L})$;

(iii) $\bigcap W \subseteq F(\mathcal{L})$, for all $W \subseteq F(\mathcal{L})$.

[1] Note that $\downarrow x = \downarrow \{x\}$ and $\uparrow x = \uparrow \{x\}$.

PROOF: Let $Z \subseteq I(\mathcal{L})$ and $W \subseteq F(\mathcal{L})$.

(i) Straightforward.

(ii) Suppose $x \in \bigcap Z$, then $x \in Z_j$ for all $Z_j \in Z$. If $A(y,x) > 0$, for some $y \in X$, then $y \in Z_j$, for all $Z_j \in Z$, and hence $y \in \bigcap Z$. Thus, $\bigcap Z \in I(\mathcal{L})$. Notice that if Z is an empty set then $\bigcap Z = X$. If $x, y \in \bigcap Z$, then $x, y \in Z_j$, for all $Z_j \in Z$, since each Z_j is an ideal, then $x \vee y \in Z_j$. Therefore, $x \vee y \in \bigcap Z$.

(iii) Similar (ii).

■

The following proposition prove the relation between the ideal $\downarrow Y$ and the principal ideal $\downarrow y$.

Proposition 5.2.6 *[Mezzomo et al., 2012a, Proposition 4.7] For all $Y \in \mathcal{P}(X)$, $\downarrow Y = \bigcup_{y \in Y} \downarrow y$ and $\uparrow Y = \bigcup_{y \in Y} \uparrow y$.*

PROOF: Let $Y \in \mathcal{P}(X)$. Then, $x \in \downarrow Y$ iff exists $y \in Y$ such that $A(x,y) > 0$ iff exists $y \in Y$ such that $x \in \downarrow y$ iff $x \in \bigcup_{y \in Y} \downarrow y$.

In an analogous manner we prove the same properties for $\uparrow Y$. ■

Proposition 5.2.7 *If (X, A) is a complete fuzzy lattice then and $Y \subseteq X$, $\downarrow Y \subseteq \downarrow \sup Y$.*

PROOF: In fact, suppose $x \in \downarrow Y$, then exists $y \in Y$ such that $A(x,y) > 0$. Therefore, because $A(y, \sup Y) > 0$, then $x \in \downarrow \sup Y$. ■

Remark 5.2.1 $\downarrow \sup Y \subseteq \downarrow Y$ *only if* $\sup Y \in Y$.

Similarly, we prove that $\uparrow Y \subseteq \uparrow \inf Y$.

Consider $I_p(\mathcal{L})$ the family of all proper ideals of a fuzzy lattice $\mathcal{L} = (X, A)$ and $F_p(\mathcal{L})$ the family of all proper filters of a fuzzy lattice $\mathcal{L}$.

Proposition 5.2.8 *[Mezzomo et al., 2012a, Proposition 4.9] Let $Z \subseteq I_p(\mathcal{L})$. Then, $\bigcup Z \neq X$.*

PROOF: Because a proper ideal do not contain the top, we have $\bigcup Z \neq X$. ■

Corollary 5.2.2 *[Mezzomo et al., 2012a, Corollary 4.3] Let $Z \subseteq I_p(\mathcal{L})$. Then, $\bigcap Z \neq X$.*

PROOF: Suppose $x \in \bigcap Z$, then $x \in Z_j$ for all $Z_j \in Z$. By definition exists $y \in X$ such that $y \notin Z_j$ for some $Z_j \in Z$. So, $y \notin \bigcap Z$. Therefore, $\bigcap Z \neq X$. ■

The proof of Proposition 5.2.8 and Corollary 5.2.2 is analogously for filters.

Definition 5.2.1 *[Mezzomo et al., 2012a, Definition 4.4] Let Y be a proper ideal (filter) of a fuzzy lattice $\mathcal{L} = (X, A)$. Y is a **prime ideal (prime filter)** if for all $x, y \in X$ and $z \in Y$, $A(x \wedge y, z) > 0$ implies either $x \in Y$ or $y \in Y$ ($A(x \vee y, z) > 0$ implies either $x \in Y$ or $y \in Y$).*

It is not hard to prove that the intersection and union of prime ideals (filters) of (X, A) is not a prime ideal (filter).

Another natural kind of ideal (filter) in a fuzzy lattice is the maximal ideal (maximal filter), defined by:

Definition 5.2.2 *We say that a proper ideal (filter) Y is a **maximal ideal (maximal filter)** of (X, A), if for each $Z \in X$ such that $Y \subseteq Z \subseteq X$, then either $Y = Z$ or $Z = X$.*

It is obvious that the intersection of maximal ideals (filters) of (X, A) is not a maximal ideal (filter) because, by Definition 5.2.2, the maximal ideal (filter) of (X, A), if exists, it is unique. Therefore, is not possible to have a family of maximal ideals (filters) of the same fuzzy lattice. The same way, we can not say anything about the union of maximal ideals (filters).

Chapter 6

Fuzzy Ideals and Fuzzy Filters of Fuzzy Lattices

In this chapter we will show the results of the paper Mezzomo et al. [2013c] where we define fuzzy ideals and fuzzy filters of fuzzy lattice and some types of fuzzy ideals and fuzzy filters.

In the paper Mezzomo et al. [2013c], we define a fuzzy ideal of (X, A) as a fuzzy set on X and we rely on a less restrictive form, that is, a fuzzy ideal is a fuzzy set of a fuzzy lattice (X, A). In addition, we define some types of fuzzy ideals and fuzzy filters of fuzzy lattice and we prove some properties analogous the classical theory of ideals (filters), such as, the class of proper fuzzy ideals (filters) is closed under fuzzy intersection. We prove also, if a bounded fuzzy lattice admits a maximal fuzzy ideal, then it is prime.

Moreover, we define a fuzzy homomorphism h from fuzzy lattices $\mathcal{L}$ and $\mathcal{M}$ and prove some results involving fuzzy homomorphism and fuzzy ideals as if h is a fuzzy monomorphism and the fuzzy image of a fuzzy set $\tilde{h}(I)$ is a fuzzy ideal, then I is a fuzzy ideal. Similarly, we prove for proper, prime and maximal fuzzy ideals. Finally, we prove that h is a fuzzy homomorphism from fuzzy lattices $\mathcal{L}$ into $\mathcal{M}$ if the inverse image of all principal fuzzy ideals of $\mathcal{M}$ is a fuzzy ideal of $\mathcal{L}$.

6.1 Definitions of Fuzzy Ideals and Filters

In this section we will show the results of the paper Mezzomo et al. [2013c] where we use the same notion of fuzzy lattice (X, A), defined by Chon [2009], used in the papers Mezzomo et al. [2012a] and Mezzomo et al. [2012b]. We define a fuzzy ideals and fuzzy filters of fuzzy lattice (X, A) as a fuzzy set I on set X as follows:

Definition 6.1.1 *[Mezzomo et al., 2013c, Definition 3.1] Let (X, A) be a fuzzy lattice. A fuzzy set I on X is a **fuzzy ideal** of (X, A) if, for all $x, y \in X$, the following conditions are verified:*

(i) If $\mu_I(y) > 0$ and $A(x, y) > 0$, then $\mu_I(x) > 0$;

(ii) If $\mu_I(x) > 0$ and $\mu_I(y) > 0$, then $\mu_I(x \vee y) > 0$.

Definition 6.1.2 *[Mezzomo et al., 2013c, Definition 3.2] Let (X, A) be a fuzzy lattice. A fuzzy set F on X is a **fuzzy filter** of (X, A) if, for all $x, y \in X$, the following conditions are verified:*

(i) If $\mu_F(y) > 0$ and $A(y, x) > 0$, then $\mu_F(x) > 0$;

(ii) If $\mu_F(x) > 0$ and $\mu_F(y) > 0$, then $\mu_F(x \wedge y) > 0$.

Example 6.1.1 *Let $\mathcal{L} = (X, A)$ be the fuzzy lattice defined in Example 3.1.1. Then, the fuzzy set $I = \{(w_1, 0.0), (z_1, 0.2), (y_1, 0.4), (x_1, 0.7)\}$ is a fuzzy ideal of $\mathcal{L}$.* ■

We defined a fuzzy ideal I of a fuzzy lattice (X, A). We have also defined the relation $S(A)$ of a fuzzy relation A in a set X as well as α-level relation A_α of a fuzzy relation A in a set X and characterize a relation on X. Then, we can think of the set of ideals obtained via α-levels as a set of ideals with degree greater than or equal to α or, the set of elements $x \in X$ and $y \in I$ such that $A(x, y) \geq \alpha$ with $\alpha \in (0, 1]$. We will denote the support of fuzzy set I by $S(I)$ and the support of a fuzzy set F by $S(F)$.

Proposition 6.1.1 *[Mezzomo et al., 2013c, Proposition 3.1] Let I be a fuzzy set on X. I is a fuzzy ideal of a fuzzy lattice (X, A) iff $S(I)$ is an ideal of $(X, S(A))$.*

PROOF:

($\Rightarrow$) Let I be a fuzzy ideal of (X, A).

(i) If $y \in S(I)$ and $(x, y) \in S(A)$, then $\mu_I(y) > 0$ and $A(x, y) > 0$. So, by hypothesis I is a fuzzy ideal of (X, A) and by Definition 6.1.1 (i), $\mu_I(x) > 0$ and therefore, $x \in S(I)$.

(ii) Suppose $x, y \in S(I)$, then $\mu_I(x) > 0$ and $\mu_I(y) > 0$. Because I is a fuzzy ideal of (X, A), by Definition 6.1.1 (ii), $\mu_I(x \vee y) > 0$. Therefore, $x \vee y \in S(I)$.

($\Leftarrow$) Suppose that $S(I)$ is an ideal of $(X, S(A))$ and let $x, y \in X$.

(i) If $\mu_I(y) > 0$ and $A(x, y) > 0$, then $y \in S(I)$ and $(x, y) \in S(A)$. Because $S(I)$ is an ideal of $(X, S(A))$, then by definition of classical ideal, $x \in S(I)$. Therefore, $\mu_I(x) > 0$.

(ii) Suppose $\mu_I(x) > 0$ and $\mu_I(y) > 0$, then $x \in S(I)$ and $y \in S(I)$. Because $S(I)$ is an ideal of $(X, S(A))$, then by definition of classical ideal, $x \vee y \in S(I)$. Therefore, $\mu_I(x \vee y) > 0$.

■

Dually, we can prove the Proposition 6.1.1 for a fuzzy filter of (X, A) as follows:

Proposition 6.1.2 *[Mezzomo et al., 2013c, Proposition 3.2] Let F be a fuzzy set on X. F is a fuzzy filter of a fuzzy lattice (X, A) iff $S(F)$ is a filter of $(X, S(A))$.*

PROOF: Analogously the Proposition 6.1.1. ■

Let A_α be the α-level set $A_\alpha = \{(x, y) \in X \times X : A(x, y) \geq \alpha\}$ for some $\alpha \in (0, 1]$ and let $I_\alpha = \{x \in I : A(x, y) \geq \alpha$ for some $y \in I\}$ be an ideal of (X, A_α).

Theorem 6.1.1 *[Mezzomo et al., 2013c, Theorem 3.1] Let I be a fuzzy set on X. I is a fuzzy ideal of fuzzy lattice $\mathcal{L} = (X, A)$ iff for each $\alpha \in (0, 1]$, I_α is an ideal of (X, A_α).*

PROOF:

($\Rightarrow$) Let I be a fuzzy ideal of (X, A), $\alpha \in (0, 1]$ and let $x, y \in I_\alpha$.

- (i) If $y \in I_\alpha$ and $(x, y) \in A_\alpha$ for some $\alpha \in (0, 1]$, then $\mu_I(y) \geq \alpha$ and $A(x, y) \geq \alpha$. Since, by hypothesis I is a fuzzy ideal, then by Definition 6.1.1 (i), $\mu_I(x) \geq \alpha$ and therefore, $x \in I_\alpha$.
- (ii) Suppose $x, y \in I_\alpha$ for some $\alpha \in (0, 1]$, then $\mu_I(x) \geq \alpha$ and $\mu_I(y) \geq \alpha$. Because I is a fuzzy ideal of (X, A), by Definition 6.1.1 (ii), $\mu_I(x \vee y) \geq \alpha$. Therefore, $x \vee y \in I_\alpha$.

($\Leftarrow$) Suppose that I_α is an ideal of (X, A_α) for each $\alpha \in (0, 1]$ and let $x, y \in X$.

- (i) If $\mu_I(y) > 0$ and $A(x, y) > 0$, then $y \in I_\alpha$ for $\alpha = A(x, y)$ and so, $(x, y) \in A_\alpha$. Because I_α is an ideal of (X, A_α), then by definition of classical ideal, $x \in I_\alpha$. Therefore, $\mu_I(x) \geq A(x, y) > 0$.
- (ii) Suppose $\mu_I(x) > 0$, $\mu_I(y) > 0$ and $\alpha = \min\{\mu_I(x), \mu_I(y)\}$. Then, $x \in I_\alpha$ and $y \in I_\alpha$. Because I_α is an ideal of (X, I_α), then by definition of classical ideal, $x \vee y \in I_\alpha$. Therefore, $\mu_I(x \vee y) \geq \min\{\mu_I(x), \mu_I(y)\} > 0$.

■

Consider the set defined by $F_\alpha = \{x \in F : A(y, x) \geq \alpha \text{ for some } y \in F\}$ be a filter of (X, A_α).

Theorem 6.1.2 *[Mezzomo et al., 2013c, Theorem 3.2] Let F be a fuzzy set on X. F is a fuzzy filter of (X, A) iff for each $\alpha \in (0, 1]$, F_α is a filter of (X, A_α).*

PROOF: Analogously the Theorem 6.1.1. ■

We will define a fuzzy ideal of product operator between bounded fuzzy lattices as:

Proposition 6.1.3 *[Mezzomo et al., 2013d, Proposition 3.1] Let $\mathcal{L} = (X, A)$ and $\mathcal{M} = (Y, B)$ be bounded fuzzy lattices, I and J be fuzzy ideals of $\mathcal{L}$ and $\mathcal{M}$, respectively. The fuzzy set*

$$\mu_{I \times J}(x, y) = \min\{\mu_I(x), \mu_J(y)\}$$

on $X \times Y$ is a fuzzy ideal of $\mathcal{L} \times \mathcal{M}$.

PROOF: According to definition of product operator on bounded fuzzy lattices and Definition 6.1.1, we need to prove that, for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$:

(i) If $\mu_{I\times J}(x_2, y_2) > 0$ and $C((x_1, y_1), (x_2, y_2)) > 0$, then $\mu_{I\times J}(x_1, y_1) > 0$;

(ii) If $\mu_{I\times J}(x_1, y_1) > 0$ and $\mu_{I\times J}(x_2, y_2) > 0$, then $\mu_{I\times J}((x_1, y_1) \vee_\times (x_2, y_2)) > 0$.

