A NEW CLASS OF FUZZY SUBSETHOOD MEASURES

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December, 2016
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For my beloved son, Theo.
The little one who truly inspired me
and keeps doing it every single day.
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“Eventually, what is likely is that in science - as in fuzzy logic - almost everything will be, or will be allowed to be, a matter of degree. This is what I see in my crystal ball.”

L.A. Zadeh
ABSTRACT

The idea of inclusion for fuzzy sets was firstly introduced by L. Zadeh in 1965 and since then many other studies proposed alternatives to indicate a degree to which a fuzzy set is included into another fuzzy set, called an inclusion degree or a subsethood measure. In this work we present a new class of fuzzy subsethood measures between fuzzy sets. We introduce a new definition of a fuzzy subsethood measure as an intersection of other axiomatizations by aggregating fuzzy implication operators. We also provide some construction methods to obtain these fuzzy subsethood measures. With our approach we recover some of the classical measures which have been discussed in the literature, as the one given by Goguen. We also show how we can use our developments to generate fuzzy entropies, fuzzy distances, penalty functions and similarity measures. Finally we study some fuzzy indexes generated from this new class of fuzzy subsethood measures.

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A ideia de inclusão para conjuntos difusos foi primeiramente introduzida por L. Zadeh em 1965 e desde então, outros estudos surgiram apresentando alternativas para indicar o quanto um conjunto difuso está incluído em outro, chamado de grau ou medida de inclusão. Neste trabalho, apresentamos uma nova classe de medida de inclusão entre conjuntos difusos. Introduzimos uma nova definição de uma medida de inclusão como uma intersecção de outras axiomatizações através da agregação de operadores de implicação. Também propusemos alguns métodos de construção para obter essas medidas de inclusão. Através da nossa proposta conseguimos recuperar uma medida de inclusão clássica da literatura, que é a medida dada por Goguen. Usamos nossas medidas também para gerar entropia difusa, distância difusa, funções pénalti e medidas de similaridade. Finalmente, estudamos índices difusos gerados a partir desta nova classe de medidas de inclusão.

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Palavras-chave: Medidas de inclusão difusas; Operador de implicação; Entropia difusa; Funções pénalti; Índices difusos.
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LIST OF ACRONYMS

DI  Decreasing-Increasing

FSM  Fuzzy subsethood measures

SD  Sinha and Dougherty

SM  Similarity measure

SUSAN  Smallest Univalue Segment Assimilating Nucleus
LIST OF SYMBOLS

$M$ aggregation function

$\sim_\alpha$ approximately equal with grade $\alpha$

$\varphi$ automorphism

$\in$ belongs to

$\wedge$ conjunction

$\vee$ disjunction

$D$ distance measure

$EQ_{DP}$ Dubois and Prade’s equality index

$E$ entropy

$L_\alpha$ equality index $L$

$REC_\alpha$ equality index $REC$

$S_\alpha$ equality index $S$

$\sigma_s$ Fan’s fuzzy subsethood measure

$\sigma_{DI}$ fuzzy DI-subsethood measure

$N$ fuzzy negation

$\sigma_G$ Goguen’s subsethood grade

$\subseteq$ inclusion relation

$\cap$ intersection

$\sigma_K$ Kitainik’s fuzzy subsethood measure
$EQ_{KY}$  Klir and Yuan’s equality index

$\sigma_{Kk}$  Kosko’s fuzzy subsethood measure

$RIC_{\sigma}$  Kosko’s index

$ROC_{\sigma}$  overlap index

$x_i$  real variables in the unit interval

$SE_{\sigma}$  semi-equality index

$\mathbb{R}$  set of real numbers

$SM$  similarity measure

$\sigma_{SD}$  Sinha and Dougherty’s fuzzy subsethood measure

$\leq$  smaller than

$\subseteq_S$  strong inclusion

$\sigma$  subsethood measure

$[0,1]$  unit interval

$\cup$  union

$\sigma_Y$  Young’s fuzzy subsethood measure

$\sigma_Z$  Zhang’s subsethood measure
Chapter 1

Introduction

The idea of measuring up to what extent a given fuzzy set is included into another was firstly given by Zadeh in [84] and has led to various axiomatizations over the years. In general, these axiomatizations propose to indicate the degree of which a fuzzy set A is included in a fuzzy set B, called an inclusion degree or a subsethood measure.

Fuzzy subsethood measures, FSM for short, between fuzzy sets have been used in different applications, for example: in mathematical morphology [57 74], in clustering [35 83], in fuzzy relational databases [56], in intelligent systems [65], in fuzzy decision making [52], in image processing [11 21], in formal concept lattice analysis [36].

Such measures arise from the partial order relation given by Zadeh [84]:

“Given \( A, B \in F(X) \) with \( X = \{x_1, \ldots, x_n\} \): \( A \subseteq B \) if and only if \( A(x) \leq B(x) \) for all \( x \in X \), where \( F(X) \) is the set of all fuzzy subsets on a universe of discourse \( X \).”

Since its introduction, this definition has been widely criticized due to its crisp nature in the following sense:

Let us consider the referential set (universe of discourse) \( X = \{0, 0.01, 0.02, \ldots, 0.99, 1\} \).
If we define the fuzzy sets $A$ and $B$ given by the membership functions:

$$A(x) = 0.9 \text{ for every } x \in [0, 1]\text{ and } B(x) = \begin{cases} 0.89 & \text{if } x = 0.5, \\ 0.91 & \text{otherwise.} \end{cases}$$

we have that Zadeh’s definition of subset relation just allows us to say that neither $A$ is included in $B$ nor $B$ is included in $A$.

Taking into account this fact, in 1980 Bandler and Kohout [4] propose the following expression to measure the subsethood grade of a set $A$ in a set $B$:

$$\sigma(A, B) = \inf_{x \in X} J(A(x), B(x)),$$

where $J : [0, 1]^2 \to [0, 1]$ is such that $J(0, 0) = J(0, 1) = J(1, 1) = 1$ and $J(1, 0) = 0$.

Bandler and Kohout’s proposal has led many authors to consider functions $\sigma : F(X) \times F(X) \to [0, 1]$, such that $\sigma(A, B)$ quantifies up to what extent a set $A$ is included in a set $B$. The conditions (axioms) which are requested to $\sigma$ depend on the application which is going to be considered.

In the literature, one can find many studies providing axiomatizations for those $\sigma$ functions. Historically, the ones which seemed relevant to our proposal were given by: Kitainik [47], Sinha and Dougherty [72], Young [83], Fan [35], Bustince [19] and Zhang [86].

Therefore, from the analysis of these axiomatizations, we can come up with the following ideas:

i) The axiom: $\sigma(A, B) = 1$ if and only if $A \leq B$ is demanded by almost all of them, except Kitainik’s and one particular case of Young.

ii) There is a great variety on the conditions that must be demanded so that $\sigma(A, B) = 0$.

iii) Some of the requested axioms are justified by the possibility of building fuzzy entropies from these measures.
1. Introduction

1.1 Motivation and goals

Our main motivation and the objective of this work is to propose a new class of FSM by aggregating fuzzy implication operators, in such a way that we can:

1. Build subsethood grades aggregating implication operators \([6, 67]\); that is, for every \(A, B \in F(X)\)

\[
\sigma(A, B) = M(I(A(x_1), B(x_1)), \ldots, I(A(x_n), B(x_n))) \tag{1.1}
\]

where \(M : [0, 1]^n \to [0, 1] (n \geq 2)\) is an aggregation function \([5, 20, 43]\) and \(I : [0, 1]^2 \to [0, 1]\) is an implication operator \([2, 16, 39, 64]\).

2. Recover Goguen’s subsethood grade (degree) \(\sigma_G [41]\), which is useful in different applications \([21]\):

\[
\sigma_G(A, B) = \frac{1}{n} \sum_{i=1}^{n} 1 \wedge (1 - A(x_i) + B(x_i)) \tag{1.2}
\]

3. Build similarity measures and distances between fuzzy sets from our axiomatization, and hence, also build fuzzy entropies.

4. Generate penalty functions from two fuzzy sets, one of them constant.

Our reasons to consider the construction proposed in Eq. (1.1) are:

i) We can use aggregation functions different from \(\inf\). It is important to mention that Bandler and Kohout consider that the choice of \(\inf\) is a very harsh criterion.

ii) We use functions \(I\) with properties very similar to those of implication operators \([39]\).

This is very important since the properties of such operators, as for instance,

\[
I(x, y) = 0 \text{ if and only if } x = 1 \text{ and } y = 0 \text{ (see } [17]) \tag{1.3}
\]
have been widely studied [2, 16, 67] and we can take advantage of these studies to build FSM.

iii) If we take as aggregation the arithmetic mean and as implication operator Lukasiewicz’s, which satisfies Eq. (1.3), we get Goguen’s subsethood degree Eq. (1.2). Note that 

\[ \sigma_G(A, B) = 0 \text{ if and only if } A = 1 \text{ and } B = 0. \]

iv) We can construct penalty functions [10, 22].

In summary, our main goal is to provide a new class of FSM. This means that, with our proposal one can: build subsethood grades by aggregating implication operators; recover Goguen’s subsethood grade, widely used in different applications; build similarity measures and distances between fuzzy sets from our axiomatization, and hence, also build fuzzy entropies; and finally generate penalty functions from two fuzzy sets, which can be used in applications like multi-criteria decision making.

We acknowledge that our proposal of fuzzy subsethood measure turned out to be more restrictive than the ones found in the literature (as it will be detailed in the fourth chapter), however, we managed to provide a simple axiomatization, demanding a reduced number of axioms. Furthermore, since we recover Goguen’s subsethood grade, our developments are useful in every application where Goguen’s subsethood grade can be applied. Besides, the fact that we can use different aggregation functions and implication operators gives us more flexibility in order to build, for instance, comparison measures that can be applied in image processing [21].

1.2 Thesis structure

This document is divided into seven chapters.

Chapter 1. Introduction. We start with the introductory chapter discussing briefly our main motivation and the goals to be achieved.
Chapter 2. Background. In the second chapter we recall some definitions and concepts used throughout the text in order to provide a self contained document.

Chapter 3. Fuzzy subsethood measures. The third chapter contains different axiomatizations of FSM found in the literature. We recalled the ones which seemed relevant and inspired our work.

Chapter 4. A new class of fuzzy subsethood measures. This chapter introduces a new axiomatization for FSM and the reasons which have led us to consider each of the proposed axioms. Then we study different methods to build our new class of FSM analyzing the properties that those FSM may have depending on the aggregation functions considered. We also generate penalty functions from our new class of FSM.

Chapter 5. Constructing measures from the new class of FSM. In Chapter 5 we study the concepts of distance and similarity measures constructed from FSM defined according to the previous chapter. We also build fuzzy entropy from those FSM and present an example in image segmentation, where our developments can be used as a reference to choose between thresholding methods.

Chapter 6. A study on indexes generated from FSM. In Chapter 6 we present a study on some indexes constructed from our new class of FSM and provide some relations between those indexes and measures investigated so far.

Chapter 7. Concluding remarks. The last chapter includes some conclusions, future works and the bibliography.
Chapter 2

Background

The idea of doing a self contained document requires a chapter explaining some of the basic concepts used throughout the work. Advanced readers may prefer to skip to the following chapter.

2.1 Fuzzy sets

We start recalling some notions and definitions that will be necessary for our subsequent developments. We are only going to deal with a finite universe of discourse (referential) set $X = \{x_1, \ldots, x_n\}$. We will also denote real variables in the unit interval by the letter $x$, but once the elements in set $X$ are only written as variables for the membership functions of the considered fuzzy sets, there is no possible misunderstanding.

We denote by 0 the fuzzy set where the membership of every element is equal to zero (i.e. the empty set) and by 1 the fuzzy set where the membership of every element is equal to 1 (i.e. set $X$).

A fuzzy set $A$ over a fuzzy set $X$ is defined by means of its membership function $A : X \to [0, 1]$. We denote by $F(X)$ the class of all fuzzy sets over the universe of discourse $X$. 
2. Background

Given two fuzzy sets \( A, B \in F(X) \), we say that \( A \) is included in \( B \), and we write \( A \leq B \), if the inequality: \( A(x) \leq B(x) \) holds for every \( x \in X \). Note that \( \leq \) defines a partial order in \( F(X) \) which extends the linear order between real numbers.

A stronger notion of inclusion given by Dubois and Prade in [30] considers that all elements belonging to \( A \) must be prototypes of \( B \), that is, \( A \subseteq_S B \) if and only if \( A(x) = 0 \) or \( B(x) = 1 \) for all \( x \in X \). Note this idea of strong inclusion \( \subseteq_S \) leads to consider that genuinely fuzzy sets can never be considered as perfectly equal, since \( A \subseteq_S B \) and \( B \subseteq_S A \) if and only if \( A(x) = B(x) \in \{0, 1\} \) for all \( x \in X \).

\( A \) is said to be weakly included in \( B \) if and only if for all \( x \), either \( A(x) \leq 0.5 \) or \( B(x) > 0.5 \) [30, 72].

The support of a fuzzy set \( A \) defined over a reference set \( X \) and denoted by \( \text{supp}(A) \) is a crisp subset of \( X \) whose elements all have non-zero membership grades in \( A \), i.e., \( \text{supp}(A) = \{x \in X, A(x) > 0\} \). Note that if \( A \) and \( B \) have disjoint support then we have \( \text{supp}(A) \cap \text{supp}(B) = \emptyset \). If \( A = 0 \) and \( B = 0 \) do not happen at the same time, we have \( \text{supp}(A) \cup \text{supp}(B) = X \).

Given a value \( e \in ]0, 1[ \), we say \( A \) refines \( B \) with respect to \( e \) (or \( A \) is a refinement of \( B \)) if \( A(x) \leq B(x) \) when \( B(x) \leq e \) and \( A(x) \geq B(x) \) when \( B(x) \geq e \).