Let I and J be fuzzy ideals of $\mathcal{L}$ and $\mathcal{M}$, respectively.

(i) Since $\mu_{I\times J}(x_2, y_2) > 0$ and $C((x_1, y_1), (x_2, y_2)) > 0$, then $\min\{\mu_I(x_2), \mu_J(y_2)\} > 0$ and $\min\{A(x_1, x_2), B(y_1, y_2)\} > 0$. So, $\mu_I(x_2) > 0$, $\mu_J(y_2) > 0$, $A(x_1, x_2) > 0$ and $B(y_1, y_2) > 0$. Hence, because I and J are fuzzy ideals, then $\mu_I(x_1) > 0$ and $\mu_J(y_1) > 0$. Therefore, $\mu_{I\times J}(x_1, y_1) = \min\{\mu_I(x_1), \mu_J(y_1)\} > 0$.

(ii) Let $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that $\mu_{I\times J}(x_1, y_1) > 0$ and $\mu_{I\times J}(x_2, y_2) > 0$. Then $\min\{\mu_I(x_1), \mu_J(y_1)\} > 0$ and $\min\{\mu_I(x_2), \mu_J(y_2)\} > 0$. So, $\mu_I(x_1) > 0$, $\mu_J(y_1) > 0$, $\mu_I(x_2) > 0$ and $\mu_J(y_2) > 0$. Thus, because I and J are fuzzy ideals, $\mu_I(x_1 \vee_{\mathcal{L}} x_2) > 0$ and $\mu_J(y_1 \vee_{\mathcal{M}} y_2) > 0$. Therefore, $\min(\mu_I(x_1 \vee_{\mathcal{L}} x_2), \mu_J(y_1 \vee_{\mathcal{M}} y_2)) > 0$, i.e., $\mu_{I\times J}(x_1 \vee_{\mathcal{L}} x_2, y_1 \vee_{\mathcal{M}} y_2) > 0$. Hence, by Lemma 4.1.1, we have that $\mu_{I\times J}((x_1, y_1) \vee_\times (x_2, y_2)) > 0$.

Therefore, the fuzzy set $\mu_{I\times J}$ is a fuzzy ideal of $\mathcal{L} \times \mathcal{M}$. ■

We will denote by $I \times J$ a fuzzy ideal of the bounded fuzzy lattice $\mathcal{L} \times \mathcal{M}$.

6.2 Types of Fuzzy Ideals and Fuzzy Filters

In this section, we define some types of fuzzy ideals and fuzzy filters and some of its consequences.

6.2.1 Main Results and Properties

In the paper Mezzomo et al. [2012a], we defined some types of ideals and filters of a fuzzy lattice and prove some of its consequences. The crisp sets $\downarrow J = \{x \in$

$X : x \leq y$ for some $y \in J\}$ and $\uparrow K = \{x \in X : x \geq y$ for some $y \in K\}$ are respectively, an ideal and a filter of X. So, given fuzzy sets I and F of a fuzzy lattice (X, A), we define the fuzzy sets $\Downarrow I$ on X and $\Uparrow F$ on X as follows:

Definition 6.2.1 *[Mezzomo et al., 2013c, Definition 4.1] Let (X, A) be a fuzzy lattice and let I be a fuzzy set on X. The fuzzy set $\Downarrow I$ is defined by $\mu_{\Downarrow I}(x) = \sup_{y \in X} \min\{\mu_I(y),\ A(x, y)\}$ for all $x \in X$. Dually, let F be a fuzzy set on X. The fuzzy set $\Uparrow F$ is defined by $\mu_{\Uparrow F}(x) = \sup_{y \in X} \min\{\mu_F(y), A(y, x)\}$ for all $x \in X$.*

Proposition 6.2.1 *[Mezzomo et al., 2013c, Proposition 4.1] Let (X, A) be a fuzzy lattice and I a fuzzy ideal of X. The fuzzy set $\Downarrow I$ of X is a fuzzy ideal of (X, A).*

Proof: Let $x, y \in X$.

(*i*) If $\mu_{\Downarrow I}(y) > 0$ and $x \in X$ such that $A(x, y) > 0$. Then, by definition, $\mu_{\Downarrow I}(y) = \sup_{z \in X} \min\{\mu_I(z), A(y, z)\} > 0$. So, there exists $z \in X$ such that $\min\{\mu_I(z), A(y, z)\} > 0$. Hence, because $A(x, y) > 0$ and $A(y, z) > 0$, then by Proposition 2.2.1, we have that $A(x, z) > 0$. Thus, $\min\{\mu_I(z), A(x, z)\} > 0$ and so, $\sup_{z \in X} \min\{\mu_I(z), A(x, z)\} > 0$. Therefore, $\mu_{\Downarrow I}(x) > 0$.

(*ii*) Suppose $\mu_{\Downarrow I}(x) > 0$ and $\mu_{\Downarrow I}(y) > 0$. By definition, $\mu_{\Downarrow I}(x) = \sup_{z \in X} \min \{\mu_I(z), A(x, z)\} > 0$ and $\mu_{\Downarrow I}(y) = \sup_{w \in X} \min\{\mu_I(w), A(y, w)\} > 0$. So, exists $z, w \in X$ such that $\mu_I(z) > 0$, $A(x, z) > 0$ and similarly, $\mu_I(w) > 0$, $A(y, w) > 0$. Because I is a fuzzy ideal, then $\mu_I(z \vee w) > 0$ and $A(x, z \vee w) > 0$, $A(y, z \vee w) > 0$. Thus, by Proposition 3.1.1 (*v*), we have that $A(x \vee y, z \vee w) > 0$ and then, $\min\{\mu_I(z \vee w), A(x \vee y, z \vee w)\} > 0$. Therefore, $\mu_{\Downarrow I}(x \vee y) = \sup_{u \in X} \min\{\mu_I(u), A(x \vee y, u)\} > 0$.

■

Proposition 6.2.2 *[Mezzomo et al., 2013c, Proposition 4.2] Let (X, A) be a fuzzy lattice and F a fuzzy set on X. The fuzzy set $\Uparrow F$ of X is a fuzzy filter of (X, A).*

PROOF: Analogously the Proposition 7.1.3. ■

Proposition 6.2.3 *[Mezzomo et al., 2013c, Proposition 4.3] Let (X, A) be a fuzzy lattice, I and J be fuzzy sets of X, then $\Downarrow I$ satisfies the following properties:*

(i) $I \subseteq$ [1] $\Downarrow I$;

(ii) $I \subseteq J \Rightarrow \Downarrow I \subseteq \Downarrow J$;

(iii) $\Downarrow\Downarrow I = \Downarrow I$.

PROOF:

(*i*) $\mu_I(x) = \min\{\mu_I(x), A(x,x)\} \leq \sup_{y\in X} \min\{\mu_I(y), A(x,y)\} = \mu_{\Downarrow I}(x)$.

(*ii*) If $I \subseteq J$ then for all $y \in X$, $\mu_I(y) \leq \mu_J(y)$. So, for all $x, y \in X$, $\min\{\mu_I(y), A(x,y)\} \leq \min\{\mu_J(y), A(x,y)\}$. Hence, $\sup_{y\in X} \min\{\mu_I(y), A(x,y)\} \leq \sup_{y\in X} \min\{\mu_J(y), A(x,y)\}$. Therefore, $\mu_{\Downarrow I}(x) \leq \mu_{\Downarrow J}(x)$.

(*iii*)

$$\begin{aligned}
\mu_{\Downarrow\Downarrow I}(x) &= \sup_{y\in X} \min\{\mu_{\Downarrow I}(y), A(x,y)\} \\
&= \sup_{y\in X} \min\{\sup_{z\in X} \min\{\mu_I(z), A(y,z)\}, A(x,y)\} \\
&= \sup_{y\in X} \{\sup_{z\in X} \min\{\mu_I(z), A(y,z), A(x,y)\} \\
&= \sup_{y,z\in X} \min\{\mu_I(z), A(x,y), A(y,z)\} \\
&= \sup_{z\in X} \min\{\mu_I(z), \sup_{y\in X} \min\{A(x,y), A(y,z)\}\} \\
&= \sup_{z\in X} \min\{\mu_I(z), A(x,z)\} \\
&= \mu_{\Downarrow I}(x)
\end{aligned}$$

■

[1] $\subseteq$ is the usual inclusion of fuzzy sets, i.e., given two fuzzy sets I and J on an universe X, then $I \subseteq J$ if $\mu_I(x) \leq \mu_J(x)$, for all $x \in X$.

Proposition 6.2.4 *[Mezzomo et al., 2013c, Proposition 4.4] Let (X, A) be a fuzzy lattice, F and G be fuzzy sets on X, then $\Uparrow F$ satisfies the following properties:*

(i) $F \subseteq \Uparrow F$;

(ii) $F \subseteq G \Rightarrow \Uparrow F \subseteq \Uparrow G$;

(iii) $\Uparrow\Uparrow F = \Uparrow F$.

Proof: Analogously the Proposition 6.2.3. ■

Proposition 6.2.5 *[Mezzomo et al., 2013c, Proposition 4.5] Let I be a fuzzy ideal such that if $\mu_I(y) > 0$ and $A(x, y) > 0$, then $\mu_I(x) \geq \mu_I(y)$. So, $I = \Downarrow I$.*

Proof: By Proposition 6.2.3 (i), we have $I \subseteq \Downarrow I$. It suffices to prove $\Downarrow I \subseteq I$, then

$$\begin{aligned}
\mu_{\Downarrow I}(x) &= \sup_{y \in X} \min\{\mu_I(y), A(x, y)\} \\
&\leq \sup_{y \in X} \min\{\mu_I(x), A(x, y)\} \quad \text{(By hypothesis)} \\
&\leq \sup_{y \in X} \min\{\mu_I(x), A(x, x)\} \\
&= \min\{\mu_I(x), A(x, x)\} \\
&= \mu_I(x).
\end{aligned}$$

■

Corollary 6.2.1 *[Mezzomo et al., 2013c, Corollary 4.1] Let I be a fuzzy ideal (filter) such that if $\mu_I(y) > 0$ and $A(x, y) > 0$, then $\mu_I(x) \geq \mu_I(y)$. Then, $\Downarrow I$ ($\Uparrow F$) is the least fuzzy ideal (filter) containing I (F).*

Proof: Let I be a fuzzy set and J be a fuzzy ideal. Suppose $I \subseteq J \subseteq \Downarrow I$, then by Proposition 6.2.3 (ii) and (iii), $\Downarrow I \subseteq \Downarrow J \subseteq \Downarrow\Downarrow I = \Downarrow I$. So, $\Downarrow I = \Downarrow J$ and by Proposition 6.2.5, $I = J$. Similarly we prove for fuzzy filters. ■

Now, fixing an element, $x \in X$, we consider singleton fuzzy sets, i.e. fuzzy sets of the form:

$$\mu_{\tilde{x}}(y) = \begin{cases} 1, & \text{if } y = x \\ 0, & \text{if } y \neq x. \end{cases} \tag{6.1}$$

Proposition 6.2.6 *[Mezzomo et al., 2013c, Proposition 4.6] Let (X, A) a fuzzy lattice. Then, for all $x \in X$, $\mu_{\Downarrow \tilde{x}}(y) = A(y, x)$.*

PROOF: Let $y \in X$, then

$$\begin{aligned} \mu_{\Downarrow \tilde{x}}(y) &= \sup_{z \in X} \min\{\mu_{\tilde{x}}(z), A(y, z)\} \\ &= \sup\{0, A(y, x)\} \quad \text{(by Equation (6.1))} \\ &= A(y, x). \end{aligned}$$

■

Proposition 6.2.7 *[Mezzomo et al., 2013c, Proposition 4.7] $\Downarrow \tilde{x}$ is a fuzzy ideal for all $x \in X$.*

PROOF: Let $y, z \in X$.

(i) Suppose $\mu_{\Downarrow \tilde{x}}(y) > 0$ and $A(z, y) > 0$. Then, by Proposition 6.2.6, $A(y, x) = \mu_{\Downarrow \tilde{x}}(y) > 0$ and by Proposition 2.2.1, $A(z, x) > 0$. So, by Proposition 6.2.6, $\mu_{\Downarrow \tilde{x}}(z) = A(z, x) > 0$.

(ii) Suppose $\mu_{\Downarrow \tilde{x}}(y) > 0$ and $\mu_{\Downarrow \tilde{x}}(z) > 0$. Then, $A(y, x) > 0$ and $A(z, x) > 0$. So, by Proposition 3.1.1 *(ii)*, we have that $\mu_{\Downarrow \tilde{x}}(y \vee z, x) > 0$. Therefore, $\mu_{\Downarrow \tilde{x}}(y \vee z) > 0$.

■

Definition 6.2.2 *[Mezzomo et al., 2013c, Definition 4.2] Let (X, A) be a fuzzy lattice and $x \in X$. The fuzzy ideal $\Downarrow \tilde{x}$ of (X, A) is called **principal fuzzy ideal** of (X, A) generated by x.*

We can establish dual results for Propositions 6.2.6 and Proposition 6.2.7 and a dual version of Definition 6.2.2 for fuzzy filters.

Proposition 6.2.8 *[Mezzomo et al., 2013c, Proposition 4.8] Let (X, A) a fuzzy lattice. Then, for all $x \in X$, $\mu_{\Uparrow\tilde{x}}(y) = A(x, y)$.*

PROOF: Let $y \in X$, then

$$\begin{aligned}
\mu_{\Uparrow\tilde{x}}(y) &= \sup_{z\in X} \min\{\mu_{\tilde{x}}(z), A(z, y)\} \\
&= \sup\{0, A(x, y)\} \quad \text{(by Equation (6.1))} \\
&= A(x, y).
\end{aligned}$$

■

Proposition 6.2.9 *[Mezzomo et al., 2013c, Proposition 4.9] $\Uparrow \tilde{x}$ is a fuzzy filter for all $x \in X$.*

PROOF: Analogously the Proposition 6.2.7. ■

Definition 6.2.3 *[Mezzomo et al., 2013c, Definition 4.3] Let $\mathcal{L} = (X, A)$ be a fuzzy lattice and $x \in X$. The fuzzy filter $\Uparrow \tilde{x}$ of (X, A) is called **principal fuzzy filter** of (X, A) generated by x.*

The family of all fuzzy ideals of a fuzzy lattice $\mathcal{L} = (X, A)$ will be denoted by $I(\mathcal{L})$. Duality, will be denoted by $F(\mathcal{L})$ the family of all fuzzy filters of $\mathcal{L}$.

Proposition 6.2.10 *[Mezzomo et al., 2013c, Proposition 4.10] Let Z be a finite subset of $I(\mathcal{L})$ $(F(\mathcal{L}))$. Then $\bigcap Z \in I(\mathcal{L})$ $(\bigcap Z \in F(\mathcal{L}))$, where $\mu_{\bigcap Z}(x) = \inf\{\mu_{Z_j}(x) : Z_j \in Z\}$.*

PROOF: In the case $Z = \emptyset$, then $\{\mu_{Z_j}(x) : Z_j \in Z\} = \emptyset$ and $\inf \emptyset = \top$. So, $\bigcap Z = \tilde{X}$ where $\mu_{\tilde{X}}(x) = 1$ for all $x \in X$ which clearly is a fuzzy ideal.
If Z is a nonempty finite set of $I(\mathcal{L})$, then:

(*i*) Suppose $x, y \in X$ such that $\mu_{\bigcap Z}(y) > 0$ and $A(x, y) > 0$. Then, for all $Z_j \in Z$, $\mu_{Z_j}(y) > 0$. Because $Z_j \in I(\mathcal{L})$, then $\mu_{Z_j}(x) > 0$. Since Z is nonempty and finite, $\mu_{\bigcap Z}(x) = \mu_{Z_j}(x)$ for some $Z_j \in Z$ and so $\mu_{\bigcap Z}(x) > 0$.

(*ii*) Suppose $x, y \in X$ such that $\mu_{\bigcap Z}(x) > 0$ and $\mu_{\bigcap Z}(y) > 0$. Then, for all $Z_j \in Z$, $\mu_{Z_j}(x) > 0$ and $\mu_{Z_j}(y) > 0$. Because $Z_j \in I(\mathcal{L})$, then $\mu_{Z_j}(x \vee y) > 0$. Since Z is nonempty and finite, $\mu_{\bigcap Z}(x \vee y) = \mu_{Z_j}(x \vee y)$ for some $Z_j \in Z$ and so $\mu_{\bigcap Z}(x \vee y) > 0$. Therefore, $\bigcap Z \in I(\mathcal{L})$.

Analogous we prove that if Z be a finite subset of $F(\mathcal{L})$, then $\bigcap Z \in F(\mathcal{L})$. ■

The following proposition prove the relation between the fuzzy ideal $\Downarrow I$ and the principal fuzzy ideal $\Downarrow \tilde{x}$.

Proposition 6.2.11 *[Mezzomo et al., 2013c, Proposition 4.11] Let I be a fuzzy ideal of (X, A), then*

$$S(\Downarrow I) = S\left(\bigcup_{x \in S(I)} \Downarrow \tilde{x}\right).$$

Proof: We denote $\bigcup_{x \in S(I)} \Downarrow \tilde{x}$ by $\mathcal{J}$ for the simplify the notation. First we will prove that $S(\Downarrow I) \subseteq S(\mathcal{J})$. In fact, suppose $z \in S(\Downarrow I)$, i.e., $\mu_{\Downarrow I}(z) > 0$. Then, by definition, $\sup_{x \in X} \min\{\mu_I(x), A(z, x)\} > 0$. So, exists at least one $x_j \in X$ such that $\min\{\mu_I(x_j), A(z, x_j)\} > 0$. Hence, $\mu_I(x_j) > 0$ and $A(z, x_j) > 0$. By Proposition 6.2.6, $\mu_{\Downarrow \tilde{x}_j}(z) = A(z, x_j) > 0$. Because $\mu_I(x_j) > 0$ then $x_j \in S(I)$. So, $\Downarrow \tilde{x}_j \subseteq \mathcal{J}$ and, therefore, $\mu_{\mathcal{J}}(z) > 0$, i.e., $z \in S(\mathcal{J})$.
Now, we will prove that $S(\mathcal{J}) \subseteq S(\Downarrow I)$. In fact, suppose $y \in X$ such that $\mu_{\mathcal{J}}(y) > 0$. So, $\mu_{\Downarrow \tilde{x}_j}(y) > 0$ for at least one $x_j \in S(I)$ and, by Proposition 6.2.6, $A(y, x_j) = \mu_{\Downarrow \tilde{x}_j}(y) > 0$. Because $x_j \in S(I)$, then $\mu_I(x_j) > 0$. So, $\min\{\mu_I(x_j), A(y, x_j)\} > 0$ and $\sup_{x_j \in S(I)} \min\{\mu_I(x_j), A(y, x_j)\} > 0$. Therefore, $\mu_{\Downarrow I}(y) > 0$, i.e., $y \in S(\Downarrow I)$. ■

By duality, the Proposition 6.2.11 can also be proved for principal fuzzy filter as follows

Proposition 6.2.12 *[Mezzomo et al., 2013c, Proposition 4.12] Let F be a fuzzy filter of (X, A), then*

$$S(\Uparrow I) = S\left(\bigcup_{x\in S(F)} \Uparrow x\right).$$

PROOF: Analogously the Proposition 6.2.11. ■

Proposition 6.2.13 *[Mezzomo et al., 2013c, Proposition 4.13] Let I be a fuzzy ideal normalized of (X, A), i.e., $\mu_I(x) = 1$ for some $x \in X$. Then*

$$\Downarrow I = \bigcup_{x\in S(I)} \Downarrow \tilde{x}.$$

PROOF: Let $z \in X$. Then,

$$\begin{aligned}
\mu_{\Downarrow I}(z) &= \sup_{x\in X} \min\{\mu_I(x), A(z,x)\} \\
&= \sup_{x\in S(I)} \{\mu_I(x) \wedge A(z,x)\} \\
&= \sup_{x\in S(I)} \{\mu_I(x) \wedge \mu_{\Downarrow\tilde{x}}(z)\} \\
&= \{\sup_{x\in S(I)} \mu_I(x)\} \wedge \{\sup_{x\in S(I)} \mu_{\Downarrow\tilde{x}}(z)\} \\
&= \sup_{x\in S(I)} \mu_{\Downarrow\tilde{x}}(z) \\
&= \mu_{\mathcal{J}}(z)
\end{aligned}$$

where $\mathcal{J} = \bigcup_{x\in S(I)} \Downarrow \tilde{x}$. Notice that $\sup_{x\in S(I)} \mu_I(x) = 1$ because I is normalized. ■

By duality, the Proposition 6.2.13 can also be proved for principal fuzzy filter as follows:

Proposition 6.2.14 *[Mezzomo et al., 2013c, Proposition 4.14] Let F be a fuzzy filter normalized of (X, A), i.e., $\mu_F(x) = 1$ for some $x \in X$. Then*

$$\Uparrow I = \bigcup_{x\in S(F)} \Uparrow \tilde{x}.$$

PROOF: Analogously the Proposition 6.2.13. ■

Proposition 6.2.15 *[Mezzomo et al., 2013c, Propositions 4.15 and 4.16] Let (X, A) be a complete fuzzy sup-lattice (inf-lattice) and I be a fuzzy set on X. Then, $\sup I$ ($\inf I$) exists and it is unique.*

PROOF: The existence of $\sup I$ is guaranteed by Definition 3.1.7. Just let us prove the uniqueness of $\sup I$. Suppose u and v are $\sup I$. Then by Definition 3.1.6, $A(v,u) > 0$ and $A(u,v) > 0$. So, by antisymmetry, $u = v$.

Analogous we prove that if (X, A) be a complete fuzzy inf-lattice and I be a fuzzy set on X, then $\inf I$ exists and it is unique. ■

Proposition 6.2.16 *[Mezzomo et al., 2013c, Proposition 4.17] Let (X, A) be a fuzzy sup-lattice. Then, exists $\top \in X$ such that, for all $x \in X$, $A(x, \top) > 0$.*

PROOF: Trivially, by Definition 6.1.1, $\tilde{X}$ is a fuzzy ideal of (X, A). Since, by hypothesis (X, A) is a fuzzy sup-lattice, then by Proposition 6.2.15 it has a supremum, denoted by $\top$. Let $x \in X$, then by definition, $\mu_{\tilde{X}}(x) = 1$. So, by Definition 3.1.6 $A(x, \top) = A(x, \sup \tilde{X}) > 0$. ■

Proposition 6.2.17 *[Mezzomo et al., 2013c, Proposition 4.18] Let (X, A) be a fuzzy inf-lattice. Then, exists $\perp \in X$ such that, for all $x \in X$, $A(\perp, x) > 0$.*

PROOF: Analogously the Proposition 6.2.16. ■

Proposition 6.2.18 *[Mezzomo et al., 2013c, Proposition 4.19] Let (X, A) be a complete fuzzy lattice and I be a fuzzy set on X. Then, $S(\Downarrow I) \subseteq S(\Downarrow \widetilde{\sup I})$.*

PROOF: Suppose $z \in S(\Downarrow I)$, i.e., $\mu_{\Downarrow I}(z) > 0$. Then, by Definition 6.2.1, $\sup\limits_{x \in X} \min\{\mu_I(x), A(z,x)\} > 0$ and therefore, exists $y \in X$ such that $\min\{\mu_I(y), A(z,y)\} > 0$. So, $A(z,y) > 0$ and by Definition 3.1.6, $A(y, \sup I) > 0$. Therefore, by Proposition 2.2.1, $A(z, \sup I) > 0$. Therefore, by Proposition 6.2.6, $0 < A(z, \sup I) = \mu_{\Downarrow \widetilde{\sup I}}(z)$, that is, $z \in S(\Downarrow \widetilde{\sup I})$. ■

Proposition 6.2.19 *[Mezzomo et al., 2013c, Proposition 4.20] Let (X, A) be a complete fuzzy lattice and F be a fuzzy set on X. Then, $S(\Uparrow F) \subseteq S(\Uparrow \widetilde{\inf F})$.*

PROOF: Analogously the Proposition 6.2.18. ■

Proposition 6.2.20 *[Mezzomo et al., 2013c, Proposition 4.21] $S(\Downarrow \widetilde{\sup I}) \subseteq S(\Downarrow I)$ only if $\sup I \in S(I)$.*

PROOF: Suppose $x \in S(\Downarrow \widetilde{\sup I})$, then, $\mu_{\Downarrow \widetilde{\sup I}}(x) > 0$. By Proposition 6.2.6, $A(x, \sup I) > 0$ and because by hypothesis, $\mu_I(\sup I) > 0$ then $\min\{\mu_I(\sup I), A(x, \sup I)\} > 0$. So, $\sup_{y \in X} \min\{\mu_I(y), A(x, y)\} > 0$. Hence, by Definition 6.2.1, $x \in S(\Downarrow I)$.
Dually, $S(\Uparrow \widetilde{\inf F}) \subseteq S(\Uparrow F)$ only if $\inf F \in S(F)$. ■

6.2.2 Proper Fuzzy Ideals and Filters

Before defining prime fuzzy ideal and prime fuzzy filter, we need define a proper fuzzy ideal of a fuzzy lattice and a proper fuzzy filter of a fuzzy lattice.

Definition 6.2.4 *[Mezzomo et al., 2013c, Definition 4.7] A nonempty fuzzy set Z on X is called **proper fuzzy set** if $\mu_Z(x) = 0$ for at least one $x \in X$. A fuzzy set Z is called improper fuzzy set if $\mu_Z(x) \neq 0$ for all $x \in X$. Fuzzy ideals which are proper (improper) fuzzy set will be called **proper (improper) fuzzy ideals**.*

In the Definition 2.1 from the paper Koguep et al. [2008] is defined that a fuzzy set I of a set X is proper if it has a non constant membership function and if $\mu_I(x) = 0$, for all $x \in X$, then I is a improper fuzzy set. Hence, the Definition 6.2.4 implies in the Definition 2.1 of the paper Koguep et al. [2008]. Notice that, by Definition 6.2.4 all the fuzzy set are either proper or improper, whereas by Definition 2.1 in Koguep et al. [2008], there are infinitely many fuzzy sets which are neither proper nor improper.

Proposition 6.2.21 *[Mezzomo et al., 2013c, Proposition 4.22] Let I be a fuzzy ideal of a bounded fuzzy lattice (X, A). I is a proper fuzzy ideal iff $\mu_I(\top) = 0$.*

PROOF: Let I a proper fuzzy ideal of (X, A).

($\Rightarrow$) By Definition 6.2.4, exists $x \in X$ such that $\mu_I(x) = 0$. Then, by Proposition 6.2.16, $A(x, \top) > 0$. So, case $\mu_I(\top) > 0$ then by Definition 6.1.1 (i), $\mu_I(x) > 0$ which is a contradiction for the hypothesis $\mu_I(x) = 0$.

($\Leftarrow$) Straightforward from definition of proper fuzzy ideals.

■

Proposition 6.2.22 *[Mezzomo et al., 2013c, Proposition 4.23] Let F be a fuzzy filter of a bounded fuzzy lattice (X, A). F is a proper fuzzy filter iff $\mu_F(\perp) = 0$.*

PROOF: Analogously the Proposition 6.2.21. ■

Let $\mathcal{I}_p(\mathcal{L})$ be the family of all proper fuzzy ideals of a fuzzy lattice and let $\mathcal{F}_p(\mathcal{L})$ be the family of all proper fuzzy filters of a fuzzy lattice.

Proposition 6.2.23 *[Mezzomo et al., 2013c, Proposition 4.24] Let $Z \subseteq \mathcal{I}_p(\mathcal{L})$ and $\widetilde{X} \in I(\mathcal{L})$. Then, $\bigcup Z \neq \widetilde{X}$.*

PROOF: By definition, $\mu_{\bigcup Z}(\top) = \sup\{\mu_{Z_j}(\top) : Z_j \in Z\} = \sup\{0\} = 0$. Therefore, $\bigcup Z \neq \widetilde{X}$. ■

Corollary 6.2.2 *[Mezzomo et al., 2013c, Corollary 4.3] The union of proper fuzzy ideals is a proper fuzzy ideal.*

PROOF: Straightforward from Proposition 7.1.2 *(2)* and Proposition 6.2.23. ■

Corollary 6.2.3 *[Mezzomo et al., 2013c, Corollary 4.4] Let $Z \subseteq \mathcal{I}_p(\mathcal{L})$ and $\widetilde{X} \in I(\mathcal{L})$. Then, $\bigcap Z \neq \widetilde{X}$.*

PROOF: Straightforward to Definition 6.2.4. ■

The Proposition 6.2.23 and Corollaries 6.2.2 and 6.2.3 holds for fuzzy filters.

6.2.3 Prime Fuzzy Ideals and Filters

Now, we will define a prime fuzzy ideal and prime fuzzy filter as follows:

Definition 6.2.5 *[Mezzomo et al., 2013c, Definition 4.8] A proper fuzzy ideal I of (X, A) is called **prime fuzzy ideal** if $\mu_I(x \wedge y) > 0$, then either $\mu_I(x) > 0$ or $\mu_I(y) > 0$, for all $x, y \in X$.*

Definition 6.2.6 *[Mezzomo et al., 2013c, Definition 4.9] A proper fuzzy filter F of (X, A) is called* ***prime fuzzy filter*** *if $\mu_F(x \vee y) > 0$, then either $\mu_F(x) > 0$ or $\mu_F(y) > 0$, for all $x, y \in X$.*

The following example show us an example of prime fuzzy ideal.