2.2 Fuzzy negations

**Definition 2.1** A fuzzy negation is a decreasing function \( N : [0, 1] \rightarrow [0, 1] \) such that \( N(0) = 1 \) and \( N(1) = 0 \). A negation \( N \) is strong if \( N(N(x)) = x \) for all \( x \in [0, 1] \).

Whenever we mention the term negation, we mean a strong negation, unless otherwise stated. Observe that every strong negation is continuous, and a continuous negation has just one single equilibrium point (i.e., a point \( e \in [0, 1] \) such that \( N(e) = e \)).
Let us take $A \in F(X)$, then the **complement** of $A$ with respect to the negation $N$ is the fuzzy set $A_N$ defined by $A_N(x) = N(A(x))$, for every $x \in X$.

Strong negations can be obtained in terms of automorphisms by means of Trillas characterization theorem [75]. Recall that an automorphism in a real interval $[a, b] \subset \mathbb{R}$ is a continuous, strictly increasing function $\varphi : [a, b] \rightarrow [a, b]$ such that $\varphi(a) = a$, $\varphi(b) = b$. In this work, we only consider automorphisms in the unit interval $[0, 1]$. Using this notion, Trillas’ theorem [75] can be stated as follows.

**Theorem 2.1** A function $N : [0, 1] \rightarrow [0, 1]$ is a strong negation if and only if there is an automorphism $\varphi$ of the unit interval such that $N(x) = \varphi^{-1}(1 - \varphi(x))$.

### 2.3 Aggregation functions

A simple way to understand aggregation functions is that they perform the combination of several inputs into a single output. Formally, a function $M : [0, 1]^n \rightarrow [0, 1]$ with $n \geq 2$ is an aggregation function if it is increasing and satisfies the boundary conditions $M(0, \ldots, 0) = 0$ and $M(1, \ldots, 1) = 1$ (see [7, 43]). In this work, however, we follow [20] and consider a more restrictive definition, namely:

**Definition 2.2** An ($n$-ary) aggregation function is a function $M : [0, 1]^n \rightarrow [0, 1]$ such that:

1. **(A1)** $M(x_1, \ldots, x_n) = 0$ if and only if $x_1 = \cdots = x_n = 0$;
2. **(A2)** $M(x_1, \ldots, x_n) = 1$ if and only if $x_1 = \cdots = x_n = 1$;
3. **(A3)** $M$ is increasing.

Alternatively, whenever $M$ is strictly increasing we will refer to the third axiom as **(A3S)**.
An aggregation function $\mathcal{M}$ is symmetric if its output does not depend on the order in which the inputs are considered. That is, if

$$(A4) \quad \mathcal{M}(x_1, \ldots, x_n) = \mathcal{M}(x_{p(1)}, \ldots, x_{p(n)}) \text{ for every permutation } p \text{ of } \{1, \ldots, n\}.$$ 

Finally, an aggregation function $\mathcal{M}$ is **idempotent** if $\mathcal{M}(x, \ldots, x) = x$ for all $x \in X$ and $\mathcal{M}$ is **sub-idempotent** if $\mathcal{M}(x, \ldots, x) \leq x$ for each $x \in [0, 1]$.

In order to simplify the notation, we will denote $\mathcal{M}(A(x_1), \ldots, A(x_n)) = \mathcal{M}(A)$ and $\mathcal{M}(x_1, \ldots, x_n) = \frac{n}{i=1} \mathcal{M}(x_i)$.

### 2.3.1 Characterization of aggregations functions

In [20], Bustince et al. characterized some properties of the aggregations functions. We recall some of their results in this subsection as they will be important afterwards.

**Proposition 2.1** \cite{20} *The mapping $\mathcal{M} : [0, 1]^n \rightarrow [0, 1]$ satisfies (A1), (A2), (A3) and (A4) if and only if*

$$\mathcal{M}(y_1, \ldots, y_n) = \frac{g(y_1, \ldots, y_n)}{g(y_1, \ldots, y_n) + h(y_1, \ldots, y_n)}$$

*for some $g, h : [0, 1]^n \rightarrow [0, 1]$ such that*

(i) $g(y_1, \ldots, y_n) = 0$ if and only if $y_i = 0$ for all $i \in 1, \ldots, n$;

(ii) $h(y_1, \ldots, y_n) = 0$ if and only if $y_i = 1$ for all $i \in 1, \ldots, n$;

(iii) $g$ is non-decreasing and $h$ is non-increasing;

(iv) $g$ and $h$ are symmetric.*
Lemma 2.1 [20] | Let $N$ be any strong negation, then $\mathcal{M} : [0, 1]^n \to [0, 1]$ satisfies (A1) – (A4) and $\mathcal{M}(y_1, \ldots, y_n) \geq 1 - \mathcal{M}(N(y_1), \ldots, N(y_n))$ if and only if Eq. (2.1) holds for some $g, h : [0, 1]^n \to [0, 1]$ satisfying (i) – (iv) from Prop. 2.1 plus the following condition:

\[(v) \ g(y_1, \ldots, y_n) \geq h(y_1, \ldots, y_n).\]

2.4 Fuzzy implications

We now recall the notion of (fuzzy) implication function. An implication function (in the sense of Fodor and Roubens [39, 37], see [1, 2, 16, 64]) is a mapping $I : [0, 1]^2 \to [0, 1]$ such that, for every $x, y, z \in [0, 1]$:

(I1) If $x \leq z$ then $I(x, y) \geq I(z, y)$ (left antitonicity);

(I2) If $y \leq z$ then $I(x, y) \leq I(x, z)$ (right isotonicity);

(I3) $I(0, x) = 1$ (left boundary condition);

(I4) $I(x, 1) = 1$ (right boundary condition);

(I5) $I(1, 0) = 0$.

Different properties can be demanded from these implication functions, mostly depending on the application (see [2] for instance). A non-exhaustive list includes the following properties:

(I6) $I(1, x) = x$ (left neutrality property);

(I7) $I(x, I(y, z)) = I(y, I(x, z))$ (exchange principle);

(I8) $I(x, y) = 1$ if and only if $x \leq y$ (ordering property);

(I9) $I(x, 0) = N(x)$ is a strong negation;
(I10) $I(x, y) \geq y$;

(I11) $I(x, x) = 1$ (identity principle);

(I12) $I(x, y) = I(N(y), N(x))$ for a given strong negation $N$ (law of contraposition);

(I13) $I$ is continuous;

(I14) If $x < 1$ then $I(x, 0) > 0$;

(I15) $I(x, y) = 0$ if and only if $x = 1$ and $y = 0$.

The relations that exist between these properties have been studied in different works, for instance in [16, 71]. Next, we recall some interesting results that will be useful to develop our study in the following chapters. Note that $I$ will not be necessarily an implication function in the sense of Fodor and Roubens, $I$ will be a function that satisfies the properties required in each case.

Proposition 2.2 [16] Let $I : [0, 1]^2 \to [0, 1]$ be a function. For all $x, y, z \in [0, 1]$, the following properties hold:

(i) $I$ satisfies (I1) if and only if $I(x \lor y, z) = I(x, z) \land I(y, z)$;

(ii) $I$ satisfies (I1) if and only if $I(x \land y, z) = I(x, z) \lor I(y, z)$;

(iii) $I$ satisfies (I2) if and only if $I(x, y \land z) = I(x, y) \land I(x, z)$;

(iv) $I$ satisfies (I2) if and only if $I(x, y \lor z) = I(x, y) \lor I(x, z)$.

In this same work, Bustince et al. also presented some interdependencies among the axioms, as follows.

Lemma 2.2 [16] Let $I : [0, 1]^2 \to [0, 1]$ be a function. If $I$ satisfies:

(i) (I1) and (I12), then $I$ satisfies (I2);
Lemma 2.3 [16] Let \( I : [0, 1]^2 \rightarrow [0, 1] \) be any function that satisfies at least one of the following items:

(i) If \( I \) satisfies \( (I_2) \) and \( (I_9) \), or

(ii) If \( I \) satisfies \( (I_1) \) and \( (I_9) \), or

(iii) If \( I \) satisfies \( (I_4) \), \( (I_9) \), \( (I_12) \) and \( (I_{15}) \), or

(iv) If \( I \) satisfies \( (I_6) \), \( (I_7) \), \( (I_9) \) and \( I(x, x) = I(0, x) \) for all \( x \in [0, 1] \), then \( I \) satisfies \( (I_{15}) \).

In [17], Bustince et al. characterized functions \( I \) that satisfy simultaneously \( (I_1) \), \( (I_8) \), \( (I_{12}) \) and \( (I_{15}) \). We recall the next theorem as it will be used afterwards.

Theorem 2.2 [17] Let \( N \) be any strong negation. For a function \( I : [0, 1]^2 \rightarrow [0, 1] \) the following statements are equivalent:
(i) \( I \) satisfies \((I1), (I8), (I12)\) with respect to \( N \) and \((I15)\):

(ii) There exists a function \( \Phi : [0, 1]^2 \rightarrow [0, 1] \), such that \( \Phi(N(x), y) = I(x, y) \) for all \( x, y \in [0, 1] \), which satisfies conditions (a)-(d) as follows:

(a) \( \Phi \) is non decreasing in the first argument,

(b) \( \Phi \) is symmetric,

(c) \( \Phi(x, y) = 0 \) if and only if \( x = y = 0 \);

(d) \( \Phi(x, y) = 1 \) if and only if \( x \geq N(y) \).
Chapter 3

Fuzzy subsethood measures

As we briefly discussed in the introductory chapter, in the literature one can find a diversity of proposals for the axiomatization of fuzzy subsethood measures (FSM). In this chapter, we recall some of them in order to acknowledge the different views and motivations which led such studies.

3.1 Kitainik’s fuzzy subsethood measure

In [47, 46] Kitainik provided an axiomatic approach to the treatment of fuzzy inclusion indicators.

**Definition 3.1** A fuzzy subsethood measure in the sense of Kitainik is a mapping \( \sigma_K : F(X) \times F(X) \to [0, 1] \) such that, for every \( A, B, C \in F(X) \)

\[
(K1) \quad \sigma_K(A, B) = \sigma_K(B_N, A_N), \text{ where } N \text{ is the standard negation}^{[1]}
\]

\[
(K2) \quad \sigma_K(A, B \land C) = \sigma_K(A, B) \land \sigma_K(A, C);
\]

\(^{[1]}\)The standard negation is: \( N(x) = 1 - x \).
(K3) For every one-to-one mapping \( s: F(X) \rightarrow F(X) \), it holds that: \( \sigma_K(A, B) = \sigma_K(s(A), s(B)) \);

(K4) \( \sigma_K \) restricted to crisp sets coincides with the usual set inclusion.

The following result was proved by Fodor and Yager in 2000 [38].

**Theorem 3.1** A mapping \( \sigma_K: F(X) \times F(X) \rightarrow [0, 1] \) satisfies (K1)-(K4) if and only if there exists an implication function \( I \) satisfying (I12) and such that, for every \( A, B \in F(X) \)

\[
\sigma_K(A, B) = \inf_{i=1}^{n} I(A(x_i), B(x_i)).
\]

Sadly, Kitainik’s FSM received little attention in the fuzzy community at that time, specially because only four requirements were demanded and almost the entire essence of Sinha and Dougherty’s approach was captured.

### 3.2 Sinha and Dougherty’s fuzzy subsethood measure

In 1993, Sinha and Dougherty [72] presented an approach with nine axioms which are the properties desired for an indicator for set inclusion. This indicator provides the degree to which a fuzzy set is a subset of another fuzzy set. Formally, we have:

**Definition 3.2** A fuzzy subsethood measure according to Sinha and Dougherty is a mapping \( \sigma_{SD}: F(X) \times F(X) \rightarrow [0, 1] \) such that, for every \( A, B, C \in F(X) \)

(SD1) \( \sigma_{SD}(A, B) = 1 \) if and only if \( A \leq B \);

(SD2) \( \sigma_{SD}(A, B) = 0 \) if and only if there exists \( x \in X \) such that \( A(x) = 1 \) and \( B(x) = 0 \);

(SD3) If \( B \leq C \), then \( \sigma_{SD}(A, B) \leq \sigma_{SD}(A, C) \);
(SD4) If $B \leq C$, then $\sigma_{SD}(C,A) \leq \sigma_{SD}(C,B)$;

(SD5) For every one-to-one mapping $s: F(X) \rightarrow F(X)$, we have: $\sigma_{SD}(A,B) = \sigma_{SD}(s(A), s(B))$;

(SD6) $\sigma_{SD}(A,B) = \sigma_{SD}(B_N,A_N)$, where $N$ is the standard negation;

(SD7) $\sigma_{SD}(B \lor C,A) = \sigma_{SD}(B,A) \land \sigma_{SD}(C,A)$;

(SD8) $\sigma_{SD}(A,B \land C) = \sigma_{SD}(A,B) \land \sigma_{SD}(A,C)$;

(SD9) $\sigma_{SD}(A,B \lor C) \geq \sigma_{SD}(A,B) \lor \sigma_{SD}(A,C)$.

Sinha and Dougherty also considered the following three optional axioms, which are useful for some applications:

(SD10) $\sigma_{SD}(A,B) + \sigma_{SD}(A_N,B_N) \geq 1$;

(SD11) If $A$ is a refinement of $B$, then $\sigma_{SD}(A \lor A_N,A \land A_N) \leq \sigma_{SD}(B \lor B_N,B \land B_N)$;

(SD12) If $A$ is weakly included in $B$, then $\sigma_{SD}(A,B) \geq \frac{1}{2}$.

Regarding the relation among these axioms, it is worth to mention that Burillo et al. [9] proved that (SD9) is a consequence of (SD3). Furthermore, Kitainik [47] also showed that FSM that satisfy (K1)-(K4) also fulfill axioms (SD3) and (SD4). Actually, except for axioms (SD1) and (SD2), we can say that the axiomatization of Sinha and Dougherty and the one given by Kitainik are equivalent. Nevertheless, note that Sinha and Dougherty extend (SD1) and (SD2) for every fuzzy set, whereas Kitainik consider only crisp sets for the analogous axioms. In any case, observe that the expression in Theorem 3.1 does not fulfill Axiom (SD2).