Example 6.2.1 *Let $X = \{v, x, y, z, w\}$ be a set and (X, A) be a fuzzy lattice such that $A(v, v) = A(x, x) = A(y, y) = A(z, z) = A(w, w) = 1.0, A(v, x) = A(v, y) = A(v, z) = A(v, w) = A(x, y) = A(x, z) = A(x, w) = A(y, z) = A(z, y) = A(y, w) = A(z, w) = 0.0, A(x, v) = 0.2, A(y, v) = 0.4, A(z, v) = 0.7, A(w, v) = 0.9, A(y, x) = 0.3,\ A(z, x) = 0.5, A(w, x) = 0.8, A(w, y) = 0.4$ and $A(w, z) = 0.1$. The graphical representations related the tabular (Table 4.1) and oriented graph (Diagram 6.1) of the prime fuzzy ideal are represented in Figure 4.1 (a) and (b), respectively.*

Consider the proper fuzzy ideal $I = \{(v, 0.0), (x, 0.3), (y, 0.7), (z, 0.6), (w, 1.0)\}$ on (X, A). It is easy checked that I is a fuzzy ideal of (X, A). Then,
$\mu_I(v \wedge x) = \mu_I(x) = 0.3,\ \mu_I(v) = 0.0$ *and* $\mu_I(x) = 0.3$;
$\mu_I(v \wedge y) = \mu_I(y) = 0.7,\ \mu_I(v) = 0.0$ *and* $\mu_I(y) = 0.7$;
$\mu_I(v \wedge z) = \mu_I(z) = 0.6,\ \mu_I(v) = 0.0$ *and* $\mu_I(z) = 0.6$;
$\mu_I(v \wedge w) = \mu_I(w) = 1.0,\ \mu_I(v) = 0.0$ *and* $\mu_I(w) = 1.0$;
$\mu_I(x \wedge y) = \mu_I(y) = 0.7,\ \mu_I(x) = 0.3$ *and* $\mu_I(y) = 0.7$;
$\mu_I(x \wedge z) = \mu_I(z) = 0.6,\ \mu_I(x) = 0.3$ *and* $\mu_I(z) = 0.6$;
$\mu_I(x \wedge w) = \mu_I(w) = 1.0,\ \mu_I(x) = 0.3$ *and* $\mu_I(w) = 1.0$;
$\mu_I(y \wedge z) = \mu_I(w) = 1.0,\ \mu_I(y) = 0.7$ *and* $\mu_I(z) = 0.7$;
$\mu_I(y \wedge w) = \mu_I(w) = 1.0,\ \mu_I(y) = 0.7$ *and* $\mu_I(w) = 1.0$;
$\mu_I(z \wedge w) = \mu_I(w) = 1.0,\ \mu_I(z) = 1.0$ *and* $\mu_I(w) = 1.0$.

Therefore, I is a prime fuzzy ideal of (X, A). But I is not a fuzzy filter because $\mu_I(x) = 0.3$ and $A(x, v) = 0.2 > 0$ but $\mu_I(v) = 0.0$. Consequently, I is not a proper fuzzy filter because $\mu_I(w) = \mu_I(\perp) \neq 0$ and therefore, I is not a prime fuzzy filter. ■

The following example show us that a intersection of the family of all prime fuzzy ideals are not a prime fuzzy ideal.

A	w	z	y	x	v
w	1.0	0.1	0.4	0.8	0.9
z	0.0	1.0	0.0	0.5	0.7
y	0.0	0.0	1.0	0.3	0.4
x	0.0	0.0	0.0	1.0	0.2
v	0.0	0.0	0.0	0.0	1.0

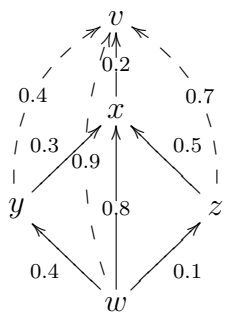

Figure 6.1: Representations of the prime fuzzy ideal of the fuzzy lattice $\mathcal{L} = (X, A)$.

Example 6.2.2 *Let $X = \{x, y, z, w\}$ be a set and (X, A) be a fuzzy lattice such that $A(x, x) = A(y, y) = A(z, z) = A(w, w) = 1.0, A(x, y) = A(x, z) = A(x, w) = A(z, y) = A(y, z) = A(z, w) = 0.0$, $A(y, x) = 0.3, A(z, x) = 0.5, A(w, x) = 0.8, A(w, y) = 0.4$ and $A(w, z) = 0.2$. We can see the fuzzy lattice in the tabular (Table 6.2) and oriented graph (Diagram 6.2) in Figure 6.2 (a) and (b), respectively.*

Consider the proper fuzzy ideals $I = \{(x, 0.0), (y, 0.5), (z, 0.0), (w, 1.0)\}$ and $J = \{(x, 0.0), (y, 0.0), (z, 0.3), (w, 1.0)\}$ on (X, A). It is easy checked that I and J are prime fuzzy ideals of (X, A) and that the fuzzy intersection $I \bigcap J = \{(x, 0.0), (y, 0.0), (z, 0.0), (w, 1.0)\}$ is a fuzzy ideal. So, $\mu_{I \bigcap J}(y \wedge z) = \mu_{I \bigcap J}(w) = 1.0$ but $\mu_{I \bigcap J}(y) = 0.0$ and $\mu_{I \bigcap J}(z) = 0.0$. Therefore, $I \bigcap J$ is not a prime fuzzy ideal. ■

A	w	z	y	x
w	1.0	0.2	0.4	0.8
z	0.0	1.0	0.0	0.5
y	0.0	0.0	1.0	0.3
x	0.0	0.0	0.0	1.0

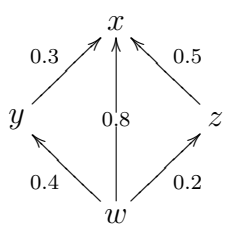

Figure 6.2: Representations of the fuzzy lattice.

Dually, we can show that the intersection of prime fuzzy filters are not a prime fuzzy filter.

Proposition 6.2.24 *[Mezzomo et al., 2013c, Propositions 4.25] Let I be a fuzzy set on X. I is a prime fuzzy ideal (filter) of a fuzzy lattice (X, A) iff $S(I)$ is an ideal (filter) of $(X, S(A))$.*

PROOF:

($\Rightarrow$) By Proposition 6.1.1, if I is a fuzzy ideal of a fuzzy lattice (X, A) then $S(I)$ is an ideal of $(X, S(A))$. By hypothesis I is a prime fuzzy ideal of $(X, S(A))$ and, by Definition 6.2.5, either $\mu_I(x) > 0$ or $\mu_I(y) > 0$. Therefore, either $x \in S(I)$ or $y \in S(I)$.

($\Leftarrow$) By Proposition 6.1.1, if $S(I)$ is an ideal of $(X, S(A))$ then I is a fuzzy ideal of a fuzzy lattice (X, A). By hypothesis $S(I)$ is an ideal of $(X, S(A))$, then by definition of classical prime ideal, either $x \in S(I)$ or $y \in S(I)$. Therefore, either $\mu_I(x) > 0$ or $\mu_I(y) > 0$.

■

Proposition 6.2.25 *[Mezzomo et al., 2013c, Propositions 4.26] Let I be a fuzzy set on X. I is a prime fuzzy ideal (filter) of a fuzzy lattice (X, A) iff for each $\alpha \in (0, 1]$, I_α is a prime ideal (filter) of (X, A_α).*

PROOF: By Theorem 6.1.1, we have that I is a fuzzy ideal of fuzzy lattice (X, A) iff for each $\alpha \in (0, 1]$, I_α is an ideal of (X, A_α). So, we just need to prove the primality.

($\Rightarrow$) Let I be a prime fuzzy ideal of (X, A), $\alpha \in (0, 1]$. Suppose $x, y \in I_\alpha$, such that $x \wedge y \in I_\alpha$, for some $\alpha \in (0, 1]$. Then, $\mu_I(x \wedge y) \geq \alpha$. By hypothesis I is a prime fuzzy ideal of (X, A_α) and, by Definition 6.2.5, either $\mu_I(x) \geq \alpha$ or $\mu_I(y) \geq \alpha$. Therefore, either $x \in I_\alpha$ or $y \in I_\alpha$.

($\Leftarrow$) Suppose that I_α is a prime ideal of (X, A_α) for each $\alpha \in (0, 1]$. Suppose $x, y \in X$ such that $\mu_I(x \wedge y) \geq \alpha$. Then, $x \wedge y \in I_\alpha$. Because I_α is a prime ideal of (X, A_α), then by definition of classical prime ideal, either $x \in I_\alpha$ or $y \in I_\alpha$. Therefore, either $\mu_I(x) \geq \alpha$ or $\mu_I(y) \geq \alpha$.

■

Dually, we can prove the Propositions 6.2.24 and 6.2.25 for prime fuzzy filters.

6.2.4 Maximal Fuzzy Ideals and Filters

Another type of fuzzy ideals (fuzzy filters) is the maximal fuzzy ideal (maximal fuzzy filter), defined by:

Definition 6.2.7 *[Mezzomo et al., 2013c, Definition 4.10] Let I be a proper fuzzy ideal of (X, A). I is called **maximal fuzzy ideal** if, for all proper fuzzy ideal J, $\mu_I(x) \geq \mu_J(x)$, for every $x \in X$. We denote the maximal fuzzy ideal by I_M.*

Definition 6.2.8 *[Mezzomo et al., 2013c, Definition 4.11] Let F be a proper fuzzy filter of (X, A). F is called **maximal fuzzy filter** if, for all proper fuzzy filter G, $\mu_F(x) \geq \mu_G(x)$, for every $x \in X$. We denote the maximal fuzzy filter by F_M.*

Remark 6.2.1 *Notice that all maximal fuzzy ideal is a proper fuzzy ideal but not all proper fuzzy ideal is a maximal fuzzy ideal. Similarly, all maximal fuzzy filter is a proper fuzzy filter but not all proper fuzzy filter is a maximal fuzzy filter.*

Proposition 6.2.26 *[Mezzomo et al., 2013c, Propositions 4.27] The maximal fuzzy ideal (filter) I_M (F_M) of (X, A), if it exists, is unique.*

Proof: Trivially, if (X, A) has a maximal fuzzy ideal then, for all $x \in X$,

$$\mu_{I_M}(x) = \begin{cases} 1, & \text{if} \quad x \neq \top \\ 0, & \text{if} \quad x = \top \end{cases}$$

So, I_M is the unique maximal fuzzy ideal of (X, A) because for each other proper fuzzy ideals of (X, A), either it is containing in I_M or it is I_M. ■

Corollary 6.2.4 *[Mezzomo et al., 2013c, Corollary 4.5] A fuzzy lattice (X, A) has a maximal fuzzy ideal iff, for all $x, y \in X \setminus \{\top\}$, $x \vee y \neq \top$.*

Proof: Straightforward. ■

Proposition 6.2.27 *[Mezzomo et al., 2013c, Proposition 4.28] The maximal fuzzy filter F_M of (X, A), if it exists, is unique.*

PROOF: Analogously the Proposition 6.2.26. ■

Theorem 6.2.1 *[Mezzomo et al., 2013c, Theorem 4.1] Let $\mathcal{L} = (X, A)$ be a bounded fuzzy lattice. If $\mathcal{L}$ admits a maximal fuzzy ideal, then it is prime. Dually, if $\mathcal{L}$ admits a maximal fuzzy filter, then it is prime.*

PROOF: Let I_M be the maximal fuzzy ideal of $\mathcal{L}$ defined in Proposition 6.2.26. If $\mu_{I_M}(x \wedge y) > 0$ then, $x \wedge y \neq \top$. So, either $x \neq \top$ or $y \neq \top$. Thus, either $\mu_{I_M}(x) > 0$ or $\mu_{I_M}(y) > 0$. Dually, we prove that if $\mathcal{L}$ admits a maximal fuzzy filter, then it is prime. ■

6.3 Fuzzy Homomorphism on Fuzzy Ideals and Filters

In section 3.2, we defined the fuzzy homomorphism between fuzzy lattices and proved some results of fuzzy homomorphism on bounded fuzzy lattices. As a continuation of this study, in this section we will prove some results involving fuzzy homomorphism, fuzzy ideals and fuzzy filters. We can note that fuzzy homomorphism can not preserve fuzzy ideals, i.e., if h is a fuzzy homomorphism and I is a fuzzy ideal of $\mathcal{L}$, then $\tilde{h}(I)$ is not necessarily a fuzzy ideal of $\mathcal{M}$. The example above illustrates this case.

Example 6.3.1 *Let $\mathcal{L}$ and $\mathcal{M}$ be the fuzzy lattices defined in Example 3.1.1 and 3.2.1, respectively, h be the fuzzy homomorphism defined in Example 3.2.1, and let $I = \{(x, 0.0), (y, 0.2), (z, 0.4), (w, 0.7)\}$ be the fuzzy ideal of $\mathcal{L}$ defined in Example 6.1.1. Then the fuzzy set $\tilde{h}(I)$=$\{(x', 0.0), (y', 0.2), (z', 0.4), (v', 0.0), (w', 0.7)\}$ is not a fuzzy ideal of $\mathcal{M}$ because $\mu_{\tilde{h}(I)}(y') > 0$ and $B(v', y') > 0$, but $\mu_{\tilde{h}(I)}(v') = 0.0$. Therefore, I is a fuzzy ideal of $\mathcal{L}$ and $\tilde{h}(I)$ is not a fuzzy ideal of $\mathcal{M}$.* ■

However, the following proposition shows that the converse occurs.

Lemma 6.3.1 *[Mezzomo et al., 2013c, Lemma 5.1] Let $\mathcal{L} = (X, A)$ and $\mathcal{M} = (Y, B)$ be bounded fuzzy lattices, $h : X \to Y$ be a fuzzy homomorphism and Z be a fuzzy set of X. Then, $\mu_{\tilde{h}(Z)}(h(x)) \geq \mu_Z(x)$. In addition, if h is injective, then $\mu_{\tilde{h}(Z)}(h(x)) = \mu_Z(x)$.*

PROOF: Let Z be a fuzzy set of X and $x \in X$. Then,

$$\begin{aligned} \mu_{\tilde{h}(Z)}(h(x)) &= \sup_{y \in X}\{\mu_Z(y) : h(y) = h(x)\} \\ &\geq \mu_Z(x). \end{aligned}$$

Trivially, because h is injective, if $h(x) = h(y)$, then $x = y$. Therefore, $\mu_{\tilde{h}(Z)}(h(x)) = \mu_Z(x)$. ■

Proposition 6.3.1 *[Mezzomo et al., 2013c, Proposition 5.7] Let $\mathcal{L} = (X, A)$ and $\mathcal{M} = (Y, B)$ be bounded fuzzy lattices, I be a fuzzy set on X and let $h : X \to Y$ be a fuzzy monomorphism from $\mathcal{L}$ into $\mathcal{M}$. Then,*

(1) If $\tilde{h}(I)$ is a fuzzy ideal of $\mathcal{M}$, then I is a fuzzy ideal of $\mathcal{L}$;

(2) If $\tilde{h}(I)$ is a proper fuzzy ideal of $\mathcal{M}$, then I is a proper fuzzy ideal of $\mathcal{L}$;

(3) If $\tilde{h}(I)$ is a prime fuzzy ideal of $\mathcal{M}$, then I is a prime fuzzy ideal of $\mathcal{L}$;

(4) If $\tilde{h}(I)$ is a maximal fuzzy ideal of $\mathcal{M}$, then I is a maximal fuzzy ideal of $\mathcal{L}$.

PROOF:

(1) First we will prove that if $\tilde{h}(I)$ is a fuzzy ideal of $\mathcal{M}$, then I is a fuzzy ideal of $\mathcal{L}$.

- (i) Suppose $y \in X$ such that $\mu_I(y) > 0$ and suppose $x \in X$ such that $A(x, y) > 0$. Because h is a fuzzy monomorphism and by Proposition 3.2.1, $B(h(x), h(y)) > 0$. By hypothesis, $\tilde{h}(I)$ is a fuzzy ideal of $\mathcal{M}$, so $\mu_{\tilde{h}(I)}(h(y)) = \sup_{y \in X}\{\mu_I(x) : h(x) = h(y)\} \geq \mu_I(y) > 0$. By Definition 6.1.1 (i), we have that $\mu_{\tilde{h}(I)}(h(x)) > 0$. So, by Lemma 6.3.1, then $\mu_I(x) = \mu_{\tilde{h}(I)}(h(x)) > 0$.
- (ii) Suppose $x, y \in X$ such that $\mu_I(x) > 0$ and $\mu_I(y) > 0$. By Lemma 6.3.1, we have that $\mu_{\tilde{h}(I)}(h(x)) = \mu_I(x) > 0$ and $\mu_{\tilde{h}(I)}(h(y)) = \mu_I(y) > 0$. By hypothesis, $\tilde{h}(I)$ is a fuzzy ideal of $\mathcal{M}$ so, by Definition 6.1.1 (ii), $\mu_{\tilde{h}(I)}(h(x) \vee_{\mathcal{M}} h(y)) > 0$. Hence, $\mu_{\tilde{h}(I)}(h(x \vee_{\mathcal{L}} y)) > 0$. By Lemma 6.3.1, $\mu_I(x \vee_{\mathcal{L}} y) = \mu_{\tilde{h}(I)}(h(x \vee_{\mathcal{L}} y)) > 0$.

(2) By *(1)* we have that if $\tilde{h}(I)$ is a fuzzy ideal of $\mathcal{M}$, then I is a fuzzy ideal of $\mathcal{L}$. By hypothesis $\tilde{h}(I)$ is a proper fuzzy ideal of $\mathcal{M}$, then there exists a $x' \in Y$ such that $\mu_{\tilde{h}(I)}(x') = 0$. Suppose $\mu_{\tilde{h}(I)}(\top_{\mathcal{M}}) > 0$, then $B(x', \top_{\mathcal{M}}) > 0$, and because $h(I)$ is a fuzzy ideal, then $\mu_{\tilde{h}(I)}(x') > 0$, that it is a contradiction. Therefore, $\mu_{\tilde{h}(I)}(\top_{\mathcal{M}}) = 0$. By Definition 3.2.1 (iv), $h(\top_{\mathcal{L}}) = \top_{\mathcal{M}}$. Thus, by Lemma 6.3.1,

$$\begin{aligned} \mu_I(\top_{\mathcal{L}}) &= \mu_{\tilde{h}(I)}(h(\top_{\mathcal{L}})) \\ &= \mu_{\tilde{h}(I)}(\top_{\mathcal{M}}) \\ &= 0 \end{aligned}$$

Therefore, I is a proper fuzzy ideal of $\mathcal{L}$.

(3) By *(2)* we have that if $\tilde{h}(I)$ is a proper fuzzy ideal of $\mathcal{M}$, then I is a proper fuzzy ideal of $\mathcal{L}$. So,

$$\begin{aligned} \mu_I(x \wedge_{\mathcal{L}} y) > 0 &\Rightarrow \mu_{\tilde{h}(I)}(h(x \wedge_{\mathcal{L}} y)) > 0 && \text{(By Lemma 6.3.1)} \\ &\Rightarrow \mu_{\tilde{h}(I)}(h(x) \wedge_{\mathcal{M}} h(y)) > 0 \\ &\Rightarrow \mu_{\tilde{h}(I)}(h(x)) > 0 \textit{ or } \mu_{\tilde{h}(I)}(h(y)) > 0 && \text{(Because } \tilde{h}(I) \text{ is prime)} \\ &\Rightarrow \mu_I(x) > 0 \textit{ or } \mu_I(y) > 0. && \text{(By Lemma 6.3.1)} \end{aligned}$$

(4) Suppose $x, y \in X \setminus \{\top\}$ such that $x \vee_{\mathcal{L}} y = \top_{\mathcal{L}}$. Then, $h(x) \vee_{\mathcal{M}} h(y) = h(x \vee_{\mathcal{L}} y) = h(\top_{\mathcal{L}}) = \top_{\mathcal{M}}$. However, because h is injective and $h(\top_{\mathcal{L}}) = \top_{\mathcal{M}}$, we have that $h(x) \neq \top_{\mathcal{M}}$ and $h(y) \neq \top_{\mathcal{M}}$, which, by Corollary 6.2.4, it is a contradiction with the hypothesis of that $\tilde{h}(I)$ is a maximal fuzzy ideal of $\mathcal{M}$. On the other hand, by Proposition 6.2.26, $\tilde{h}(I) = I_{\mathcal{M}}$, and so

$$\mu_{\tilde{h}(I)}(x') = \begin{cases} 1, & \text{if } \; x' \neq \top_{\mathcal{M}} \\ 0, & \text{if } \; x' = \top_{\mathcal{M}} \end{cases}$$

Hence, by Lemma 6.3.1,

$$\mu_I(x) = \mu_{\tilde{h}(I)}(h(x)) = \begin{cases} 1, & \text{if} \quad h(x) \neq \top_{\mathcal{M}} \\ 0, & \text{if} \quad h(x) = \top_{\mathcal{M}} \end{cases}$$

And, because h is injective, we have that

$$\mu_I(x) = \begin{cases} 1, & \text{if} \quad x \neq \top_{\mathcal{L}} \\ 0, & \text{if} \quad x = \top_{\mathcal{L}} \end{cases}$$

Therefore, by Proposition 6.2.26, I is a maximal fuzzy ideal of $\mathcal{L}$.

Dually, we prove that, since h is a fuzzy monomorphism, if $\tilde{h}(F)$ is a fuzzy filter of $\mathcal{M}$, then F is a fuzzy filter of $\mathcal{L}$. Moreover, if $\tilde{h}(F)$ is a proper, prime and maximal fuzzy filter, then F is a proper, prime and maximal fuzzy filter, respectively. ■

Proposition 6.3.2 *[Mezzomo et al., 2013c, Proposition 5.8] Let $\mathcal{L} = (X, A)$ and $\mathcal{M} = (Y, B)$ be bounded fuzzy lattices, I be a fuzzy set on X and let $h : X \to Y$ be a fuzzy isomorphism from $\mathcal{L}$ into $\mathcal{M}$ such that $A(x, y) = B(h(x), h(y))$. If $\tilde{h}(I)$ is a principal fuzzy ideal of $\mathcal{M}$, then I is a principal fuzzy ideal of $\mathcal{L}$.*

Proof: By Proposition 6.3.1, we have that if $\tilde{h}(I)$ is a fuzzy ideal of $\mathcal{M}$, then I is a fuzzy ideal of $\mathcal{L}$. By hypothesis, $\tilde{h}(I)$ is a principal fuzzy ideal of $\mathcal{M}$. By Definition 6.2.2, there exists $y' \in Y$ such that $\tilde{h}(I) = \Downarrow \widetilde{y'}$. Hence, by Proposition 6.2.6, $\mu_{h(I)}(x') = \mu_{\Downarrow \widetilde{y'}}(x') = B(x', y')$ for all $x' \in Y$. Because h is surjective, then there exists $y \in X$ such that $h(y) = y'$. So

$$\begin{aligned}
\mu_I(x) &= \mu_{\tilde{h}(I)}(h(x)) && \text{(By Lemma 6.3.1)} \\
&= \mu_{\Downarrow \widetilde{y'}}(h(x)) && \text{(By hyphotesis)} \\
&= \mu_{\Downarrow \widetilde{h(y)}}(h(x)) && \\
&= B(h(x), h(y)) && \text{(By Proposition 6.2.6)} \\
&= A(x, y) && \text{(By hypothesis)} \\
&= \mu_{\Downarrow \tilde{y}}(x). && \text{(By Proposition 6.2.6)}
\end{aligned}$$

Dually, we prove that if $\tilde{h}(F)$ is a principal fuzzy filter of $\mathcal{M}$, then F is a principal fuzzy filter of $\mathcal{L}$. ■

The condition of h to be a fuzzy homomorphism is not sufficient for principal, proper, prime and fuzzy ideals of $\mathcal{M}$ are mapped on principal, proper, prime and fuzzy ideals on $\mathcal{L}$, respectively. In the following theorem we will prove that, if h is a fuzzy homomorphism, then the inverse image of principal fuzzy ideal is a fuzzy ideal. Dually, the same is true for fuzzy filters.

Theorem 6.3.1 *[Mezzomo et al., 2013c, Theorem 5.1] Let $\mathcal{L} = (X, A)$ and $\mathcal{M} = (Y, B)$ be bounded fuzzy lattices and let $h : X \to Y$ be a map. Then, h is a fuzzy homomorphism if the fuzzy inverse image induced by h, of all principal fuzzy ideals of $\mathcal{M}$ is a fuzzy ideal of $\mathcal{L}$.*

Proof: Suppose that h is a fuzzy homomorphism and $\Downarrow \widetilde{y'}$ is a principal fuzzy ideal generated by $y' \in Y$.

(*i*) Suppose $x, y \in X$ such that $\mu_{\overleftarrow{\tilde{h}}(\Downarrow\widetilde{y'})}(y) > 0$ and $A(x, y) > 0$. So, by Definition 3.2.3, $\mu_{\Downarrow\widetilde{y'}}(h(y)) > 0$. Because h is a fuzzy homomorphism, $B(h(x), h(y)) > 0$. Because $\Downarrow \widetilde{y'}$ is a fuzzy ideal, then $\mu_{\Downarrow\widetilde{y'}}(h(x)) > 0$. Therefore, by Definition 3.2.3, $\mu_{\overleftarrow{\tilde{h}}(\Downarrow\widetilde{y'})}(x) > 0$.

(*ii*) Suppose $x, y \in X$ such that $\mu_{\overleftarrow{\tilde{h}}(\Downarrow\widetilde{y'})}(x) > 0$ and $\mu_{\overleftarrow{\tilde{h}}(\Downarrow\widetilde{y'})}(y) > 0$. So, by Definition 3.2.3, we have that $\mu_{\Downarrow\widetilde{y'}}(h(x)) > 0$ and $\mu_{\Downarrow\widetilde{y'}}(h(y)) > 0$. By hypothesis $\Downarrow \widetilde{y'}$ is a fuzzy ideal of $\mathcal{M}$. So, by Definition 6.1.1 (*ii*), $\mu_{\Downarrow\widetilde{y'}}(h(x) \vee_{\mathcal{M}} h(y)) > 0$. Hence, $\mu_{\Downarrow\widetilde{y'}}(h(x \vee_{\mathcal{L}} y)) > 0$. Therefore, by Definition 3.2.3, $\mu_{\overleftarrow{\tilde{h}}(\Downarrow\widetilde{y'})}(x \vee_{\mathcal{L}} y) > 0$.

Therefore, if h is a fuzzy homomorphism, then the fuzzy inverse image of all fuzzy principal ideals of $\mathcal{M}$ is a fuzzy ideal of $\mathcal{L}$. ■

Theorem 6.3.2 *[Mezzomo et al., 2013c, Theorem 5.2] Let $\mathcal{L} = (X, A)$ and $\mathcal{M} = (Y, B)$ be bounded fuzzy lattices and let $h : X \to Y$ be a map. Then, h is a fuzzy homomorphism if the fuzzy inverse image induced by h, of all principal fuzzy filters of $\mathcal{M}$ is a fuzzy filter of $\mathcal{L}$.*

PROOF: Analogous to the Theorem 6.3.1. ■

Remark 6.3.1 *Notice that if h is a fuzzy isomorphism and I is a fuzzy ideal of $\mathcal{L}$, then $h(I)$ is a fuzzy ideal of $\mathcal{M}$. Moreover, if I is a proper, principal, prime or maximal fuzzy ideal, then $h(I)$ is, respectively, a proper, principal, prime or maximal fuzzy ideal. Dually, the same is true for fuzzy filters.*

Chapter 7

α-Ideals and Fuzzy α-Ideals of Fuzzy Lattices

In this chapter we will show the results of the papers Mezzomo et al. [2013a] and Mezzomo et al. [2013d] where we use the fuzzy partial order relation notion defined by Zadeh [1971], and fuzzy lattices defined by Chon [2009]. In Mezzomo et al. [2013a], we propose the notions of α-ideals and α-filters of a fuzzy lattice and characterize them by using its support and its level set. Observe that Definition 6.1.1 can be generalized in order to embrace the notions of *ideals/filters with degree of possibility greater than or equal to* α; it is enough to generalize the first and third requirements to: "If $x \in X$, $y \in Y$ and $A(y, x) > \alpha$, then $x \in Y$, for $\alpha \in (0, 1]$".

In paper Mezzomo et al. [2013d], we characterize a fuzzy ideal on operation of product between bounded fuzzy lattices $\mathcal{L}$ and $\mathcal{M}$ and define fuzzy α-ideals of fuzzy lattices. Moreover, we characterize a fuzzy α-ideal on operation of product between bounded fuzzy lattices $\mathcal{L}$ and $\mathcal{M}$ and prove that given a fuzzy α-ideal H_α of $\mathcal{L} \times \mathcal{M}$, there exist fuzzy α-ideals I_α of $\mathcal{L}$ and J_α of $\mathcal{M}$ such that $H_\alpha \subseteq I_\alpha \times J_\alpha$.

7.1 α-Ideals and α-Filters of Fuzzy Lattices

In this section, we propose the notions of α-ideals and α-filters of a fuzzy lattice and characterize them by using its support and its level set.