Table 3.1 sums up the equivalence between the axioms of Kitainik and the ones given by Sinha and Dougherty.
3. Fuzzy subsethood measures

<table>
<thead>
<tr>
<th>Kitainik’s Axiom</th>
<th>Equivalent to</th>
</tr>
</thead>
<tbody>
<tr>
<td>(K1) $\sigma_K(A, B) = \sigma_K(B_N, A_N)$</td>
<td>(SD6)</td>
</tr>
<tr>
<td>(K2) $\sigma_K(A, B \land C) = \sigma_K(A, B) \land \sigma_K(A, C)$</td>
<td>(SD8)</td>
</tr>
<tr>
<td>(K3) $\sigma_K(A, B) = \sigma_K(s(A), s(B))$</td>
<td>(SD5)</td>
</tr>
<tr>
<td>(K4) $\sigma_K$ restricted to crisp sets coincides with the usual set inclusion for crisp sets</td>
<td>(SD1) and (SD2)</td>
</tr>
</tbody>
</table>

Table 3.1: Equivalence between Kitainik’s and Sinha and Dougherty’s axioms

3.3 Young’s fuzzy subsethood measure

In [83], Young took into account the axioms of Sinha and Dougherty but offered a new set of axioms as an alternative. Her main proposal was to present an axiomatization that was able to connect fuzzy subsethood with fuzzy entropy (which indicates the degree to which a set is fuzzy). Young’s FSM were presented as follows.

**Definition 3.3** Let $N$ be a strong negation and $e$ be the equilibrium point of $N$. A fuzzy subsethood measure in the sense of Young is a mapping $\sigma_Y: F(X) \times F(X) \rightarrow [0, 1]$ such that, for every $A, B, C \in F(X)$:

(Y1) $\sigma_Y(A, B) = 1$ if and only if $A \leq B$;

(Y2) If $e \leq A$, then $\sigma_Y(A, A_N) = 0$ if and only if $A = 1$;

(Y3) If $A \leq B \leq C$ then $\sigma_Y(C, A) \leq \sigma_Y(B, A)$, and if $A \leq B$ then $\sigma_Y(C, A) \leq \sigma_Y(C, B)$.

It is important to say that Young states that axiom (Y1) is not necessary for constructing fuzzy entropies from this definition. For this reason, she also proposes the concept of weak FSM. Further studies on these measures can be found in [13].

\[^2\]In Young’s original work, the negation considered was the standard one. In [19] this concept was generalized for an arbitrary strong negation.
Definition 3.4 A weak fuzzy subsethood measure (in the sense of Young) is a mapping
\( \sigma_{\text{WY}} : F(X) \times F(X) \rightarrow [0, 1] \) which, for every \( A, B, C \in F(X) \), it satisfies (Y2) and (Y3) and
such that there exist \( A, B \in F(X) \) with \( A \leq B \) but \( \sigma_{\text{WY}}(A, B) < 1 \).

3.4 Fan’s fuzzy subsethood measure

From the point of view of set-theoretic approach and from fuzzy implication operator,
in 1999 Fan et al. [35] presented some new definitions of FSM. They proposed the next
definition of a weak fuzzy *-subsethood measure so that appropriate boundary conditions
are provided.

Definition 3.5 A weak fuzzy *-subsethood measure is a mapping
\( \sigma_* : F(X) \times F(X) \rightarrow [0, 1] \) such that, for every \( A, B, C \in F(X) \):

\(
(*1) \quad \sigma_*(0, 0) = \sigma_*(0, 1) = \sigma_*(1, 1) = 1;
\)

\(
(*2) \quad \sigma_*(1, 0) = 0;
\)

\(
(*3) \quad \text{If } A \leq B \leq C, \text{ then } \sigma_*(C, A) \leq \sigma_*(B, A) \text{ and } \sigma_*(C, A) \leq \sigma_*(C, B).
\)

If we compare Def. 3.5 with the one given by Young, Def. 3.3, we can see that the
difference is that Young demands \( \sigma_Y \) to be increasing in the second argument (Y3).

3.5 Fuzzy DI-subsethood measure

Aiming at constructing measures to compare images from subsethood degrees Bustince
et al. introduced fuzzy DI-subsethood measures in [19]. The idea was to obtain fuzzy sub-
sethood measures aggregating implication operators.
Definition 3.6 We say that $\sigma_{DI} : F(X) \times F(X) \to [0, 1]$ is a fuzzy DI-subsethood measure on $X$, if $\sigma_{DI}$ satisfies:

(DI1) $\sigma_{DI}(A, B) = 1$ if and only if $A \leq B$;

(DI2) $\sigma_{DI}(A, A_N) = 0$ if and only if $A = X$;

(DI3) If $A \leq B$, then $\sigma_{DI}(A, C) \geq \sigma_{DI}(B, C)$ and $\sigma_{DI}(C, A) \leq \sigma_{DI}(C, B)$;

The name fuzzy DI-subsethood measure was inspired on axiom (DI3), where it requires that $\sigma_{DI}$ is decreasing in the first argument and increasing in the second.

In [19], Theorem 6, it was proved that every fuzzy subsethood measure in the sense of Young (Definition 3.3) is a fuzzy $\ast$-subsethood measure in the sense of Fan et al. (Definition 3.5). And in Theorem 10 of the same work it was proved that every fuzzy DI-subsethood measure is a fuzzy subsethood measure in the sense of Def. 3.3 as we can see in Figure 3.1.

![Diagram](image)

Figure 3.1: Relationships between some FSM

3.6 Other fuzzy subsethood measures

In [86], Zhang et al. presented the concept of hybrid monotonic FSM ($\sigma_Z$), substituting axiom (Y3) of Young’s definition (Def. 3.3) by axiom (DI3). Their main appeal on this proposal was to overcome the lack of transitivity property in most of the definitions of FSM found in the literature.
Back in the 70’s and 80’s, many researchers have contributed to relax the strictness of Zadeh’s definition on subsethood measures and led to the most recent studies on this topic. Different tools and approaches were used, and to name a few authors we can cite for instance, Bandler and Kohout [4] and Willmott [79] that developed measures using fuzzy implication operators, or Sanchez [69] and Kosko [50] who connected subsethood with fuzzy entropy.

3.7 General considerations

In order to fulfill our goal of defining a new class of FSM, first we have to establish which are the desirable properties our proposal should have. As mentioned before, we are interested in FSM that are constructed by aggregating fuzzy implication operators. That is why understanding a little bit of these axiomatizations was relevant and discussed in this chapter. Moreover, this study led us to define a new class of FSM as we will see in the following chapter.
Chapter 4

A new class of fuzzy subsethood measures

In this chapter we present our proposal for a new class of FSM. We explain the reasons which have led us to it and we study the relations of our definition with other subsethood measures. We present two methods for constructing those FSM and we also provide other constructions fixing the aggregation functions, including the use of automorphisms. Finally we generate penalty functions using the new class of FSM.

**Definition 4.1** A function $\sigma : F(X) \times F(X) \to [0, 1]$ is called a fuzzy subsethood measure, if $\sigma$ satisfies the following properties:

(a) $\sigma(A, B) = 1$ if and only if $A \leq B$;

(b) $\sigma(A, B) = 0$ if and only if, for every $x \in X$, $A(x) = 1$ and $B(x) = 0$;

(c) If $A \leq B$, then $\sigma(A, C) \geq \sigma(B, C)$ and $\sigma(C, A) \leq \sigma(C, B)$.

**Example 4.1**

1) Goguen’s subsethood degree, Eq. (1.2).
4. A new class of fuzzy subsethood measures

2)
\[
\sigma(A, B) = \frac{1}{n} \sum_{i=1}^{n} \begin{cases} 
1 & \text{if } A(x_i) \leq B(x_i), \\
\frac{1 - A(x_i) \lor B(x_i)}{2} & \text{if } A(x_i) > B(x_i).
\end{cases}
\]

3)
\[
\sigma(A, B) = \begin{cases} 
1 & \text{if } A \leq B, \\
0 & \text{if } A = 1 \text{ and } B = 0, \\
\frac{1}{2} & \text{otherwise}.
\end{cases}
\]

Our proposal of a new class of FSM is based on axioms (a), (b) and (c) which are justified as follows.

A. We acknowledge that axiom (a) must always be maintained. In the same way as V. Young, Dubois and Prade, Sinha and Dougherty, Willmott, Bandler and Kohout, etc. we consider that, for Zadeh’s fuzzy subsethood relation, this axiom must hold. If axiom (a) is not demanded, then we would be working with weak subsethood measures (see [13, 83]), which we do not consider in our proposal.

B. Regarding axiom (b), and as we have already said, we follow Willmott [79] and Bandler and Kohout’s ideas, and we try to get FSM by means of an appropriate aggregation of implication operators; that is, by means of the following equation:
\[
\sigma(A, B) = \max_{i=1}^{n}(I(A(x_i), B(x_i)))
\]

(i) In this setting, one of the most widely used FSM, Goguen’s one Eq. (1.2), fulfills axiom (b).

(ii) We use functions I with specific properties in the construction presented in Eq. (4.1). So, if I verifies (15) \(I(x, y) = 0\) if and only if \(x = 1\) and \(y = 0\), then we have that the construction in Eq. (4.1) always fulfills axiom (b).

(iii) Furthermore, axiom (b) allows us to ensure that for the fuzzy subsethood measure \(\sigma(A \lor B, A \land B)\), the following property holds:
Similarity between A and B is equal to zero if and only if A and B are complementary crisp sets.

This property is required to similarity measures between fuzzy sets in the sense of [21]. So, by demanding this axiom, we can construct not only fuzzy entropies, but also similarities and distances from the new class of FSM, according to Definition 4.1.

C. With respect to axiom (c), we have the same reasons as those which led Sinha and Dougherty to introduce axioms (SD3) and (SD4), as well as the considerations by Dubois et al. [29] and the ideas leading to justify the definition of DI-measures in [19]. That is, the idea comes from the transitivity of set inclusion in the sense of Zadeh. So, (SD3) and (SD4) demand, respectively, for the subsethood measure to be increasing in the second argument and decreasing in the first, which we merged into a single axiom (c) in Def. 4.1.

4.1 Relation with other definitions

We discuss in this section how our definition is related to other definitions found in the literature. We start with the following result.

**Theorem 4.1** Every fuzzy subsethood measure on X is a fuzzy DI-subsethood measure on X.

**Proof.** We only need to take $B$ as the complementary of $A$ in axiom (b). ■

**Example 4.2** The converse of Theorem 4.1 does not hold, in general. Consider the following fuzzy DI-subsethood measure

$$
\sigma(A, B) = \frac{1}{n} \sum_{i=1}^{n} \begin{cases} 
1 & \text{if } A(x_i) \leq B(x_i), \\
0 & \text{if } A(x_i) = 1 \text{ and } B(x_i) \neq 1, \\
\frac{1}{2} & \text{otherwise}. 
\end{cases}
$$
If we take $A(x) = 1$ and $B(x) = \frac{1}{2}$ for every $x \in X$, then $\sigma(A, B) = 0$. Therefore, it is not a fuzzy subsethood measure in the sense of Definition 4.1.

As a consequence, the following corollary also holds.

**Corollary 4.1** Every fuzzy subsethood measure on $X$ is a fuzzy subsethood measure on $X$ in the sense of V. Young and in the sense of J. Fan.

**Proof.** Straightforward. ■

Of course, from Definition 4.1 it is clear that the FSM on $X$ fulfill axioms $(SD1)$, $(SD3)$ and $(SD4)$.

**Corollary 4.2** Every fuzzy subsethood measure on $X$ satisfies axioms $(SD9)$, $(SD11)$ and the inequalities:

\[
\begin{align*}
\sigma(B \land C, A) &\geq \sigma(B, A) \lor \sigma(C, A) \\
\sigma(B \lor C, A) &\leq \sigma(B, A) \land \sigma(C, A) \\
\sigma(A, B \land C) &\leq \sigma(B, A) \land \sigma(C, A).
\end{align*}
\]

**Proof.** Since $\sigma$ is increasing in the second component, we know that $B \land C \leq B$, then $\sigma(A, B \land C) \geq \sigma(A, B)$. Moreover, $B \land C \leq C$, then $\sigma(A, B \land C) \geq \sigma(A, C)$. Therefore $\sigma(A, B \land C) \geq \sigma(A, B) \land \sigma(A, C)$.

If $B \leq C$, then $B \land C = C$, so $\sigma(A, B \land C) = \sigma(A, C) \geq \sigma(A, B) \land \sigma(A, C)$. Therefore $\sigma(A, B) \leq \sigma(A, C)$. Thus we have that $\sigma$ satisfies $(SD9)$.

From axiom $(c)$ of Def. 4.1 it is clear that $(SD11)$ and the inequalities above are obtained as a direct consequence. ■

Therefore, for all $A, B, C \in F(X)$ axioms $(SD7)$ and $(SD8)$ do not hold and the inequalities according to Corollary 4.2 do.
Remark 4.1 Every fuzzy subsethood measure on $X$ is a fuzzy subsethood measure on $X$ in the sense of Zhang.

Observe that Zhang’s properties are the same as Young’s except $(Y3)$ which is replaced by $(DI3)$. Nevertheless, we observe that in Def. 4.1 axiom $(a)$ is the same as $(Y1)$ which is equal to $(DI1)$; and similarly $(b)$ and $(c)$ correspond to $(DI2)$ and $(DI3)$, respectively. However, the converse does not always hold as we can see in the next example.

Example 4.3 A function $\sigma_Z(A, B) = 1$, for $A = B = 0$, and

$$\sigma_Z(A, B) = \frac{|B|}{|A \cup B|}, \text{ otherwise}$$

where $|A| = \sum_{x \in X} A(x)$ is the cardinal number of the fuzzy set $A$, is a fuzzy subsethood measure in the sense of Zhang which is not a fuzzy DI-subsethood measure, and consequently, it is not a fuzzy subsethood measure in the sense of Definition 4.1.

Figure 4.1 shows the relationships between the measures proposed by Fan ($\sigma_\ast$), Young ($\sigma_Y$), Zhang ($\sigma_Z$) and Bustince ($\sigma_{DI}$) and our proposal ($\sigma$), according to Definition 4.1.

![Figure 4.1: Relationships between $\sigma$ and other FSM.](image-url)

Observe that our axiomatization is the most restrictive compared to the others. However, we understand that it is an efficient axiomatization as: it demands only three axioms; it
recovers Goguen’s subsethood grade, in such a way that our developments are useful in every application where Goguen’s subsethood grade can be applied; and it also gives us more flexibility in order to build, for example, comparison measures that can be applied in image processing, once different aggregation functions and implication operators can be used.

4.2 Building FSM

In this section, we present two main methods for constructing FSM in the sense of Definition 4.1 and also other types of constructions.

4.2.1 First construction

Proposition 4.1 Take a strictly increasing function $\mathcal{M}$ that verifies (A1). Take functions $g, h : [0, 1]^2 \rightarrow [0, 1]$ such that

\begin{align*}
  i. & \quad g(x, y) \leq h(x, y) \text{ for all } x, y \in [0, 1]; \\
  ii. & \quad g(x, y) = h(x, y) \text{ if and only if } x \leq y; \\
  iii. & \quad g(x, y) = 0 \text{ if and only if } x = 1 \text{ and } y = 0; \\
  iv. & \quad \text{If } x \leq y, \text{ then}
\begin{align*}
  g(z, x) & \leq g(z, y) \\
  g(y, z) & \leq g(x, z) \\
  h(z, y) & \leq h(z, x) \\
  h(x, z) & \leq h(y, z).
\end{align*}
\end{align*}

Then, the mapping $\sigma : F(X) \times F(X) \rightarrow [0, 1]$ given by

\[
\sigma(A, B) = \frac{\mathcal{M}(g(A(x_i), B(x_i)))}{\mathcal{M}(h(A(x_i), B(x_i)))},
\]
is a fuzzy subsethood measure on $X$.

**Proof.** First of all assume that $h(A(x_i), B(x_i)) = 0$. Then, from (ii.) we have $g(A(x_i), B(x_i)) = 0$ if and only if $A(x_i) \leq B(x_i)$. But from (iii.) we have that $A(x_i) = 1$ and $B(x_i) = 0$ given $g(A(x_i), B(x_i)) = 0$. So we have a contradiction.

Now let us prove the result.

(a) (Necessity) If $\sigma(A, B) = 1$, then $\frac{n}{i=1} (g(A(x_i), B(x_i))) = \frac{n}{i=1} (h(A(x_i), B(x_i)))$.

By hypothesis we know that $g(x, y) \leq h(x, y)$. Since $\mathcal{M}$ is strictly increasing, we have that if there exists a $x_i$ such that $g(A(x_i), B(x_i)) < h(A(x_i), B(x_i))$, then $\frac{n}{i=1} (g(A(x_i), B(x_i))) < \frac{n}{i=1} (h(A(x_i), B(x_i)))$. Therefore for all $i \in \{1, \ldots, n\}$ we have $g(A(x_i), B(x_i)) = h(A(x_i), B(x_i))$.

From (i.) it follows that $A(x_i) \leq B(x_i)$ for all $i \in \{1, \ldots, n\}$.

(Sufficiency) If $A(x_i) \leq B(x_i)$ for all $i \in \{1, \ldots, n\}$, then by (ii.) we have that $g(A(x_i), B(x_i)) = h(A(x_i), B(x_i))$

for all $i \in \{1, \ldots, n\}$. Therefore, it follows that $\sigma(A, B) = 1$.

(b) Since $\mathcal{M}$ satisfies (A1) we have that $\sigma(A, B) = 0$ if and only if $\frac{n}{i=1} (g(A(x_i), B(x_i))) = 0$, i.e., if and only if $g(A(x_i), B(x_i)) = 0$ for all $i \in \{1, \ldots, n\}$, and if and only if $A(x_i) = 1$ and $B(x_i) = 0$ for all $i \in \{1, \ldots, n\}$.

(c) If $A \leq B$, then from (iv.) we have

$$
\sigma(A, C) = \frac{\frac{n}{i=1} (g(A(x_i), C(x_i)))}{\frac{n}{i=1} (h(A(x_i), C(x_i)))} \geq \frac{\frac{n}{i=1} (g(B(x_i), C(x_i)))}{\frac{n}{i=1} (h(B(x_i), C(x_i)))} \geq \frac{\frac{n}{i=1} (g(B(x_i), C(x_i)))}{\frac{n}{i=1} (h(B(x_i), C(x_i)))} = \sigma(B, C).
$$
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\[ \sigma(C, A) = \frac{\mathcal{M}(g(C(x_i), A(x_i)))}{\mathcal{M}(h(C(x_i), A(x_i)))} = \frac{\mathcal{M}(g(C(x_i), B(x_i)))}{\mathcal{M}(h(C(x_i), B(x_i)))} \leq \frac{\mathcal{M}(g(C(x_i), B(x_i)))}{\mathcal{M}(h(C(x_i), B(x_i)))} \leq \frac{\mathcal{M}(g(C(x_i), A(x_i)))}{\mathcal{M}(h(C(x_i), A(x_i)))} \]

**Corollary 4.3** In the setting of Proposition 4.1, the following items hold.

1. If \( \mathcal{M} \) is symmetric then for every injective mapping \( s : F(X) \to F(X) \) and for every \( A, B \in F(X) \) it holds axiom (SD5); that is, \( \sigma(A, B) = \sigma(s(A), s(B)) \), where \( s(A)(x) = A(s(x)) \).

2. If \( g(x, y) = g(N(y), N(x)) \) and \( h(x, y) = h(N(y), N(x)) \) then \( \sigma(A, B) = \sigma(B_N, A_N) \), where \( N \) is a strong negation (axiom (SD6)).

3. If \( g(x, y) + g(x, N(y)) \geq 1 \) and \( \mathcal{M}(x_i) + \frac{\mathcal{M}(1-x_i)}{\mathcal{M}(1)} \geq 1 \), then \( \sigma(A, B) + \sigma(A, B_N) \geq 1 \) (axiom (SD10)).

4. Suppose that \( g(x, y) = g(N(y), N(x)) \), \( g(x, y) \geq y \) and \( \mathcal{M} \) is idempotent. If \( A \) is weakly included in \( B \), then \( \sigma(A, B) \geq e \), where \( e \) is the equilibrium point of \( N \) (axiom (SD12)).

This first construction leads us to the second one, as follows.

### 4.2.2 Second construction

Following the works by Sanchez [69], Bandler and Kohout [4], L. Kitainik [47, 46], E. Ruspini [68] and Willmott [79, 80, 81], we desire to build FSM by aggregating implication-like operators. Thus,

**Proposition 4.2** Let \( \sigma : F(X) \times F(X) \to [0, 1] \) be given by:

\[ \sigma(A, B) = \frac{\mathcal{M}(I(A(x_i), B(x_i)))}{\mathcal{M}(I(A(x_i), A(x_i)))} \]
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where \( \mathcal{M} : [0, 1]^n \rightarrow [0, 1] \) is an aggregation function and \( I : [0, 1]^2 \rightarrow [0, 1] \) is a function that satisfies (I1), (I2), (I8) and (I15). Then \( \sigma \) is a fuzzy subsethood measure on \( X \).

**Proof.** (a) (Necessity) Suppose that \( \sigma(A, B) = 1 = \sum_{i=1}^{n} (I(A(x_i), B(x_i))) \). Then, from (A2), \( I(A(x_i), B(x_i)) = 1 \) for all \( i \in \{1, \ldots, n\} \). Since \( I \) satisfies (I8), we have \( A(x_i) \leq B(x_i) \) for every \( i \in \{1, \ldots, n\} \).

(Sufficiency) If \( A(x_i) \leq B(x_i) \) for all \( i \in \{1, \ldots, n\} \), then by (I8) we have that \( I(A(x_i), B(x_i)) = 1 \). Since \( \mathcal{M} \) satisfies (A2) we see that

\[
\sigma(A, B) = \sum_{i=1}^{n} (I(A(x_i), B(x_i))) = \mathcal{M}(1, \ldots, 1) = 1.
\]

(b) (Necessity) If \( \sigma(A, B) = 0 = \sum_{i=1}^{n} (I(A(x_i), B(x_i))) \), then since \( \mathcal{M} \) satisfies (A1) we have \( I(A(x_i), B(x_i)) = 0 \). Therefore, by (I15) \( A(x_i) = B(x_i) = 1 \) for all \( i \in \{1, \ldots, n\} \).

(Sufficiency) If \( A(x_i) = B(x_i) = 1 \) for all \( i \in \{1, \ldots, n\} \), then we have by (I15) that \( I(A(x_i), B(x_i)) = 0 \). As \( \mathcal{M} \) satisfies (A1) we have

\[
\sigma(A, B) = \sum_{i=1}^{n} (I(A(x_i), B(x_i))) = \mathcal{M}(0, \ldots, 0) = 0.
\]

(c) If \( A(x_i) \leq B(x_i) \) for all \( i \in \{1, \ldots, n\} \) then, as \( I \) satisfies (I1), we have \( I(A(x_i), C(x_i)) \geq I(B(x_i), C(x_i)) \). Since \( \mathcal{M} \) satisfies (A3) we have

\[
\sigma(A, C) = \sum_{i=1}^{n} (I(A(x_i), C(x_i))) \geq \sum_{i=1}^{n} (I(B(x_i), C(x_i))) = \sigma(B, C).
\]

On the other hand, as (I2) holds, \( I(C(x_i), A(x_i)) \leq I(C(x_i), B(x_i)) \). Since \( \mathcal{M} \) satisfies (A3) we have

\[
\sigma(C, A) = \sum_{i=1}^{n} (I(C(x_i), A(x_i))) \leq \sum_{i=1}^{n} (I(C(x_i), B(x_i))) = \sigma(C, B).
\]

Observe that (I15) follows as a consequence of other properties of the implication operators [16 [17]. Thus, in the following corollary, we replace the requirement of this property by some other ones.
Corollary 4.4 Let \( N \) be a strong negation and let \( \sigma : F(X) \times F(X) \to [0, 1] \), be given by Eq. (4.2), for all \( A, B \in F(X) \), where \( \mathcal{M} : [0, 1]^n \to [0, 1] \) is an aggregation function and \( I : [0, 1]^2 \to [0, 1] \) is a function that satisfies:

i) \((I1), (I2), (I6), (I8), (I9)\) or

ii) \((I2), (I7), (I8), (I9)\).

So, under these conditions, \( \sigma \) is a fuzzy subsethood measure on \( X \).

Proof. It is enough to take into account Lemmas 2.2 and 2.3 and Proposition 4.2.

i) If \( I \) satisfies \((I2), (I6) \) and \((I9)\), then \( I \) satisfies \((I15)\).

ii) If \( I \) satisfies \((I7) \) and \((I9)\), then \( I \) satisfies \((I12)\). If \( I \) satisfies \((I9) \) and \((I12)\), then it satisfies \((I6)\) and if \( I \) satisfies \((I2) \) and \((I12)\), then it fulfills \((I1)\). Therefore \( I \) satisfies \((I15)\).

Bearing in mind Prop. 4.2, it is satisfied that \( \sigma \) is a fuzzy subsethood measure on \( X \).

4.2.2.1 Relation between the new class of \( \text{FSM} \) and Sinha and Dougherty’s fuzzy sub-
sethood measure

In [72], Sinha and Dougherty understood that fuzzification of set inclusion for fuzzy sets is developed in terms of an indicator for set inclusion. The idea was that such indicator provided the degree to which a fuzzy set is a subset of another fuzzy set. Therefore, they established the desired properties of those indicators for fuzzified set inclusion. Their investigation resulted in a very general class of indicators based on the union operation, and, most importantly, in a complete measure-theoretical characterization of this class.