7.1.1 Definitions and Some Results

We define α-ideals and α-filters of a fuzzy lattice as follows:

Definition 7.1.1 *[Mezzomo et al., 2013a, Definitions 4.1 and 4.2] Let (X, A) be a fuzzy lattice, $\alpha \in (0, 1]$ and $Y \subseteq X$. Y is an α-**ideal** of (X, A),*

(i) If $x \in X$, $y \in Y$ and $A(x, y) \geq \alpha$, then $x \in Y$;

(ii) If $x, y \in Y$, then $x \vee y \in Y$.

*On the other hand, Y is an α-**filter** of (X, A),*

(iii) If $x \in X$, $y \in Y$ and $A(y, x) \geq \alpha$, then $x \in Y$;

(iv) If $x, y \in Y$, then $x \wedge y \in Y$.

Proposition 7.1.1 *[Mezzomo et al., 2013a, Proposition 4.1] If $\alpha \leq \beta$, then any α-ideal Y is a β-ideal.*

Proof: Let Y be a α-ideal and $\alpha \leq \beta$. Then for any $x \in X$, if $A(x, y) \geq \beta$, then $A(x, y) \geq \alpha$, so by Definition 7.1.1 (i), $x \in Y$. On the other hand, if $x, y \in Y$, then by Definition 7.1.1 (ii), $x \vee y \in Y$. Therefore, Y is a β-ideal of (X, A). ■

Dually, we prove that if $\alpha \leq \beta$, then any β-filter is an α-filter.

Remark 7.1.1 *Notice that the set X of a fuzzy lattice (X, A) is an α-ideal, for all $\alpha \in (0, 1]$. Dually, the set X of a fuzzy lattice (X, A) is an α-filter, for all $\alpha \in (0, 1]$.*

Corollary 7.1.1 *[Mezzomo et al., 2013a, Corollary 4.1] All ideal (filter) in the sense of Definition 5.1.1 is an α-ideal (α-filter).*

Proof: Straightforward from Proposition 7.1.1. ■

Proposition 7.1.2 *[Mezzomo et al., 2013a, Propositions 4.2 and 4.3] Let $\alpha \in (0, 1]$. If Y is an ideal (filter) of the lattice $(X, S(A))$, then for all $\alpha \in (0, 1]$, Y is an α-ideal (α-filter) of the fuzzy lattice (X, A).*

PROOF: Let Y be an ideal of $(X, S(A))$ and $y \in Y$. Consider α fixed. If $A(x, y) \geq \alpha$, then $(x, y) \in S(A)$ and so, because Y is an ideal, $x \in Y$. So, trivially satisfy the condition (i) of Definition 7.1.1 and the condition (ii) is satisfied because it does not depend on the value of α.

Analogously we prove that if Y is a filter of the lattice $(X, S(A))$, then for all $\alpha \in (0, 1]$, Y is an α-filter of fuzzy lattice (X, A). ■

Proposition 7.1.3 *[Mezzomo et al., 2013a, Proposition 4.5] Let (X, A) be a fuzzy lattice, $\alpha \in (0, 1]$ and $Y \subseteq X$. If $(Y, A|_{Y \times Y})$ is a fuzzy sup-lattice, then the set $\downarrow Y_\alpha = \{x \in X : A(x, y) \geq \alpha$ for some $y \in Y\}$ is an α-ideal of (X, A).*

PROOF:

(i) Let $\alpha \in (0, 1]$, $z \in \downarrow Y_\alpha$ and $w \in X$ such that $A(w, z) \geq \alpha$. Because $z \in \downarrow Y_\alpha$, then exists $x \in Y$ such that $A(z, x) \geq \alpha$, and by Proposition 2.2.2, $A(w, x) \geq \alpha$. Therefore, $w \in \downarrow Y_\alpha$.

(ii) Let $\alpha \in (0, 1]$. Suppose $x, y \in \downarrow Y_\alpha$, then exist $z_1, z_2 \in Y$ such that $A(x, z_1) \geq \alpha$ and $A(y, z_2) \geq \alpha$. So, $A(x, z_1 \vee z_2) \geq \alpha$ and $A(y, z_1 \vee z_2) \geq \alpha$ and by Proposition 3.1.1 (ii), $A(x \vee y, z_1 \vee z_2) \geq \alpha$. By hypothesis $(Y, A|_{Y \times Y})$ is a fuzzy sup-lattice, then $z_1 \vee z_2 \in Y$. Hence, $A(x \vee y, z) \geq \alpha$, for some $z \in Y$, and therefore, $x \vee y \in \downarrow Y_\alpha$.

■

Remark 7.1.2 *Let (X, A) be a fuzzy lattice, $\alpha \in (0, 1]$ and $Y \subseteq X$. If $(Y, A|_{Y \times Y})$ is a fuzzy sup-lattice, then $(\downarrow Y_\alpha, A|_{Y \times Y})$ is also a fuzzy sup-lattice of (X, A). Therefore, $\downarrow\downarrow Y_\alpha$ is an α-ideal of (X, A).*

Proposition 7.1.4 *[Mezzomo et al., 2013a, Proposition 4.6] Let (X, A) be a fuzzy lattice, $\alpha \in (0, 1]$ and $Y \subseteq X$. If $(Y, A|_{Y \times Y})$ is a fuzzy inf-lattice, then the set $\uparrow Y_\alpha = \{x \in X : A(y, x) \geq \alpha$ for any $y \in Y\}$ is an α-filter of (X, A).*

PROOF: Analogous to Proposition 7.1.3. ■

7.1.2 Properties of α-Ideals and α-Filters

Proposition 7.1.5 *[Mezzomo et al., 2013a, Proposition 4.7] Let (X, A) be a fuzzy lattice, $\alpha \in (0, 1]$ and $Y \subseteq X$. If $(Y, A|_{Y\times Y})$ is a fuzzy sup-lattice, then $\downarrow Y_\alpha$ satisfies the following properties:*

(i) $Y \subseteq\downarrow Y_\alpha$;

(ii) $Y \subseteq W \Rightarrow\downarrow Y_\alpha \subseteq\downarrow W_\alpha$;

(iii) $\downarrow\downarrow Y_\alpha =\downarrow Y_\alpha$.

PROOF:

(i) If $y \in Y$ then, because $A(y, y) = 1$, we have that $A(y, y) \geq \alpha$, for any $\alpha \in (0, 1]$. Therefore, $y \in\downarrow Y_\alpha$.

(ii) Let $\alpha \in (0, 1]$. Suppose $Y \subseteq W$ and $y \in\downarrow Y_\alpha$, then by definition, exists $z \in Y$ such that $A(z, y) \geq \alpha$. Because $Y \subseteq W$, then $z \in W$ and $A(z, y) \geq \alpha$. So $y \in\downarrow W_\alpha$.

(iii) $(\Rightarrow)$ Let $\alpha \in (0, 1]$. Suppose $y \in\downarrow\downarrow Y_\alpha$, then exists $x \in\downarrow Y_\alpha$ such that $A(y, x) \geq \alpha$. Since $x \in\downarrow Y_\alpha$, then exists $z \in Y$ such that $A(x, z) \geq \alpha$. So, by Proposition 2.2.2, $A(y, z) \geq \alpha$. Therefore, $y \in\downarrow Y_\alpha$. Hence, $\downarrow\downarrow Y_\alpha \subseteq\downarrow Y_\alpha$.

$(\Leftarrow)$ Straightforward from (i).

■

Proposition 7.1.6 *[Mezzomo et al., 2013a, Proposition 4.8] Let (X, A) be a fuzzy lattice, $\alpha \in (0, 1]$ and $Y \subseteq X$. If $(Y, A|_{Y\times Y})$ is a fuzzy inf-lattice, then $\uparrow Y_\alpha$ satisfies the following properties:*

(i) $Y \subseteq\uparrow Y_\alpha$;

(ii) $Y \subseteq W \Rightarrow\uparrow Y_\alpha \subseteq\uparrow W_\alpha$;

(iii) $\uparrow\uparrow Y_\alpha =\uparrow Y_\alpha$.

PROOF: Analogous to Proposition 7.1.5. ■

Proposition 7.1.7 *Let (X, A) be a fuzzy lattice, $\alpha \in (0, 1]$ and $Y \subseteq X$. If $(Y, A|_{Y\times Y})$ is a fuzzy sup(inf)-lattice, then $\downarrow Y_\alpha$ ($\uparrow Y_\alpha$) is the least α-ideal (α-filter) containing Y.*

PROOF: Suppose that there exists an α-ideal Z such that $Y \subseteq Z \subseteq \downarrow Y_\alpha$ and suppose $x \in \downarrow Y_\alpha$ and $x \notin Z$. If $x \in \downarrow Y_\alpha$, then $A(x, y) \geq \alpha$ for some $y \in Y$ and so, $y \in Z$. Thus, because Z is an α-ideal, then $x \in Z$, that is a contradiction. ■

Corollary 7.1.2 *Let (X, A) be a fuzzy lattice, $\alpha \in (0, 1]$ and $Y \subseteq X$. If $(Y, A|_{Y\times Y})$ is a fuzzy sup(inf)-lattice and if Y is an α-ideal (filter) of (X, A), then $\downarrow Y_\alpha = Y$ ($\uparrow Y_\alpha = Y$).*

The family of all α-ideals of a fuzzy lattice $\mathcal{L} = (X, A)$, for some $\alpha \in (0, 1]$, will be denoted by $Y_\alpha(\mathcal{L})$.

Proposition 7.1.8 *Let $\alpha \in (0, 1]$ and Z be a subset of $Y_\alpha(\mathcal{L})$. Then, $\bigcap Z \in Y_\alpha(\mathcal{L})$.*

PROOF: Let $\alpha \in (0, 1]$ and $Z \subseteq Y_\alpha(\mathcal{L})$.

(*i*) Suppose $x \in \bigcap Z$ and $A(y, x) \geq \alpha$, then $x \in Z_j$ for all $Z_j \in Z$. Because $A(y, x) \geq \alpha$, then $y \in Z_j$ for each $Z_j \in Z$. So $y \in \bigcap Z$.

(*ii*) Suppose $x, y \in \bigcap Z$. Then for all $Z_j \in Z$ we have that $x, y \in Z_j$. Because $Z_j \in Y_\alpha(\mathcal{L})$, then $x \vee y \in Z_j$, for all $Z_j \in Z$. So, $x \vee y \in \bigcap Z$.

Therefore, $\bigcap Z \in Y_\alpha(\mathcal{L})$. Notice that if Z is an empty set then $\bigcap Z = X$. ■

We will define a kind of α-ideals of fuzzy lattice called principal α-ideal generated by $x \in X$.

Proposition 7.1.9 *Let (X, A) be a fuzzy lattice, $\alpha \in (0, 1]$ and $Y \subseteq X$. If $(Y, A|_{Y\times Y})$ is a fuzzy sup(inf)-lattice, then the set defined by $\downarrow x_\alpha = \{y \in X : A(y, x) \geq \alpha$, for some $\alpha \in (0, 1]\}$ is an α-ideal called **principal α-ideal** of (X, A) generated by x. Dually, the set defined by $\uparrow x_\alpha = \{y \in X : A(x, y) \geq \alpha$, for some $\alpha \in (0, 1]\}$ is an α-filter called **principal α-filter** of (X, A) generated by x.*

PROOF: Straightforward, by the fact that $\{x\}$ is a sup-lattice and so, $\downarrow x_\alpha = \downarrow \{x\}_\alpha$, and by Definition 7.1.1. ■

The following propositions prove the relation between an α-ideal $\downarrow Y_\alpha$ and principal α-ideals $\downarrow y_\alpha$.

Proposition 7.1.10 *[Mezzomo et al., 2013a, Propositions 4.11 and 4.12] Let (X, A) be a fuzzy lattice, $\alpha \in (0, 1]$ and $Y \subseteq X$. If $(Y, A|_{Y\times Y})$ is a fuzzy sup(inf)-lattice, then $\downarrow Y_\alpha = \bigcup_{y\in Y} \downarrow y_\alpha$ ($\uparrow Y_\alpha = \bigcup_{y\in Y} \uparrow y_\alpha$).*

PROOF: $x \in \downarrow Y_\alpha$ iff exists $y \in Y$ such that $A(x, y) \geq \alpha$ iff exists $y \in Y$ such that $x \in \downarrow y_\alpha$ iff $x \in \bigcup_{y\in Y} \downarrow y_\alpha$.
Dually, we prove $\uparrow Y_\alpha = \bigcup_{y\in Y} \uparrow y_\alpha$. ■

7.2 Fuzzy α-Ideals of Fuzzy Lattices

In this section we will define a fuzzy α-ideals of fuzzy lattice and fuzzy α-ideals on product of bounded fuzzy lattices. In addition, we prove that α-ideals of the product are equal to the product of α-ideals on bounded fuzzy lattices.

7.2.1 Definitions and Some Results

Definition 7.2.1 *[Mezzomo et al., 2013d, Definition 3.1] Let (X, A) be bounded fuzzy lattice and $\alpha \in (0, 1]$. A fuzzy set $\mathcal{I}_\alpha$ on X is a **fuzzy α-ideal** of (X, A) if, for all $x, y \in X$,*

(i) If $\mu_{\mathcal{I}_\alpha}(y) \geq \alpha$ and $A(x, y) > 0$, then $\mu_{\mathcal{I}_\alpha}(x) \geq \alpha$;

(ii) If $\mu_{\mathcal{I}_\alpha}(x) \geq \alpha$ and $\mu_{\mathcal{I}_\alpha}(y) \geq \alpha$, then $\mu_{\mathcal{I}_\alpha}(x \vee y) \geq \alpha$.

Proposition 7.2.1 *Let (X, A) be a fuzzy lattice, $\alpha \in (0, 1]$ and $\mathcal{I}_\alpha$ be a fuzzy set on X. If (X, A) is a sup-complete fuzzy lattice, then the fuzzy set $\mu_{\Downarrow\mathcal{I}_\alpha}(x) = \sup_{y\in X}\{\mu_{\mathcal{I}_\alpha}(y) : A(x, y) > 0$ and $\mu_{\mathcal{I}_\alpha}(y) \geq \alpha\}$ is a fuzzy α-ideal of (X, A).*

PROOF: Let $x, y \in X$.

(*i*) If $\mu_{\Downarrow \mathcal{I}_\alpha}(y) \geq \alpha$ and $x \in X$ such that $A(x,y) > 0$. Then, by definition, $\mu_{\Downarrow \mathcal{I}_\alpha}(y) = \sup_{z \in X}\{\mu_{\mathcal{I}_\alpha}(z) : A(y,z) > 0 \text{ and } \mu_{\mathcal{I}_\alpha}(z) \geq \alpha\} \geq \alpha$. So, there exists $z \in X$ such that $\mu_{\mathcal{I}_\alpha}(z) \geq \alpha$ and $A(y,z) > 0$. Since $A(x,y) > 0$ and $A(y,z) > 0$, then by Proposition 2.2.1, we have that $A(x,z) > 0$. Thus, $\sup_{z \in X}\{\mu_{\mathcal{I}_\alpha}(z) : A(x,z) > 0 \text{ and } \mu_{\mathcal{I}_\alpha}(z) \geq \alpha\} \geq \alpha$. Therefore, $\mu_{\Downarrow \mathcal{I}_\alpha}(x) \geq \alpha$.