In Table 4.1 our intent was to compare the twelve \((SD)\) axioms with our new class of \( \text{FSM} \) \((\sigma)\). The strength of our results is easily perceived, once with only three axioms we managed to fulfill all of the axioms given by Sinha and Dougherty.
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<table>
<thead>
<tr>
<th>Axioms of Sinha and Dougherty</th>
<th>The new class of FSM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(SD1)$ $\sigma_{SD}(A, B) = 1$ if and only if $A \leq B.$</td>
<td>$\sigma$ satisfies $(SD1), (a).$</td>
</tr>
<tr>
<td>$(SD2)$ $\sigma_{SD}(A, B) = 0$ if and only if there exists $x \in X$ such that $A(x) = 1$ and $B(x) = 0.$</td>
<td>It is replaced by $(b)$: that is, $\sigma(A, B) = 0$ if and only if $A = 1$ and $B = 0.$</td>
</tr>
<tr>
<td>$(SD3)$ If $B \leq C,$ then $\sigma_{SD}(A, B) \leq \sigma_{SD}(A, C).$</td>
<td>It coincides with the second conditions of $(c).$</td>
</tr>
<tr>
<td>$(SD4)$ If $B \leq C,$ then $\sigma_{SD}(C, A) \leq \sigma_{SD}(B, A).$</td>
<td>It coincides with the first condition of $(c).$</td>
</tr>
<tr>
<td>$(SD5)$ $\sigma_{SD}(A, B) = \sigma_{SD}(s(A), s(B)).$</td>
<td>Under the conditions of Proposition $4.2$ if $\mathcal{M}$ satisfies $(A4),$ then $\sigma$ satisfies $(SD5).$</td>
</tr>
<tr>
<td>$(SD6)$ $\sigma_{SD}(A, B) = \sigma_{SD}(B_N, A_N).$</td>
<td>Under the conditions of Proposition $4.2$ if $I$ satisfies $(I12),$ then $\sigma$ satisfies $(SD6).$</td>
</tr>
<tr>
<td>$(SD7)$ $\sigma_{SD}(B \lor C, A) = \sigma_{SD}(B, A) \land \sigma_{SD}(C, A).$</td>
<td>$\sigma$ satisfies the following inequality: $\sigma_{SD}(B \lor C, A) \leq \sigma_{SD}(B, A) \land \sigma_{SD}(C, A).$</td>
</tr>
<tr>
<td>$(SD8)$ $\sigma_{SD}(A, B \land C) = \sigma_{SD}(A, B) \land \sigma_{SD}(A, C).$</td>
<td>$\sigma$ satisfies the following inequality: $\sigma(A, B \land C) \leq \sigma(A, B) \land \sigma(A, C).$</td>
</tr>
<tr>
<td>$(SD9)$ $\sigma_{SD}(A, B \lor C) \geq \sigma_{SD}(A, B) \lor \sigma_{SD}(A, C).$</td>
<td>$\sigma$ satisfies $(SD9).$</td>
</tr>
<tr>
<td>$(SD10)$ $\sigma_{SD}(A, B) + \sigma_{SD}(A, B_N) \geq 1.$</td>
<td>Under the conditions of Proposition $4.2$ if $I$ satisfies $(I10)$ $a \mathcal{M}(x_1, \ldots, x_n) + \mathcal{M}(N(x_1), \ldots, N(x_n)) \geq 1,$ then $\sigma$ satisfies $(SD10).$</td>
</tr>
<tr>
<td>$(SD11)$ If $A$ is a refinement of $B,$ then $\sigma_{SD}(A \lor A_N, A \land A_N) \leq \sigma_{SD}(B \lor B_N, B \land B_N).$</td>
<td>$\sigma$ satisfies $(SD11).$</td>
</tr>
<tr>
<td>$(SD12)$ If $A$ is weakly included in $B,$ then $\sigma_{SD}(A, B) \geq \epsilon.$</td>
<td>Under the conditions of Proposition $4.2$ if $I$ satisfies $(I10), (I12)$ and $\mathcal{M}$ is idempotent, then $\sigma$ satisfies $(SD12).$</td>
</tr>
</tbody>
</table>

Table 4.1: Relation between FSM $\sigma$ and $\sigma_{SD}.$

4.2.3 Other constructions

We analyze now the properties that FSM may have depending on the requirements that are demanded and/or on the aggregation functions that are being considered.

The following result shows us whenever FSM $\sigma$ fulfill axioms $(SD5), (SD6), (SD10)$ and $(SD12).$

**Corollary 4.5** Let $N$ be a strong negation. Under the same conditions of Proposition $4.2$ the following items hold:

i) If $\mathcal{M}$ is symmetric, then $\sigma$ satisfies $(SD5);$
ii) If $I$ satisfies $(I12)$, then $\sigma$ satisfies $(SD6)$;

iii) If $M$ satisfies $M(x_1, \ldots, x_n) + M(N(x_1), \ldots, N(x_n)) \geq 1$ and $I$ satisfies $(I10)$ then, $\sigma$ satisfies $(SD10)$;

iv) If $I$ satisfies $(I10)$ and $(I12)$ and $M$ is idempotent, then $\sigma$ satisfies $(SD12)$.

Proof.

i) Straightforward.

ii) Bearing in mind that $I$ satisfies $(I12)$, we have

$$\sigma(B_N, A_N) = \bigvee_{i=1}^{n}(I(N(B(x_i))), N(A(x_i))) = \bigvee_{i=1}^{n}(I(A(x_i), B(x_i))) = \sigma(A, B).$$

iii) $\sigma(A, B) + \sigma(A, B_N) =

$$\bigvee_{i=1}^{n}(I(A(x_i), B(x_i))) + \bigvee_{i=1}^{n}(I(A(x_i), N(B(x_i)))) \geq

\bigvee_{i=1}^{n}(I(A(x_i), B(x_i))) + 1 - \bigvee_{i=1}^{n}(N(I(A(x_i), N(B(x_i))))).$$

If $A(x_i) \leq B(x_i)$, regarding that $I$ satisfies $(I8)$ we have

$$I(A(x_i), B(x_i)) = 1 \geq N(I(A(x_i), N(B(x_i)))).$$

If $A(x_i) > B(x_i)$, then as $I$ satisfies $(I10)$ and $N$ is a strong negation we have

$$I(A(x_i), B(x_i)) \geq B(x_i)$$

$$I(A(x_i), N(B(x_i))) \geq N(B(x_i))$$

$$N(I(A(x_i), N(B(x_i)))) \leq B(x_i)$$

therefore $N(I(A(x_i), N(B(x_i)))) \leq B(x_i) \leq I(A(x_i), B(x_i))$. Lastly, we only need to recall that $M$ satisfies $(A3)$. 
iv) If \( A(x_i) \leq e \), then as \( I \) satisfies \((I1), (I10)\) and \((I12)\) we have

\[
I(A(x_i), B(x_i)) \geq I(e, B(x_i)) = I(N(B(x_i)), e) \geq e.
\]

If \( B(x_i) > e \), then \( I(A(x_i), B(x_i)) \geq B(x_i) > e \).

Taking into account that \( \mathcal{M} \) satisfies \((A3)\) and is idempotent, we have

\[
\sigma(A, B) = \frac{1}{n} \sum_{i=1}^{n} \left\{ \begin{array}{ll}
1 & \text{if } A(x_i) \leq B(x_i), \\
1 - A(x_i) \lor B(x_i) & \text{if } A(x_i) > B(x_i).
\end{array} \right.
\]

Example 4.4 The following three FSM satisfy axioms \((SD5), (SD6), (SD10)\) and \((SD12)\). In every case, we consider the standard negation \( N(x) = 1 - x \), for all \( x \in [0, 1] \).

1) If \( \mathcal{M}(x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i \), then

\[
\sigma(A, B) = \frac{1}{n} \sum_{i=1}^{n} \left\{ \begin{array}{ll}
1 & \text{if } A(x_i) \leq B(x_i), \\
1 - A(x_i) \lor B(x_i) & \text{if } A(x_i) > B(x_i).
\end{array} \right.
\]

2) If \( \mathcal{M}(x_1, \ldots, x_n) = \frac{n}{\sum_{i=1}^{n} x_i} \) then,

\[
\sigma(A, B) = \frac{\sqrt[n]{\sum_{i=1}^{n} x_i}}{\sqrt[n]{\sum_{i=1}^{n} (1 - A(x_i) + B(x_i))}}.
\]

3) If \( \mathcal{M}(x_1, \ldots, x_n) = \left( \frac{1}{n} \sum_{i=1}^{n} x_i \right)^{1/\lambda} \), \( \lambda > 1 \) then,

\[
\sigma(A, B) = \left( \frac{1}{n} \sum_{i=1}^{n} \left\{ \begin{array}{ll}
1 & \text{if } A(x_i) \leq B(x_i), \\
1 - A(x_i) \lor B(x_i) & \text{if } A(x_i) > B(x_i)
\end{array} \right. \right)^{1/\lambda}.
\]

In the two following corollaries we show FSM constructed from functions \( \mathcal{M} \) and \( I \) that
satisfy the conditions of Proposition 4.2 and we use for $\mathcal{M}$, the constructions studied previously.

**Corollary 4.6** Let $g, h : [0, 1]^n \rightarrow [0, 1]$ be such that

i) $g(y_1, \ldots, y_n) = 0$ if and only if $y_i = 0$ for all $i \in \{1, \ldots, n\}$;

ii) $h(y_1, \ldots, y_n) = 0$ if and only if $y_i = 1$ for all $i \in \{1, \ldots, n\}$;

iii) $g$ is non decreasing and $h$ is non increasing;

iv) $g$ and $h$ are symmetric.

and let $I : [0, 1]^2 \rightarrow [0, 1]$ be such that it satisfies (I1), (I2), (I8) and (I15). Under these conditions,

$$
\sigma(A, B) = \frac{g(I(A(x_1), B(x_1)), \ldots, I(A(x_n), B(x_n)))}{g(I(A(x_1), B(x_1)), \ldots, I(A(x_n), B(x_n))) + h(I(A(x_1), B(x_1)), \ldots, I(A(x_n), B(x_n)))}
$$

is a fuzzy subsethood measure on $X$.

**Proof.** By Proposition 2.1 we have that: $\mathcal{M} : [1, 0]^n \rightarrow [0, 1]$ satisfies (A1), (A2) and (A3) if and only if

$$
\mathcal{M}(y_1, \ldots, y_n) = \frac{g(y_1, \ldots, y_n)}{g(y_1, \ldots, y_n) + h(y_1, \ldots, y_n)}
$$

where $g, h : [0, 1]^n \rightarrow [0, 1]$ satisfy i), ii), iii) and iv).

Taking into account that the expression of the statement is obtained by replacing $\mathcal{M}$ with the value above; that is,

$$
\sigma(A, B) = \frac{n}{\prod_{i=1}^{n} I(A(x_i), B(x_i))} = \frac{g(I(A(x_1), B(x_1)), \ldots, I(A(x_n), B(x_n)))}{g(I(A(x_1), B(x_1)), \ldots, I(A(x_n), B(x_n))) + h(I(A(x_1), B(x_1)), \ldots, I(A(x_n), B(x_n)))}
$$

it results that $\mathcal{M}$ and $I$ satisfy the conditions of Proposition 4.2. ■
Example 4.5 In the following examples we take $I(x, y) = 1 \land (1 - x + y)$ and the standard negation.

1) Let $g$ and $h$ be given by:

$$g(y_1, \ldots, y_n) = \frac{\frac{1}{n} \sum y_i}{\left(\frac{1}{n} \sum (1 - y_i)^2\right)^{\frac{1}{2}}}.$$  

Then, the resulting fuzzy subsethood measure is:

$$\sigma(A, B) = \frac{\frac{1}{n} \sum 1 \land (1 - A(x_i) + B(x_i))}{\frac{1}{n} \sum 1 \land (1 - A(x_i) + B(x_i)) + \left(\frac{1}{n} \sum (0 \lor (A(x_i) - B(x_i)))^2\right)^{\frac{1}{2}}}.$$  

2) Let $g$ and $h$ be given by:

$$g(y_1, \ldots, y_n) = \frac{\lambda \sum y_i + (1 - \lambda) \sum y_i}{1 - (\lambda \sum y_i + (1 - \lambda) \sum y_i)}$$

where $\lambda \in (0, 1)$. Then, the resulting fuzzy subsethood measure is:

$$\sigma(A, B) = \lambda \sum 1 \land (1 - A(x_i) + B(x_i)) + (1 - \lambda) \sum 1 \land (1 - A(x_i) + B(x_i))$$

Corollary 4.7 Under the same conditions of Corollary 4.6, if $N$ is a strong negation, $g(y_1, \ldots, y_n) \geq h(N(y_1), \ldots, N(y_n))$ and $I$ satisfies (I10), then

$$\sigma(A, B) = \frac{g(I(A(x_1), B(x_1)), \ldots, I(A(x_n), B(x_n)))}{g(I(A(x_1), B(x_1)), \ldots, I(A(x_n), B(x_n))) + h(I(A(x_1), B(x_1)), \ldots, I(A(x_n), B(x_n)))}$$

is a fuzzy subsethood measure on $X$ that satisfies axiom (SD10).

Proof. Bearing in mind Lemma 2.1 and taking into account that $M$ under these conditions
4. A new class of fuzzy subsethood measures

satisfies (A1)-(A3) and the conditions of item i) of Corollary 4.6 and also that $I$ satisfies (I10), then by Corollary 4.6 we have that

$$\sigma(A, B) = \frac{\bigvee_{i=1}^{n} I(A(x_i), B(x_i))}{g(I(A(x_1), B(x_1)), \ldots, I(A(x_n), B(x_n)))}$$

is a fuzzy subsethood measure on $X$ that satisfies (SD10).

**Example 4.6** In the following examples we take $I(x, y) = 1 \land (1 - x + y)$ and the standard negation.

1) Let $g$ and $h$ be given by:

$$g(y_1, \ldots, y_n) = \frac{\bigvee_{i=1}^{n} y_i}{\bigvee_{i=1}^{n} (1 - y_i)}.$$

Then, the resulting fuzzy subsethood measure is:

$$\sigma(A, B) = \frac{\bigvee_{i=1}^{n} 1 \land (1 - A(x_i) + B(x_i))}{\bigvee_{i=1}^{n} 1 \land (1 - A(x_i) + B(x_i)) + \bigvee_{i=1}^{n} 0 \lor (A(x_i) - B(x_i))}.$$

2) Let $g$ and $h$ be given by:

$$g(y_1, \ldots, y_n) = \frac{\bigvee_{i=1}^{n} y_i}{\left(\frac{1}{n} \sum_{i=1}^{n} (1 - y_i)^2\right)^{\frac{1}{2}}}.$$

Then, the resulting fuzzy subsethood measure is:
\[ \sigma(A, B) = \frac{\bigvee_{i=1}^{n} 1 \land (1 - A(x_i) + B(x_i))}{\bigvee_{i=1}^{n} 1 \land (1 - A(x_i) + B(x_i)) + \left( \frac{1}{n} \sum_{i=1}^{n} (0 \lor (A(x_i) - B(x_i)))^2 \right)^{\frac{1}{2}}}. \]

### 4.2.3.1 Characterizing FSM on X with a fixed \( \mathcal{M} \)

**Theorem 4.2** Let \( \sigma : F(X) \times F(X) \to [0, 1] \) be given by Eq. (4.2), for all \( A, B \in F(X) \), where \( \mathcal{M} : [0, 1]^n \to [0, 1] \) is an idempotent aggregation function and \( I : [0, 1]^2 \to [0, 1] \) is a function. Then, \( \sigma \) is a fuzzy subsethood measure on \( X \) if and only if \( I \) satisfies (I1), (I2), (I8) and (I15).

**Proof.** (Necessity) (I1). Let \( p, q, r \in [0, 1] \) such that \( p \leq q \). Consider the following sets:

- \( A = \{ (x, A(x) = p) : x \in X \} \)
- \( B = \{ (x, B(x) = q) : x \in X \} \)
- \( C = \{ (x, C(x) = r) : x \in X \} \),

evidently \( A \leq B \). Since \( \mathcal{M} \) is idempotent, we have

\[
I(p, r) = \mathcal{M}(I(p, r)) = \mathcal{M}(I(A(x_i), C(x_i))) = \sigma(A, C) \geq \sigma(B, C)
\]

\[
= \frac{n}{\bigvee_{i=1}^{n} 1 \land (1 - A(x_i) + B(x_i)) + \left( \frac{1}{n} \sum_{i=1}^{n} (0 \lor (A(x_i) - B(x_i)))^2 \right)^{\frac{1}{2}}}
\]

\[
= \frac{n}{\bigvee_{i=1}^{n} 1 \land (1 - A(x_i) + B(x_i)) + \left( \frac{1}{n} \sum_{i=1}^{n} (0 \lor (A(x_i) - B(x_i)))^2 \right)^{\frac{1}{2}}}
\]

\[
= I(q, r).
\]

(I2). Similar to the one above.

(I8). Let us take the sets \( A = \{ (x, A(x) = p) : x \in X \} \) and \( B = \{ (x, B(x) = q) : x \in X \} \).

Since \( \mathcal{M} \) satisfies (A2) and it is idempotent we have that if \( I(p, q) = 1 \), then \( I(p, q) = 1 = \mathcal{M}(I(p, q)) = \sigma(A, B) \). Considering that, by hypothesis, \( \sigma \) satisfies (a) we have \( A \leq B \), that is \( p \leq q \).
If \( p \leq q \) then \( A \leq B \), so \( \sigma(A, B) = 1 = \sum_{i=1}^{n}(I(p, q)) \), since \( \mathcal{M} \) satisfies (A2) \( I(p, q) = 1 \).

Finally, if \( I(p, q) = 0 \), bearing in mind that \( \mathcal{M} \) is idempotent and satisfies (A1) we have 
\[
\sigma(A, B) = \sum_{i=1}^{n}(I(p, q)) = \mathcal{M}(0, \ldots, 0) = 0.
\]

As \( \sigma \) satisfies (b) we have \( A = 1 \) and \( B = 0 \); that is, \( A = \{\langle x, A(x) = p = 1 \rangle : x \in X \} \) and \( B = \{\langle x, B(x) = q = 0 \rangle : x \in X \} \). If \( p = 1 \) and \( q = 0 \), then \( A = 1 \) and \( B = 0 \).

(Sufficiency) Evident, we only need to recall Proposition 4.2.

\[\square\]

**Theorem 4.3** Let \( N \) be a strong negation and let \( \sigma : F(X) \times F(X) \rightarrow [0, 1] \) be given by Eq. (4.2) for all \( A, B \in F(X) \), where \( \mathcal{M} : [0, 1]^{n} \rightarrow [0, 1] \) is an idempotent aggregation function and \( I : [0, 1]^{2} \rightarrow [0, 1] \) is a function. Under these conditions the following items are equivalent:

i) \( \sigma \) is a fuzzy subsethood measure on \( X \) that satisfies (SD6);

ii) \( I \) satisfies (I1), (I8), (I12) with respect to \( N \) and (I15);

iii) \( \sigma(A, B) = \sum_{i=1}^{n}(\Phi(N(A(x_{i})), B(x_{i}))) \) where \( \Phi : [0, 1]^{2} \rightarrow [0, 1] \) is a function such that:

(a’) is non decreasing in the first argument;

(b’) is symmetric;

(c’) \( \Phi(x, y) = 0 \) if and only if \( x = y = 0 \);

(d’) \( \Phi(x, y) = 1 \) if and only if \( x \geq N(y) \).

**Proof.** i)\(\Rightarrow\)ii) Similar to the necessary condition of Theorem 4.2. It only needs to be proven that \( I \) satisfies (I12) with respect to \( N \). We know by Lemma 2.2 that if \( I \) satisfies (I1) and (I12), then it also satisfies (I2). By hypothesis, \( \sigma \) satisfies (SD6), therefore if \( p, q \in [0, 1] \), then we can take the following sets \( A = \{\langle x, A(x) = p \rangle : x \in X \} \) and \( B = \{\langle x, B(x) = q \rangle : x \in X \} \). Since \( \mathcal{M} \) is idempotent we have
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\[ I(N(q), N(p)) = \prod_{i=1}^{n} I(N(q), N(p)) \]
\[ = \sigma(B_N, A_N) = \sigma(A, B) \]
\[ = \prod_{i=1}^{n} I(p, q) \]
\[ = I(p, q). \]

ii)⇒iii) By Theorem 2.2 we know:

\[ \sigma(A, B) \prod_{i=1}^{n} I(A(x_i), B(x_i)) = \prod_{i=1}^{n} \Phi(N(A(x_i)), B(x_i)). \]

iii)⇒i) (a) If \( \sigma(A, B) = 1 \), as \( M \) satisfies (A2) we have

\[ \sigma(A, B) = 1 = \prod_{i=1}^{n} \Phi(N(A(x_i)), B(x_i)), \]

so \( \Phi(N(A(x_i)), B(x_i)) = 1 \) for all \( i \in \{1, \ldots, n\} \). Considering item (d') we have \( N(A(x_i)) \geq N(B(x_i)) \); that is, \( A(x_i) \leq B(x_i) \).

If \( A(x_i) \leq B(x_i) \) for all \( i \in \{1, \ldots, n\} \) we have \( N(A(x_i)) \geq N(B(x_i)) \). By (d') we know that \( \Phi(N(A(x_i)), B(x_i)) = 1 \). Since \( M \) satisfies (A2) we have \( 1 = \prod_{i=1}^{n} \Phi(N(A(x_i)), B(x_i)) = \sigma(A, B) \).

(b) If \( \sigma(A, B) = 0 = \prod_{i=1}^{n} \Phi(N(A(x_i)), B(x_i)) \). Taking into account that \( M \) satisfies (A1) we know that \( \Phi(N(A(x_i)), B(x_i)) = 0 \) for all \( i \in \{1, \ldots, n\} \). Bearing in mind (c') we have \( N(A(x_i)) = B(x_i) = 0 \), therefore \( A(x_i) = 1 \) and \( B(x_i) = 0 \) for all \( i \in \{1, \ldots, n\} \).

If \( A(x_i) = 1 \) and \( B(x_i) = 0 \) for all \( i \in \{1, \ldots, n\} \), then \( N(A(x_i)) = B(x_i) = 0 \). By (c') we have \( \Phi(N(A(x_i)), B(x_i)) = 0 \). Since \( M \) satisfies (A1) it results that

\[ 0 = M(0, \ldots, 0) = \prod_{i=1}^{n} \Phi(N(A(x_i)), B(x_i)) = \sigma(A, B). \]

(c) If \( A \leq B \), then \( A(x_i) \leq B(x_i) \); that is, \( N(A(x_i)) \geq N(B(x_i)) \) for all \( i \in \{1, \ldots, n\} \).
We know that \( \Phi \) satisfies (a’), therefore \( \Phi(N(A(x_i)), C(x_i)) \geq \Phi(N(B(x_i)), C(x_i)) \). As \( \mathcal{M} \) satisfies (A3) we have
\[
\sigma(A, C) = \mathcal{M}(\Phi(N(A(x_i)), C(x_i))) \geq \mathcal{M}(\Phi(N(B(x_i)), C(x_i))) = \sigma(B, C).
\]

Since \( \Phi \) is symmetric and non decreasing in the first argument, it is not non decreasing either in the second. Besides, \( \mathcal{M} \) satisfies (A3), therefore
\[
\sigma(C, A) = \mathcal{M}(\Phi(N(C(x_i)), A(x_i))) \leq \mathcal{M}(\Phi(N(C(x_i)), B(x_i))) = \sigma(C, B).
\]

Lastly, since \( N \) is a strong negation and \( \Phi \) satisfies (b’) we have that
\[
\sigma(B_N, A_N) = \mathcal{M}(\Phi(B(x_i), N(A(x_i)))) = \mathcal{M}(\Phi(N(A(x_i)), B(x_i))) = \sigma(A, B).
\]

**Example 4.7** The function
\[
\Phi(x, y) = \begin{cases} 
1 & \text{if } x \geq N(y), \\
\frac{x \lor y}{2} & \text{if } x < N(y).
\end{cases}
\]
satisfies conditions a’)-d’). If we take \( \mathcal{M}(x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i \), then
\[
\frac{1}{n} \sum_{i=1}^{n} \begin{cases} 
1 & \text{if } A(x_i) \leq B(x_i), \\
\frac{N(A(x_i)) \lor B(x_i)}{2} & \text{if } A(x_i) > B(x_i).
\end{cases}
\]

**4.2.3.2 Construction of FSM on \( X \) from a fixed \( \mathcal{M} \) and functions \( I \) generated from functions \([0, 1]^2 \rightarrow \mathbb{R}^+\)**

In the following theorem we show the way of constructing FSM using functions \( \mathcal{M} \), functions from \([0, 1]^2 \rightarrow [0, 1]\) and functions \([0, 1]^2 \rightarrow \mathbb{R}^+\). That is, the fuzzy subsethood measure may have some properties that varies depending on the aggregation functions considered.
Theorem 4.4 Let $N$ be a strong negation and let $\sigma : F(X) \times F(X) \to [0, 1]$ be given by Eq. (4.2) for all $A, B \in F(X)$, where $\mathcal{M} : [0, 1]^n \to [0, 1]$ is an idempotent function that satisfies (A1), (A2), (A3), such that $\mathcal{M}(x_1, \ldots, x_n) \geq \frac{1}{n} \sum_{i=1}^{n} x_i$ and $I : [0, 1]^2 \to [0, 1]$ is an implication function. Under these conditions the following items are equivalent:

i) $\sigma$ is a fuzzy subsethood measure on $X$ that satisfies (SD6) and (SD10).

ii) $I$ satisfies (I1), (I8), (I12), (I15) and

$$I(x, y) + I(x, N(y)) \geq 1.$$  

iii)

$$\sigma(A, B) = \frac{\sum_{i=1}^{n} G(N(A(x_i)), B(x_i))}{G(N(A(x_i)), B(x_i)) + H(N(A(x_i)), B(x_i))},$$

where functions $G : [0, 1]^2 \to [0, 1]$ and $H : [0, 1]^2 \to \mathbb{R}^+$ are such that:

(a’) $G(x, y) \geq H(x, N(y))$ for all $x, y \in [0, 1]$;

(b’) $G(x, y) = 0$ if and only if $x = y = 0$;

(c’) $H(x, y) = 0$ if and only if $x \geq N(y)$;

(d’) $G$ is non decreasing in both arguments and $H$ is non increasing in both arguments;

(e’) $G$ and $H$ are symmetric;

(f’) $H(0, 0) = 1$.

Proof. i)$\Rightarrow$ii) Similar to the one done in the necessary condition of Theorem 4.2 and in the first part of Theorem 4.3. Let us see that $I$ satisfies $I(p, q) + I(p, N(q)) \geq 1$ for all $p, q \in [0, 1]$.

Let $p, q \in [0, 1]$, let us take sets $A = \{ (x_i, A(x_i) = p) : x_i \in X \}$ and $B = \{ (x_i, B(x_i) = q) : x_i \in X \}$. 


Bearing in mind that $\mathcal{M}$ is idempotent and $\sigma$ satisfies (SD10) we have

$$I(p, q) + I(p, N(q)) = \mathcal{M}(I(p, q)) + \mathcal{M}(I(p, N(q)))$$

$$= \mathcal{M}(I(A(x_i), B(x_i))) + \mathcal{M}(I(A(x_i), N(B(x_i))))$$

$$= \sigma(A, B) + \sigma(A, B_n) \geq 1.$$ 

ii)⇒iii) Taking into account that $\sigma(A, B) = \mathcal{M}(I(A(x_i), B(x_i)))$ we have

$$\sigma(A, B) = \mathcal{M}\left(\frac{G(N(A(x_i)), B(x_i))}{G(N(A(x_i)), B(x_i)) + H(N(A(x_i)), B(x_i))}\right)$$

where functions $G : [0, 1] \to [0, 1]$ and $H : [0, 1] \to \mathbb{R}^+$ are such that (a’), (b’), (c’), (d’), (e’) and (f’) hold.

iii)⇒i) (a) If $\sigma(A, B) = 1 = \mathcal{M}\left(\frac{G(N(A(x_i)), B(x_i))}{G(N(A(x_i)), B(x_i)) + H(N(A(x_i)), B(x_i))}\right)$, then by (A2) we have

$$\frac{G(N(A(x_i)), B(x_i))}{G(N(A(x_i)), B(x_i)) + H(N(A(x_i)), B(x_i))} = 1$$

for all $i \in \{1, \ldots, n\}$. That is,

$$G(N(A(x_i)), B(x_i)) = G(N(A(x_i)), B(x_i)) + H(N(A(x_i)), B(x_i)),$$

then $H(N(A(x_i)), B(x_i)) = 0$. Taking into account that $H$ satisfies (e’) we have $N(A(x_i)) \geq N(B(x_i))$; that is $A(x_i) \leq B(x_i)$.

If $A(x_i) \leq B(x_i)$ for all $i \in \{1, \ldots, n\}$, then $N(A(x_i)) \geq N(B(x_i))$. By (c’) we have $H(N(A(x_i)), B(x_i)) = 0$. Therefore

$$\frac{G(N(A(x_i)), B(x_i))}{G(N(A(x_i)), B(x_i)) + H(N(A(x_i)), B(x_i))} = 1.$$

Bearing in mind that $\mathcal{M}$ satisfies (A2) we have that

$$\sigma(A, B) = \mathcal{M}\left(\frac{G(N(A(x_i)), B(x_i))}{G(N(A(x_i)), B(x_i)) + H(N(A(x_i)), B(x_i))}\right) = \mathcal{M}(1, \ldots, 1) = 1.$$
(b) If \(\sigma(A, B) = 0 = \mathcal{M}\left(\frac{G(N(A(x_i)), B(x_i))}{G(N(A(x_i)), B(x_i)) + H(N(A(x_i)), B(x_i))}\right)\).