(*ii*) Suppose $\mu_{\Downarrow \mathcal{I}_\alpha}(x) \geq \alpha$ and $\mu_{\Downarrow \mathcal{I}_\alpha}(y) \geq \alpha$. By definition, $\mu_{\Downarrow \mathcal{I}_\alpha}(x) = \sup_{z \in X}\{\mu_{\mathcal{I}_\alpha}(z) : A(x,z) > 0 \text{ and } \mu_{\mathcal{I}_\alpha}(z) \geq \alpha\} \geq \alpha$ and $\mu_{\Downarrow \mathcal{I}_\alpha}(y) = \sup_{w \in X}\{\mu_{\mathcal{I}_\alpha}(w) : A(y,w) > 0 \text{ and } \mu_{\mathcal{I}_\alpha}(w) \geq \alpha\} \geq \alpha$. So, exists $z \in X$ such that $\mu_{\mathcal{I}_\alpha}(z) \geq \alpha$ and $A(x,z) > 0$. Similarly, exists $w \in X$ such that $\mu_{\mathcal{I}_\alpha}(w) \geq \alpha$ and $A(y,w) > 0$. Because (X, A) is a sup-complete fuzzy lattice, then by Definition 3.1.7, if $\mu_{\mathcal{I}_\alpha}(z) \geq \alpha$ and $\mu_{\mathcal{I}_\alpha}(w) \geq \alpha$, then $\mu_{\mathcal{I}_\alpha}(z \vee w) \geq \alpha$, and because $A(x,z) > 0$ and $A(y,w) > 0$, then $A(x, z \vee w) > 0$ and $A(y, z \vee w) > 0$. Thus, by Proposition 3.1.1 (*v*), we have that $A(x \vee y, z \vee w) > 0$ and then, $\sup_{u \in X}\{\mu_{\mathcal{I}_\alpha}(u) : A(x \vee y, u) > 0 \text{ and } \mu_{\mathcal{I}_\alpha}(u) \geq \alpha\} \geq \alpha$, for some $u \in X$. Therefore, $\mu_{\Downarrow \mathcal{I}_\alpha}(x \vee y) \geq \alpha$.

■

Proposition 7.2.2 *Let (X, A) be a fuzzy lattice, $\mathcal{I}_\alpha$ and $\mathcal{J}_\alpha$ be fuzzy sets of X, then $\Downarrow \mathcal{I}_\alpha$ satisfies the following properties:*

(i) $\mathcal{I}_\alpha \subseteq$ [1] $\Downarrow \mathcal{I}_\alpha$;

(ii) $\mathcal{I}_\alpha \subseteq \mathcal{J}_\alpha \Rightarrow \Downarrow \mathcal{I}_\alpha \subseteq \Downarrow \mathcal{J}_\alpha$;

(iii) $\Downarrow\Downarrow \mathcal{I}_\alpha = \Downarrow \mathcal{I}_\alpha$.

PROOF:

(*i*) $\mu_{\mathcal{I}_\alpha}(x) \leq \sup_{y \in X}\{\mu_{\mathcal{I}_\alpha}(y) : A(x,y) > 0 \text{ and } \mu_{\mathcal{I}_\alpha}(y) \geq \alpha\} = \mu_{\Downarrow \mathcal{I}_\alpha}(x)$.

[1] $\subseteq$ is the usual inclusion for fuzzy sets, i.e., given two fuzzy sets $\mathcal{I}_\alpha$ and $\mathcal{J}_\alpha$ on an universe X, then $\mathcal{I}_\alpha \subseteq \mathcal{J}_\alpha$ if $\mu_{\mathcal{I}_\alpha}(x) \leq \mu_{\mathcal{J}_\alpha}(x)$, for all $x \in X$.

(*ii*) If $\mathcal{I}_\alpha \subseteq \mathcal{J}_\alpha$ then for all $y \in X$, $\mu_{\mathcal{I}_\alpha}(y) \leq \mu_{\mathcal{J}_\alpha}(y)$. So, for all $x, y \in X$, $\sup_{y\in X}\{\mu_{\mathcal{I}_\alpha}(y) : A(x,y) > 0 \text{ and } \mu_{\mathcal{I}_\alpha}(y) \geq \alpha\} \leq \sup_{y\in X}\{\mu_{\mathcal{J}_\alpha}(y) : A(x,y) > 0 \text{ and } \mu_{\mathcal{J}_\alpha}(y) \geq \alpha\}$. Hence, $\mu_{\Downarrow\mathcal{I}_\alpha}(x) \leq \mu_{\Downarrow\mathcal{J}_\alpha}(x)$. Therefore, $\Downarrow \mathcal{I}_\alpha \subseteq \Downarrow \mathcal{J}_\alpha$.

(*iii*)

$$\begin{aligned}
\mu_{\Downarrow\Downarrow\mathcal{I}_\alpha}(x) &= \sup_{y\in X}\{\mu_{\Downarrow\mathcal{I}_\alpha}(y) : A(x,y) > 0 \text{ and } \mu_{\Downarrow\mathcal{I}_\alpha}(y) \geq \alpha\} \\
&= \sup_{y\in X}\{\sup_{z\in X}\{\mu_{\mathcal{I}_\alpha}(z) : A(y,z) > 0 \text{ and } \mu_{\mathcal{I}_\alpha}(z) \geq \alpha\}, A(x,y) > 0\} \\
&= \sup_{y,z\in X}\{\mu_{\mathcal{I}_\alpha}(z) : A(x,y) > 0, A(y,z) > 0 \text{ and } \mu_{\mathcal{I}_\alpha}(z) \geq \alpha\} \\
&= \sup_{z\in X}\{\mu_{\mathcal{I}_\alpha}(z) : A(x,z) > 0 \text{ and } \mu_{\mathcal{I}_\alpha}(z) \geq \alpha\} \quad \text{(By Proposition 2.2.2)} \\
&= \mu_{\Downarrow\mathcal{I}_\alpha}(x)
\end{aligned}$$

■

Proposition 7.2.3 *Let (X, A) be a sup-complete fuzzy lattice, $\alpha \in (0, 1]$ and $\mathcal{I}_\alpha$ be a fuzzy set on X. Then $\Downarrow \mathcal{I}_\alpha$ is the least fuzzy α-ideal containing $\mathcal{I}_\alpha$.*

Proof: Suppose that there exists a fuzzy α-ideal $\mathcal{J}_\alpha$ such that $\mathcal{I}_\alpha \subseteq \mathcal{J}_\alpha \subseteq \Downarrow \mathcal{I}_\alpha$ and suppose $\mu_{\Downarrow\mathcal{I}_\alpha}(x) \geq \alpha$ and $\mu_{\mathcal{J}_\alpha}(x) < \alpha$. If $\mu_{\Downarrow\mathcal{I}_\alpha}(x) \geq \alpha$, then $\sup_{y\in X}\{\mu_{\mathcal{I}_\alpha}(y) : A(x,y) > 0 \text{ and } \mu_{\mathcal{I}_\alpha}(y) \geq \alpha\} \geq \alpha$ and so, $\mu_{\mathcal{I}_\alpha}(y) \geq \alpha$, for some $y \in X$. Because $\mathcal{J}_\alpha \subseteq \Downarrow \mathcal{I}_\alpha$ and $\mu_{\mathcal{I}_\alpha}(y) \geq \alpha$, then $\mu_{\mathcal{J}_\alpha}(y) \geq \alpha$, that is a contradiction. ■

The family of all fuzzy α-ideals of a fuzzy lattice $\mathcal{L} = (X, A)$, for some $\alpha \in (0, 1]$, will be denoted by $\mathcal{I}_\alpha(\mathcal{L})$.

Proposition 7.2.4 *Let $\alpha \in (0, 1]$ and J be a subset of $\mathcal{I}_\alpha(\mathcal{L})$. Then, $\bigcap J \in \mathcal{I}_\alpha(\mathcal{L})$.*

Proof: Let $\alpha \in (0, 1]$ and $J \subseteq \mathcal{I}_\alpha(\mathcal{L})$.

(*i*) Suppose $\mu_{\bigcap J}(x) \geq \alpha$ and $A(y, x) > 0$, then $\mu_{J_i}(x) \geq \alpha$ for all $J_i \in J$. Because $A(y, x) > 0$, then $\mu_{J_i}(y) \geq \alpha$ for each $J_i \in J$. So $\mu_{\bigcap J}(y) \geq \alpha$.

(*ii*) Suppose $\mu_{\bigcap J}(x) \geq \alpha$ and $\mu_{\bigcap J}(y) \geq \alpha$. Then for all $J_i \in J$ we have that $\mu_{J_i}(x) \geq \alpha$ and $\mu_{J_i}(y) \geq \alpha$. Because $J_i \in \mathcal{I}_\alpha(\mathcal{L})$, then $\mu_{J_i}(x \vee y) \geq \alpha$, for all $J_i \in J$. So, $\mu_{\bigcap Z}(x \vee y) \geq \alpha$.

Therefore, $\bigcap J \in \mathcal{I}_\alpha(\mathcal{L})$. ■

7.2.2 Fuzzy α-Ideals and Product Operator

Similarly from Proposition 6.1.3, we define a fuzzy α-ideal $\mathcal{I}_\alpha \times \mathcal{J}_\alpha$ of $\mathcal{L} \times \mathcal{M}$ as:

Theorem 7.2.1 *[Mezzomo et al., 2013d, Theorem 4.1] Let $\mathcal{L} = (X, A)$ and $\mathcal{M} = (Y, B)$ be bounded fuzzy lattices, $\mathcal{I}_\alpha$ and $\mathcal{J}_\alpha$ be fuzzy α-ideals of $\mathcal{L}$ and $\mathcal{M}$, respectively. The fuzzy set*

$$\mu_{\mathcal{I}_\alpha \times \mathcal{J}_\alpha}(x, y) = \min\{\mu_{\mathcal{I}_\alpha}(x), \mu_{\mathcal{J}_\alpha}(y)\}$$

is a fuzzy α-ideal of $\mathcal{L} \times \mathcal{M}$, denoted by $\mathcal{I}_\alpha \times \mathcal{J}_\alpha$.

Proof: According to the definition of product operator of bounded fuzzy lattices and Definition 7.2.1, we need to prove that, for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$:

(*i*) If $\mu_{\mathcal{I}_\alpha \times \mathcal{J}_\alpha}(x_2, y_2) \geq \alpha$ and $C((x_1, y_1), (x_2, y_2)) > 0$, then $\mu_{\mathcal{I}_\alpha \times \mathcal{J}_\alpha}(x_1, y_1) \geq \alpha$;

(*ii*) If $\mu_{\mathcal{I}_\alpha \times \mathcal{J}_\alpha}(x_1, y_1) \geq \alpha$ and $\mu_{\mathcal{I}_\alpha \times \mathcal{J}_\alpha}(x_2, y_2) \geq \alpha$, then $\mu_{\mathcal{I}_\alpha \times \mathcal{J}_\alpha}((x_1, y_1) \vee_\times (x_2, y_2)) \geq \alpha$.

Let $\mathcal{I}_\alpha$ and $\mathcal{J}_\alpha$ be fuzzy α-ideals of $\mathcal{L}$ and $\mathcal{M}$, respectively.

(*i*) Since $\mu_{\mathcal{I}_\alpha \times \mathcal{J}_\alpha}(x_2, y_2) \geq \alpha$ and $C((x_1, y_1), (x_2, y_2)) > 0$, then $\min\{\mu_{\mathcal{I}_\alpha}(x_2), \mu_{\mathcal{J}_\alpha}(y_2)\} \geq \alpha$ and $\min\{A(x_1, x_2), B(y_1, y_2)\} > 0$. So, $\mu_{\mathcal{I}_\alpha}(x_2) \geq \alpha$, $\mu_{\mathcal{J}_\alpha}(y_2) \geq \alpha$, $A(x_1, x_2) > 0$ and $B(y_1, y_2) > 0$. Hence, because $\mathcal{I}_\alpha$ and $\mathcal{J}_\alpha$ are fuzzy α-ideals, then $\mu_{\mathcal{I}_\alpha}(x_1) \geq \alpha$ and $\mu_{\mathcal{J}_\alpha}(y_1) \geq \alpha$. Therefore, $\mu_{\mathcal{I}_\alpha \times \mathcal{J}_\alpha}(x_1, y_1) = \min\{\mu_{\mathcal{I}_\alpha}(x_1), \mu_{\mathcal{J}_\alpha}(y_1)\} \geq \alpha$.

(*ii*) Let $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that $\mu_{\mathcal{I}_\alpha \times \mathcal{J}_\alpha}(x_1, y_1) \geq \alpha$ and $\mu_{\mathcal{I}_\alpha \times \mathcal{J}_\alpha}(x_2, y_2) \geq \alpha$. Then $\min\{\mu_{\mathcal{I}_\alpha}(x_1), \mu_{\mathcal{J}_\alpha}(y_1)\} \geq \alpha$ and $\min\{\mu_{\mathcal{I}_\alpha}(x_2), \mu_{\mathcal{J}_\alpha}(y_2)\} \geq \alpha$. So, $\mu_{\mathcal{I}_\alpha}(x_1) \geq \alpha$, $\mu_{\mathcal{J}_\alpha}(y_1) \geq \alpha$, $\mu_{\mathcal{I}_\alpha}(x_2) \geq \alpha$ and $\mu_{\mathcal{J}_\alpha}(y_2) \geq \alpha$. Thus, because $\mathcal{I}_\alpha$

and $\mathcal{J}_\alpha$ are fuzzy α-ideals, $\mu_{\mathcal{I}_\alpha}(x_1 \vee_{\mathcal{L}} x_2) \geq \alpha$ and $\mu_{\mathcal{J}_\alpha}(y_1 \vee_{\mathcal{M}} y_2) \geq \alpha$. Therefore, $\min(\mu_{\mathcal{I}_\alpha}(x_1 \vee_{\mathcal{L}} x_2), \mu_{\mathcal{J}_\alpha}(y_1 \vee_{\mathcal{M}} y_2)) \geq \alpha$, i.e., $\mu_{\mathcal{I}_\alpha \times \mathcal{J}_\alpha}(x_1 \vee_{\mathcal{L}} x_2, y_1 \vee_{\mathcal{M}} y_2) \geq \alpha$. Hence, by Lemma 4.1.1, we have that $\mu_{\mathcal{I}_\alpha \times \mathcal{J}_\alpha}((x_1, y_1) \vee_\times (x_2, y_2)) \geq \alpha$.

Therefore, the fuzzy function $\mu_{\mathcal{I}_\alpha \times \mathcal{J}_\alpha}(x, y)$ is a fuzzy α-ideal of $\mathcal{L} \times \mathcal{M}$. ■

Theorem 7.2.2 *[Mezzomo et al., 2013d, Theorem 4.2] Let $\mathcal{L} = (X, A)$ and $\mathcal{M} = (Y, B)$ be bounded fuzzy lattices and H_α be a fuzzy α-ideal of $\mathcal{L} \times \mathcal{M}$. Then, the fuzzy set $I_{H_\alpha} \subseteq \mathcal{L}$ defined by $\mu_{I_{H_\alpha}}(x) = \sup\{\mu_{H_\alpha}(x, y) : y \in Y\}$ is a fuzzy α-ideal. Similarly, the fuzzy set $J_{H_\alpha} \subseteq \mathcal{M}$ defined by $\mu_{J_{H_\alpha}}(y) = \sup\{\mu_{H_\alpha}(x, y) : x \in X\}$ is a fuzzy α-ideal.*

Proof: Let H_α be a fuzzy α-ideal of $\mathcal{L} \times \mathcal{M}$ and $x, z \in X$.

(i) If $\mu_{I_{H_\alpha}}(z) \geq \alpha$, then $\mu_{I_{H_\alpha}}(z) = \sup\{\mu_{H_\alpha}(z, y) : y \in Y\} \geq \alpha$, and so, $\mu_{H_\alpha}(z, \sup_{y \in Y} y) \geq \alpha$. So, there exists $y_0 \in Y$ such that $\mu_{H_\alpha}(z, y_0) \geq \alpha$. Because $A(x, z) > 0$, then $C((x, y_0), (z, y_0)) > 0$. Because H_α is a fuzzy α-ideal, $\mu_{H_\alpha}(x, y_0) \geq \alpha$. Because $\mu_{H_\alpha}(x, y_0) \in \{\mu_{H_\alpha}(x, y) : y \in Y\}$, we have that

$$\begin{aligned} \mu_{I_{H_\alpha}}(x) &= \sup\{\mu_{H_\alpha}(x, y) : y \in Y\} \\ &\geq \mu_{H_\alpha}(x, y_0) \\ &\geq \alpha. \end{aligned}$$

(ii) If $\mu_{I_{H_\alpha}}(x) \geq \alpha$ and $\mu_{I_{H_\alpha}}(z) \geq \alpha$, then $\sup\{\mu_{H_\alpha}(x, y) : y \in Y\} \geq \alpha$ and $\sup\{\mu_{H_\alpha}(z, y) : y \in Y\} \geq \alpha$. So, $\mu_{H_\alpha}(x, \sup_{y \in Y} y) \geq \alpha$ and $\mu_{H_\alpha}(z, \sup_{y \in Y} y) \geq \alpha$. Thus, there exist $y_0, y_1 \in Y$ such that $\mu_{H_\alpha}(x, y_0) \geq \alpha$ and $\mu_{H_\alpha}(z, y_1) \geq \alpha$. Because H_α is a fuzzy α-ideal, then $\mu_{H_\alpha}((x, y_0) \vee_{\mathcal{L} \times \mathcal{M}} (z, y_1)) \geq \alpha$ and so, $\mu_{H_\alpha}(x \vee_{\mathcal{L}} z, y_0 \vee_{\mathcal{M}} y_1) \geq \alpha$. Since, $\mu_{H_\alpha}(x \vee_{\mathcal{L}} z, y_0 \vee_{\mathcal{M}} y_1) \in \{\mu_{H_\alpha}(x \vee_{\mathcal{L}} z, y) : y \in Y\}$ then, we have that

$$\begin{aligned} \mu_{I_{H_\alpha}}(x \vee_{\mathcal{L}} z) &= \sup\{\mu_{H_\alpha}(x \vee_{\mathcal{L}} z, y) : y \in Y\} \\ &\geq \mu_{H_\alpha}(x \vee_{\mathcal{L}} z, y_0 \vee_{\mathcal{M}} y_1) \\ &\geq \alpha. \end{aligned}$$

Therefore, we have that the fuzzy set I_{H_α} is a fuzzy α-ideal of $\mathcal{L}$. Similarly, we prove that J_{H_α} is a fuzzy α-ideal of $\mathcal{M}$. ■

Proposition 7.2.5 *[Mezzomo et al., 2013d, Proposition 4.1] Let $\mathcal{L} = (X, A)$ and $\mathcal{M} = (Y, B)$ be bounded fuzzy lattices. Given a fuzzy α-ideal $H_\alpha \subseteq \mathcal{L} \times \mathcal{M}$, there exist fuzzy α-ideals $I_{H_\alpha} \subseteq \mathcal{L}$ and $J_{H_\alpha} \subseteq \mathcal{M}$ such that $H_\alpha \subseteq I_{H_\alpha} \times J_{H_\alpha}$.*

Proof: Let H_α be a fuzzy α-ideal of $\mathcal{L} \times \mathcal{M}$. Then, H_α is a fuzzy subset of $\mathcal{L} \times \mathcal{M}$ that satisfies the conditions from Definition 7.2.1. By Theorem 7.2.2, the fuzzy sets $I_{H_\alpha} \subseteq \mathcal{L}$ and $J_{H_\alpha} \subseteq \mathcal{M}$ defined by $\mu_{I_{H_\alpha}}(x) = \sup\{\mu_{H_\alpha}(x, y) : y \in Y\}$ and $\mu_{J_{H_\alpha}}(y) = \sup\{\mu_{H_\alpha}(x, y) : x \in X\}$, respectively, are fuzzy α-ideals. By Theorem 7.2.1, we have that $I_{H_\alpha} \times J_{H_\alpha}$ is a fuzzy α-ideal of $\mathcal{L} \times \mathcal{M}$. Since $\mu_{H_\alpha}(x, y) \in \{\min\{\mu_{H_\alpha}(x, y'), \mu_{H_\alpha}(x', y)\} : x' \in X \text{ and } y' \in Y\}$, then

$$\begin{aligned}
\mu_{I_{H_\alpha} \times J_{H_\alpha}}(x, y) &= \min\{\mu_{I_{H_\alpha}}(x), \mu_{J_{H_\alpha}}(y)\} \\
&= \min\{\sup\{\mu_{H_\alpha}(x, y') : y' \in Y\}, \sup\{\mu_{H_\alpha}(x', y) : x' \in X\}\} \\
&= \sup\{\min\{\mu_{H_\alpha}(x, y'), \mu_{H_\alpha}(x', y)\} : x' \in X \text{ and } y' \in Y\} \\
&\geq \mu_{H_\alpha}(x, y)
\end{aligned}$$

Therefore, we have that $H_\alpha \subseteq I_{H_\alpha} \times J_{H_\alpha}$. ■

Proposition 7.2.6 *[Mezzomo et al., 2013d, Proposition 4.2] Let $\mathcal{L} = (X, A)$ and $\mathcal{M} = (Y, B)$ be bounded fuzzy lattices. Given the fuzzy α-ideals $\mathcal{I}_\alpha \subseteq \mathcal{L}$ and $\mathcal{J}_\alpha \subseteq \mathcal{M}$, there exists a fuzzy α-ideal $H_\alpha \subseteq \mathcal{L} \times \mathcal{M}$ such that $\mathcal{I}_\alpha \times \mathcal{J}_\alpha = I_{H_\alpha} \times J_{H_\alpha}$.*

Proof: Analogously from Proposition 7.2.5. ■

Chapter 8

Remarks and Further Works

In this chapter we make a brief discussion of the results achieved in this thesis and the prospects for future work.

As we have seen, the concept of a fuzzy set was first introduced by Zadeh [1965] and, in Zadeh [1971], fuzzy ordering was defined as a generalization of the concept of ordering, that is, a fuzzy ordering is a fuzzy relation that is transitive. In particular, a fuzzy partial ordering is a fuzzy ordering that is reflexive and antisymmetric. Thenceforward, several different notions of fuzzy order relations have been given, for example Belohlávek [2004]; Bodenhofer and Kung [2004]; Fodor and Roubens [1994]; Gerla [2004]; Yao and Lu [2009] and the Zadeh's notion in Zadeh [1971] has been widely considered in recent years as we can find in Amroune and Davvaz [2011]; Beg [2012]; Chon [2009]; Mezzomo et al. [2012b]; Seselja and Tepavcevic [2010]. In the same way, one should observe that the concept of fuzzy partial order, fuzzy partially ordered set, fuzzy lattice and fuzzy ideal can be found in several other forms in the literature. In 2009, Chon [2009] considering the notion of fuzzy order of Zadeh [1971], introduced a new notion of fuzzy lattice and studied the level sets of fuzzy lattices. He also introduced the notions of distributive and modular fuzzy lattices and considered some basic properties of fuzzy lattices. Our choice for using the fuzzy lattice defined by Chon [2009] is because their notion of fuzzy lattice is very similar to the usual notion of lattice as a partial order, i.e., a fuzzy relation is a fuzzy partial order relation if it is reflexive, antisymmetric and transitive. The fuzzy reflexivity, antisymmetry and transitivity notion used by Chon, was first defined by Zadeh [1971].

Extending these previous studies, we began our work investigating the properties and characteristics of the fuzzy lattice defined by Chon [2009]. Furthermore, as expected, we expected that the fuzzy lattice could be able to preserve the main properties that the classical lattice. Thenceforth, we defined some types of this fuzzy lattice and prove some properties besides those already described by Chon.

As the main contributions of this work, a new notion of ideals (filters) are proposing, including fuzzy ideals (fuzzy filters) and α-ideals (α-filters) of fuzzy lattices defined by Chon [2009] and developing a new theory involving some properties analogous the classical theory of ideals (filters). In addition, it aims at investigating the behavior of the operations on bounded fuzzy lattices. They are conceived as extensions of their analogous operations on the classical theory, by using the fuzzy partial order relation and the definition of fuzzy lattices.

In paper Mezzomo et al. [2012a], we defined ideals and filters of fuzzy lattice (X, A) as a classical set $Y \subseteq X$. In both definitions, it was proposed a discussion of some kinds of ideals and filters. The intersection of families for each class of such ideals and filters together with preserved properties were also studied. However, some results showed that not all properties of ideals of classical lattices are preserved by this ideal as, for example, the union of ideals of fuzzy lattices can not be an ideal.

After this early studies, emerged the necessity to define fuzzy ideal of (X, A), which was proposed in Mezzomo et al. [2013c], as a fuzzy set on X and we rely on a less restrictive form, that is, a fuzzy ideal is a fuzzy set on fuzzy lattices (X, A). In this work, we define some types of fuzzy ideals and fuzzy filters of fuzzy lattice and we prove some properties analogous to the classical theory of ideals (filters), such as, the class of proper fuzzy ideals (filters) is closed under fuzzy union and fuzzy intersection. We also prove that if a bounded fuzzy lattice admits a maximal fuzzy ideal, then it is prime. Here, as it occurs in paper Mezzomo et al. [2012a] and as expected, some properties of fuzzy ideals of fuzzy lattices could not be able to preserve the properties of ideals of classical lattices. In addition, we define a fuzzy homomorphism h from fuzzy lattices $\mathcal{L}$ and $\mathcal{M}$ and prove some results involving fuzzy homomorphism and fuzzy ideals as if h is a fuzzy monomorphism and the fuzzy image of a fuzzy set $\tilde{h}(I)$ is a fuzzy ideal, then I is a fuzzy ideal. Similarly, we prove for proper, prime and maximal fuzzy ideals. Finally, we prove

that h is a fuzzy homomorphism from fuzzy lattices $\mathcal{L}$ into $\mathcal{M}$ if the inverse image of all principal fuzzy ideals of $\mathcal{M}$ is a fuzzy ideal of $\mathcal{L}$.

In sequence, the operations of product and collapsed sum on bounded fuzzy lattice were defined in Mezzomo et al. [2013b] as an extension of the classical theory. Furthermore, the product and collapsed sum on bounded fuzzy lattices were stated as fuzzy posets, and, consequently, as bounded fuzzy lattices. Extending these previous studies about operators on fuzzy lattices, in work Mezzomo et al. [2013e] we focus on the lifting, opposite, interval and intuitionist operations on bounded fuzzy lattices. They are conceived as extensions of their analogous operations on the classical theory, by using the fuzzy partial order relation and the definition of fuzzy lattices, as conceived by Chon. In addition, we prove that lifting, opposite, interval and intuitionist on (complete) bounded fuzzy lattices are (complete) bounded fuzzy lattices introducing new results from both operators, product and collapsed sum, which had already been defined in our previous paper Mezzomo et al. [2013b].

Thenceforward, in paper Mezzomo et al. [2013a], it emerged the necessity to define α-ideals and α-filters of fuzzy lattices, characterize an α-ideal of fuzzy lattice, using its support and its level set, and study some properties analogous to the classical theory of α-ideals and α-filters, such as, the class of α-ideals and α-filters are closed under union and intersection.

Lastly, in paper Mezzomo et al. [2013d], we define fuzzy α-ideals of fuzzy lattices defined by Chon. Moreover, we characterize a fuzzy α-ideal on operation of product between bounded fuzzy lattices $\mathcal{L}$ and $\mathcal{M}$ and prove that given a fuzzy α-ideal H_α of $\mathcal{L} \times \mathcal{M}$, there exist fuzzy α-ideals I_α of $\mathcal{L}$ and J_α of $\mathcal{M}$ such that $H_\alpha \subseteq I_\alpha \times J_\alpha$.

The results obtained in paper Mezzomo et al. [2013d] are preliminaries. There are several others results involving fuzzy α-ideals that can be explored, including the study of which properties related to the classical theory of ideals are applied to fuzzy α-ideals. We can define some types of fuzzy α-ideals of fuzzy lattices and prove some properties, such as, the class of proper fuzzy α-ideals are closed under fuzzy intersection. Also, define fuzzy α-ideals on operators of collapsed sum, lifting, opposite, interval and intuitionistic of bounded fuzzy lattices and study theirs consequences.

Due to lack of studies on fuzzy lattices and fuzzy ideals in the literature and considering the new notions of the kinds of fuzzy ideals introduced in this work, we believe that we have great potential for producing new scientific articles in this area of knowledge. One of the most promising ideas could be the investigation of fuzzy ideals for another operations among fuzzy lattices and its consequences. We want to deepen the studies about fuzzy α-ideals of fuzzy lattices and study their properties and consequences. As future work, we consider the idea of Palmeira and Bedregal [2012]; Palmeira et al. [2013a,b] to extend fuzzy ideals and fuzzy filters from a fuzzy lattice to a sup-lattice. Thus, for further research, we hope to think of building bounded interval fuzzy lattice, using the idea of Bedregal and Santos [2006], from bounded fuzzy lattices. Finally, we intend to conduct this study for other types of fuzzy lattices as Boolean fuzzy lattices.

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