By (A1) we have that \(\frac{G(N(A(x_i)), B(x_i))}{G(N(A(x_i)), B(x_i)) + H(N(A(x_i)), B(x_i))} = 0\) for all \(i \in \{1, \ldots, n\}\). Since

\[
G(N(A(x_i)), B(x_i)) + H(N(A(x_i)), B(x_i)) \neq 0,
\]

then \(G(N(A(x_i)), B(x_i)) = 0\). By (b') we have \(N(A(x_i)) = B(x_i) = 0\); that is, \(A(x_i) = 1\) and \(B(x_i) = 0\) for all \(i \in \{1, \ldots, n\}\).

If \(A(x_i) = 1\) and \(B(x_i) = 0\) for all \(i \in \{1, \ldots, n\}\), then \(N(A(x_i)) = B(x_i) = 0\), therefore by (b') \(G(N(A(x_i)), B(x_i)) = 0\). Then

\[
\frac{G(N(A(x_i)), B(x_i))}{G(N(A(x_i)), B(x_i)) + H(N(A(x_i)), B(x_i))} = 0
\]

for all \(i \in \{1, \ldots, n\}\). Since \(\mathcal{M}\) satisfies (A1) we have

\[
\sigma(A, B) = \mathcal{M}\left(\frac{G(N(A(x_i)), B(x_i))}{G(N(A(x_i)), B(x_i)) + H(N(A(x_i)), B(x_i))}\right) = \mathcal{M}(0, \ldots, 0) = 0.
\]

(c) If \(A \leq B\), then \(A(x_i) \leq B(x_i)\); that is \(N(A(x_i)) \geq N(B(x_i))\) for all \(i \in \{1, \ldots, n\}\). By (d') \(G\) is non decreasing in both arguments and \(H\) is no increasing in both arguments, therefore

\[
G(N(A(x_i)), C(x_i)) \geq G(N(B(x_i)), C(x_i)) \text{ and } H(N(B(x_i)), C(x_i)) \geq H(N(A(x_i)), C(x_i)).
\]

Therefore,

\[
G(N(A(x_i)), C(x_i))H(N(B(x_i)), C(x_i)) \geq G(N(B(x_i)), C(x_i))H(N(A(x_i)), C(x_i)); \text{ that is,}
\]

\[
G(N(A(x_i)), C(x_i))G(N(B(x_i)), C(x_i)) + G(N(A(x_i)), C(x_i))H(N(B(x_i)), C(x_i)) \geq
\]

\[
G(N(A(x_i)), C(x_i))G(N(B(x_i)), C(x_i)) + G(N(B(x_i)), C(x_i))H(N(A(x_i)), C(x_i)).
\]

Then

\[
G(N(A(x_i)), C(x_i))[G(N(B(x_i)), C(x_i)) + H(N(B(x_i)), C(x_i))] \geq
\]

\[
G(N(B(x_i)), C(x_i))[G(N(A(x_i)), C(x_i)) + H(N(A(x_i)), C(x_i))],
\]
therefore,

\[
\frac{G(N(A(x_i)), C(x_i))}{G(N(A(x_i)), C(x_i)) + H(N(A(x_i)), C(x_i))} \geq \frac{G(N(B(x_i)), C(x_i))}{G(N(B(x_i)), C(x_i)) + H(N(B(x_i)), C(x_i))},
\]

that is, bearing in mind that \( M \) satisfies (A3) we have that \( \sigma(A, C) \geq \sigma(B, C) \).

If \( A \leq B \), then \( A(x_i) \leq B(x_i) \). As \( G \) and \( H \) fulfill (d’) we have that

\[
G(N(C(x_i)), A(x_i)) \leq G(N(C(x_i)), B(x_i)) \quad \text{and} \quad H(N(C(x_i)), B(x_i)) \leq H(N(C(x_i)), A(x_i)).
\]

Therefore,

\[
G(N(C(x_i)), A(x_i))H(N(C(x_i)), B(x_i)) \leq G(N(C(x_i)), B(x_i))H(N(C(x_i)), A(x_i));
\]

that is,

\[
G(N(C(x_i)), A(x_i))G(N(C(x_i)), B(x_i)) + G(N(C(x_i)), A(x_i))H(N(C(x_i)), B(x_i)) \leq
\]

\[
G(N(C(x_i)), A(x_i))G(N(C(x_i)), B(x_i)) + G(N(C(x_i)), B(x_i))H(N(C(x_i)), A(x_i)).
\]

Then

\[
G(N(C(x_i)), A(x_i))[G(N(C(x_i)), B(x_i)) + H(N(C(x_i)), B(x_i))] \leq
\]

\[
G(N(C(x_i)), B(x_i))[G(N(C(x_i)), A(x_i)) + H(N(C(x_i)), A(x_i))],
\]

therefore,

\[
\frac{G(N(C(x_i)), A(x_i))}{G(N(C(x_i)), A(x_i)) + H(N(C(x_i)), A(x_i))} \leq \frac{G(N(C(x_i)), B(x_i))}{G(N(C(x_i)), B(x_i)) + H(N(C(x_i)), B(x_i))};
\]

that is \( \sigma(C, A) \leq \sigma(C, B) \).
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Since $G$ and $H$ fulfill (e') and $N$ is a strong negation we have that

$$\sigma(B_N, A_N) = \frac{\prod_{i=1}^{n} G(B(x_i), N(A(x_i)))}{G(B(x_i), N(A(x_i))) + H(B(x_i), N(A(x_i)))}$$

We know by (a') that $G(A(x_i), B(x_i)) \geq H(A(x_i), N(B(x_i)))$ for all $i \in \{1, \ldots, n\}$, then bearing in mind that $N$ is involutive we have $G(N(A(x_i)), B(x_i)) \geq H(N(A(x_i)), N(B(x_i)))$ and $G(N(A(x_i)), N(B(x_i))) \geq H(N(A(x_i)), B(x_i))$, therefore

$$G(N(A(x_i)), B(x_i))G(N(A(x_i)), N(B(x_i))) \geq H(N(A(x_i)), B(x_i))H(N(A(x_i)), N(B(x_i))),$$

then

$$G(N(A(x_i)), B(x_i))G(N(A(x_i)), N(B(x_i))) + G(N(A(x_i)), B(x_i))H(N(A(x_i)), N(B(x_i))) \geq$$

$$G(N(A(x_i)), B(x_i))H(N(A(x_i)), N(B(x_i))) + H(N(A(x_i)), B(x_i))H(N(A(x_i)), N(B(x_i))),$$

that is:

$$G(N(A(x_i)), B(x_i)) [G(N(A(x_i)), N(B(x_i))) + H(N(A(x_i)), N(B(x_i)))] \geq$$

$$H(N(A(x_i)), N(B(x_i))) [G(N(A(x_i)), B(x_i)) + H(N(A(x_i)), B(x_i))].$$

By (b') and (f') we have that $G(N(A(x_i)), N(B(x_i))) + H(N(A(x_i)), N(B(x_i))) \neq 0$ and $G(N(A(x_i)), B(x_i)) + H(N(A(x_i)), B(x_i)) \neq 0$. Therefore

$$\frac{G(N(A(x_i)), B(x_i))}{G(N(A(x_i)), B(x_i)) + H(N(A(x_i)), B(x_i))} \geq$$

$$\frac{G(N(A(x_i)), N(B(x_i)))}{H(N(A(x_i)), N(B(x_i))) + G(N(A(x_i)), N(B(x_i)))} =$$

$$1 - \frac{G(N(A(x_i)), N(B(x_i))) + H(N(A(x_i)), N(B(x_i)))}{G(N(A(x_i)), N(B(x_i))) + H(N(A(x_i)), N(B(x_i)))}.$$
that is:
\[
\frac{G(N(A(x_i)), B(x_i))}{G(N(A(x_i)), B(x_i)) + H(N(A(x_i)), B(x_i))} + \frac{G(N(A(x_i)), N(B(x_i)))}{G(N(A(x_i)), N(B(x_i))) + H(N(A(x_i)), N(B(x_i)))} \geq 1.
\]

Therefore, since \(\mathcal{M}(x_1, \ldots, x_n) \geq \frac{1}{n} \sum_{i=1}^{n} x_i\), we have

\[
\sigma(A, B) + \sigma(A, B_N) = \frac{1}{n} \sum_{i=1}^{n} \left( \begin{array}{l}
\frac{G(N(A(x_i)), B(x_i))}{G(N(A(x_i)), B(x_i)) + H(N(A(x_i)), B(x_i))} \\
+ \frac{G(N(A(x_i)), N(B(x_i)))}{G(N(A(x_i)), N(B(x_i))) + H(N(A(x_i)), N(B(x_i)))}
\end{array} \right) \\
\geq \frac{1}{n} \sum_{i=1}^{n} \left( \begin{array}{l}
\frac{G(N(A(x_i)), B(x_i))}{G(N(A(x_i)), B(x_i))} \\
+ \frac{G(N(A(x_i)), N(B(x_i)))}{G(N(A(x_i)), N(B(x_i))) + H(N(A(x_i)), N(B(x_i)))}
\end{array} \right)
\geq 1.
\]

**Example 4.8** Let us take:

\[
G(x, y) = \begin{cases} 
1 & \text{if } x \geq 1 - y, \\
 x \lor y & \text{if } x < 1 - y.
\end{cases}
H(x, y) = \begin{cases} 
0 & \text{if } x \geq 1 - y, \\
1 - x \land 1 - y & \text{if } x < 1 - y.
\end{cases}
\]

Under these conditions, if we consider \(\mathcal{M}(x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i\) and \(N\) the standard negation then:

\[
\sigma(A, B) = \frac{1}{n} \sum_{i=1}^{n} \left( \begin{array}{l}
1 & \text{if } A(x_i) \leq B(x_i), \\
1 - A(x_i) \lor B(x_i) & \text{if } A(x_i) > B(x_i).
\end{array} \right)
\]

The next corollary shows a construction of FSM on \(X\) from functions \(f : [0, 1] \to \mathbb{R}\) and \(I : [0, 1]^2 \to \mathbb{R}^+\).
Corollary 4.8 Under the same conditions of Theorem 4.4 if \( f : [0, 1] \to \mathbb{R} \) is a continuous, strictly increasing and convex function\(^1\) then

\[
\sigma(A, B) = f^{-1}\left( \frac{1}{n} \sum_{i=1}^{n} f \left( \frac{G(N(A(x_i)), B(x_i))}{G(N(A(x_i)), B(x_i)) + H(N(A(x_i)), B(x_i))} \right) \right)
\]

is a fuzzy subsethood measure on \( X \) that satisfies axioms (SD6) and (SD10).

**Proof.** We need to take into account Theorem 4.4 and item (a) in Section 6.1 of [20] by which if \( f \) is convex we have

\[
M_k(x_1, \ldots, x_n) > \frac{1}{n} \sum_{i=1}^{n} x_i.
\]

And also consider Theorem 5 in [20], by which:

\[
M(x_1, \ldots, x_n) = f^{-1}\left( \frac{1}{n} \sum_{i=1}^{n} f(x_i) \right).
\]

\( \square \)

**Example 4.9** This example presents a construction of a fuzzy subsethood measure on \( X \) from functions \( f : [0, 1] \to \mathbb{R} \) and functions \( I \) generated from functions \([0, 1]^2 \to \mathbb{R}^+\).

1) Take \( f = x^\lambda \), for \( \lambda > 1 \). Then:

\[
\sigma(A, B) = \left( \frac{1}{n} \sum_{i=1}^{n} \left( \begin{cases} 1 & \text{if } A(x_i) \leq B(x_i) \\ 1 - A(x_i) \lor B(x_i) & \text{if } A(x_i) > B(x_i) \end{cases} \right)^\lambda \right)^\frac{1}{\lambda}.
\]

4.2.3.3 Using automorphisms to construct FSM on \( X \) with a fixed \( \mathcal{M} \)

We consider now those FSM for which the identity \( \sigma(1, A) = \mathcal{M}(A) \) holds. Note that \( \sigma(1, A) \) is a probability function when \( A \) is a crisp set, if and only if \( \mathcal{M} \) is the arithmetic mean.

**Theorem 4.5** Let \( N \) be a strong negation and let \( \sigma : F(X) \times F(X) \to [0, 1] \), be given by Eq. (4.2) for all \( A, B \in F(X) \), where \( \mathcal{M} : [0, 1]^n \to [0, 1] \) is an idempotent aggregation function and \( I : [0, 1]^2 \to [0, 1] \) is a function that satisfies (I7) and (I13). Under these conditions, the following statements are equivalent:

\(^1\)A function is said to be convex in an interval if and only if for all \( x \) and \( y \) on the interval we have: \( f(tx + (1-t)y) < tf(x) + (1-t)f(y) \), for \( 0 < t < 1 \).
i) \( \sigma \) is a fuzzy subsethood measure on \( X \) that satisfies (SD6), \( \sigma(1, A) = \mathcal{M}(A) \) and (SD12);

ii) \( I \) satisfies (I2), (I6), (I8) and (I12) with respect to \( N \);

iii) There exists an automorphism \( \varphi \) of the unit interval such that

\[
\begin{align*}
\sigma(A, B) &= \frac{1}{n} \sum_{i=1}^{n} (\varphi(N(A(x_i))) + \varphi(B(x_i)) \land 1)) \\
N(x) &= \varphi^{-1}(1 - \varphi(x)) \text{ for all } x \in [0, 1].
\end{align*}
\]

**Proof.** i)\(\Rightarrow\)ii) (I2). Let \( p, q, r \in [0, 1] \) such that \( p \leq q \). Let us take the following sets:

\[
\begin{align*}
A &= \{ \langle x, A(x) = p \rangle : x \in X \} \\
B &= \{ \langle x, B(x) = q \rangle : x \in X \} \\
C &= \{ \langle x, C(x) = r \rangle : x \in X \}.
\end{align*}
\]

Evidently \( A \leq B \). Since \( \sigma \) satisfies (c) and \( \mathcal{M} \) is idempotent, then by Eq. (4.2) we have

\[
\begin{align*}
I(r, p) &= \mathcal{M}(I(r, p), \ldots, I(r, p)) = \sigma(C, A) \leq \\
\sigma(C, B) &= \mathcal{M}(I(r, q), \ldots, I(r, q)) = I(r, q).
\end{align*}
\]

(I6). Under the same conditions of Theorem 4.4 and bearing in mind that \( \sigma(1, A) = \mathcal{M}(A) \), we have: \( I(1, p) = \mathcal{M}(I(1, p), \ldots, I(1, p)) = \sigma(1, A) = \mathcal{M}(A) = \mathcal{M}(p, \ldots, p) = p. \)

(I8). Let \( p, q \in [0, 1] \) such that \( p \leq q \). Let us take the following sets: \( A = \{ \langle x, A(x) = p \rangle : x \in X \} \) and \( B = \{ \langle x, B(x) = q \rangle : x \in X \} \). Evidently, \( A \leq B \). Since \( \sigma \) satisfies (a) we have: \( I(p, q) = \mathcal{M}(I(p, q), \ldots, I(p, q)) = \sigma(A, B) = 1. \)

Let \( p, q \in [0, 1] \), let us take the following sets: \( A = \{ \langle x, A(x) = p \rangle : x \in X \} \) and \( B = \{ \langle x, B(x) = q \rangle : x \in X \} \), if \( I(p, q) = 1. \) Then, since \( \sigma \) satisfies (a) and \( \mathcal{M} \) satisfies (A2) we have: \( \sigma(A, B) = \mathcal{M}(I(p, q), \ldots, I(p, q)) = \mathcal{M}(1, \ldots, 1) = 1 \), then \( A \leq B \), therefore \( p \leq q. \)
(I12). By hypothesis \( \sigma \) satisfies \((SD6)\), then

\[
I(N(p), N(p)) = \mathcal{M}(I(N(q), N(p)), \ldots, I(N(q), N(p)))
\]

\[
= \sigma(B_N, A_N)
\]

\[
= \sigma(A, B)
\]

\[
= \mathcal{M}(I(p, q), \ldots, I(p, q))
\]

\[
= I(p, q).
\]

ii) \( \Rightarrow \) iii) By Lemma 2.2 we have that since \( I \) satisfies \((I6)\) and \((I12)\), it also satisfies \((I9)\). Besides, since \( I \) satisfies \((I8)\), it satisfies \((I11)\). Then, \( I \) satisfies among others the following properties: \((I2)\), \((I7)\), \((I9)\), \((I11)\) and \((I13)\). Besides, Theorem 7 in [16] states that:

There exists an automorphism \( \varphi \) of the unit interval such that

\[
I(x, y) = \varphi^{-1}\left((\varphi(N(x)) + \varphi(y)) \land 1\right)
\]

and

\[
N(x) \geq \varphi^{-1}(1 - \varphi(x)).
\]

Since \( I \) satisfies \((I8)\), by Corollary 4 in [16] we have \( N(x) = \varphi^{-1}(1 - \varphi(x)) \). Therefore, under our conditions

\[
\left\{
\begin{align*}
\sigma(A, B) &= \mathcal{M}^{n}_{i=1}\left(\varphi^{-1}\left((\varphi(N(A(x_i))) + \varphi(B(x_i))) \land 1\right)\right) \\
\text{and} \\
N(x) &= \varphi^{-1}(1 - \varphi(x)) \text{ for all } x \in [0, 1].
\end{align*}
\right.
\]

iii) \( \Rightarrow \) i) (a) If \( \sigma(A, B) = 1 \), since \( \mathcal{M} \) satisfies \((A2)\) we have

\[
\varphi^{-1}\left((\varphi(N(A(x_i))) + \varphi(B(x_i))) \land 1\right) = 1
\]

for all \( i \in \{1, \ldots, n\} \). Besides \( N(x) = \varphi^{-1}(1 - \varphi(x)) \), therefore \( \varphi(N(A(x_i))) + \varphi(B(x_i)) = 1 - \varphi(A(x_i)) + \varphi(B(x_i)) \geq 1 \), then \( \varphi(B(x_i)) \geq \varphi(A(x_i)) \), therefore \( A \leq B \).
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If \( A \leq B \), then \( \varphi(A(x_i)) \leq \varphi(B(x_i)) \), therefore \( 1 - \varphi(A(x_i)) + \varphi(B(x_i)) \geq 1 \); that is, \( \varphi^{-1}(N(\varphi(A(x_i)))) + \varphi(B(x_i)) \geq 1 \), then

\[
(\varphi^{-1}(N(\varphi(A(x_i)))) + \varphi(B(x_i))) \land 1 = 1.
\]

Since \( \mathcal{M} \) satisfies (A2) we have

\[
\sigma(A, B) = \mathcal{M}(\varphi^{-1}((\varphi(N(A(x_i))) + \varphi(B(x_i))) \land 1)) = \mathcal{M}(\varphi^{-1}(1)) = 1.
\]

(b) If \( \sigma(A, B) = 0 = \mathcal{M}(\varphi^{-1}((\varphi(N(A(x_i))) + \varphi(B(x_i))) \land 1)) \), since \( \mathcal{M} \) satisfies (A1) we have \( \varphi(N(A(x_i))) + \varphi(B(x_i)) = 0 \), then \( \varphi(N(A(x_i))) = 0 \) and \( \varphi(B(x_i)) = 0 \) for all \( i \in \{1, \ldots, n\} \), therefore \( 0 = N(A(x_i)) \) and \( B(x_i) = 0 \); that is, \( A(x_i) = 1 \) and \( B(x_i) = 0 \).

If \( A = \{\langle x, A(x) = 1 \rangle : x \in X\} \), then \( B = \{\langle x, B(x) = 0 \rangle : x \in X\} \). As \( \mathcal{M} \) satisfies (A1) we have

\[
\sigma(A, B) = \sigma(1, 0) = \mathcal{M}(\varphi^{-1}((\varphi(0) + \varphi(0)) \land 1)) = \mathcal{M}(0, \ldots, 0) = 0.
\]

(c) If \( A \leq B \), then \( A_N \geq B_N \). Since \( \mathcal{M} \) satisfies (A3) we have

\[
\sigma(A, C) = \mathcal{M}(\varphi^{-1}((\varphi(N(A(x_i))) + \varphi(C(x_i))) \land 1)) \geq \mathcal{M}(\varphi^{-1}((\varphi(N(B(x_i))) + \varphi(C(x_i))) \land 1)) = \sigma(B, C).
\]

\[
\sigma(C, A) = \mathcal{M}(\varphi^{-1}((\varphi(N(C(x_i))) + \varphi(A(x_i))) \land 1)) \leq \mathcal{M}(\varphi^{-1}((\varphi(N(C(x_i))) + \varphi(B(x_i))) \land 1)) = \sigma(C, B).
\]

With respect to axiom (SD6) we have

\[
\sigma(B_N, A_N) = \mathcal{M}(\varphi^{-1}((\varphi(N(N(B(x_i))))) + \varphi(N(A(x_i)))) \land 1)) = \mathcal{M}(\varphi^{-1}((\varphi(N(A(x_i))) + \varphi(B(x_i))) \land 1)) = \sigma(A, B).
\]
On the other hand,

\[
\sigma(1, A) = \bigwedge_{i=1}^{n} (\varphi^{-1} (\varphi(N(1)) + \varphi(A(x_i))) \land 1)) \\
= \bigwedge_{i=1}^{n} (\varphi^{-1} (\varphi(0) + \varphi(A(x_i))) \land 1)) \\
= \bigwedge_{i=1}^{n} (\varphi^{-1} (\varphi(A(x_i))) \land 1)) \\
= \bigwedge_{i=1}^{n} (\varphi^{-1} (\varphi(A(x_i)))) \\
= M(A).
\]

Lastly, \((SD12)\) holds because:

1) If \(A(x_i) \leq e\), then \(N(A(x_i)) \geq e\), therefore \(\varphi(N(A(x_i))) \geq \varphi(e)\). Thus

\[
\varphi^{-1} (\varphi(N(A(x_i))) + \varphi(B(x_i))) \land 1) \geq \\
\varphi^{-1} (\varphi(e) + \varphi(B(x_i))) \land 1) \geq \\
\varphi^{-1}(\varphi(e)) = e.
\]

2) If \(B(x_i) > e\), then

\[
\varphi^{-1} (\varphi(N(A(x_i))) + \varphi(B(x_i))) \land 1) \geq \\
\varphi^{-1} (\varphi(N(A(x_i))) + \varphi(e)) \land 1) \geq \\
\varphi^{-1}(\varphi(e)) = e.
\]

Since \(\mathcal{M}\) satisfies \((A3)\) and is idempotent we have that in these conditions; that is, if \(A\) is weakly included in \(B\), then

\[
\sigma(A, B) = \varphi^{-1} (\varphi(N(A(x_i))) + \varphi(B(x_i))) \land 1) \geq \mathcal{M}(e, \ldots, e) = e.
\]

**Example 4.10**  A) We consider the arithmetic mean as the aggregation function for the following examples.

1) \(\varphi(x) = x, N(x) = 1 - x\), \(\sigma_g\) given in Eq. [1.2].
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2) \( \varphi(x) = x^2, N(x) = (1 - x^2)^{\frac{1}{2}}, \)

\[
\sigma(A, B) = \frac{1}{n} \sum_{i=1}^{n} \left( 1 \wedge \left(1 - A^2(x_i) + B^2(x_i)\right) \right)^{\frac{1}{2}}.
\]

B) In the following example we take the automorphism \( \varphi(x) = x \) (and hence the negation \( N(x) = 1 - x \)) and the aggregation:

\[
\mathcal{M}(x_1, \ldots, x_n) = \begin{cases} 
\vee (x_1, \ldots, x_n) & \text{when for all } i \in \{1, \ldots, n\} \ x_i \in [0, \alpha], \\
\wedge (x_1, \ldots, x_n) & \text{when for all } i \in \{1, \ldots, n\} \ x_i \in [\alpha, 1], \\
\alpha & \text{elsewhere.}
\end{cases}
\]

\[
\sigma(A, B) = \begin{cases} 
\bigvee_{i=1}^{n} \left(1 - A(x_i) + B(x_i)\right) & \text{if } 1 \wedge \left(1 - A(x_i) + B(x_i)\right) \in [0, \alpha], \\
\bigwedge_{i=1}^{n} \left(1 - A(x_i) + B(x_i)\right) & \text{if } 1 \wedge \left(1 - A(x_i) + B(x_i)\right) \in [\alpha, 1], \\
\alpha & \text{elsewhere.}
\end{cases}
\]

### 4.2.3.4 Construction of FSM on \( X \) from functions \( f : [0, 1] \rightarrow \mathbb{R} \) and automorphisms

**Corollary 4.9** The following items hold.

i) If a function \( f : [0, 1] \rightarrow \mathbb{R} \) is continuous and strictly increasing and \( \varphi \) is an automorphism of the unit interval, with \( N(x) = \varphi^{-1}(1 - \varphi(x)) \) for all \( x \in [0, 1] \), then

\[
\sigma(A, B) = f^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} f \left( \varphi^{-1}((\varphi(N(A(x_i))) + \varphi(B(x_i))) \wedge 1) \right) \right) \quad (4.3)
\]

is a fuzzy subsethood measure on \( X \) that satisfies axioms (SD6) and (SD12).

ii) Under the same conditions of the previous item, if \( f \) is convex and \( N(x) = 1 - x \) for all \( x \in [0, 1] \), then \( \sigma \) satisfies (SD10) and also Eq. (4.3) holds.
Proof. i) Since \( f \) is strictly increasing and continuous and by Theorem 4 in [20], which states that:

\[
M(x_1, \ldots, x_n) = f^{-1}\left(\frac{1}{n} \sum_{i=1}^{n} f(x_i)\right)
\]

we have that it satisfies (A1)-(A3) and it is symmetric and idempotent. Setting this \( M \) in item iii) of the Theorem 4.5 we have proven that our expression is a fuzzy subsethood measure on \( X \) that satisfies (SD6).

ii) We need to recall Theorem 4.5, the comments made in [20] subsection 6.1 item (b) and item iv) of Corollary 4.5.  

\[\text{Corollary 4.10} \]

The following items hold.

i) If \( \varphi \) is an automorphism of the unit interval, with \( N(x) = \varphi^{-1}(1 - \varphi(x)) \) for all \( x \in [0, 1] \), then

\[
\sigma(A, B) = \varphi^{-1}\left(\frac{1}{n} \sum_{i=1}^{n} (\varphi(N(A(x_i))) + \varphi(B(x_i))) \land 1\right)
\]  \hspace{1cm} (4.4)

is a fuzzy subsethood measure on \( X \) that satisfies (SD6) and (SD12).

ii) If \( \varphi \) is an automorphism of the unit interval such that \( \varphi(x) + \varphi(y) = 1 \) if and only if \( x + y = 1 \), with \( N(x) = 1 - x \) for all \( x \in [0, 1] \), then \( \sigma \) given by Eq. (4.4) is a fuzzy subsethood measure on \( X \) that also satisfies (SD10).

Proof.

i) Analogous to Corollary 4.9.

ii) We only need to recall Theorem 7 in [20] and item iv) of Corollary 4.5.  

\[\blacksquare\]
4.3 Fuzzy subsethood measures and penalty functions

Penalty functions [10,22] have been defined as a measure of deviation from a consensus value, or a penalty for not having a consensus, in situations where several input values are aggregated into a single output. In other words, if there is a disagreement, the penalty is applied in a way that the more inputs disagree with the output, the larger (in general) is the penalty. Penalty functions can be used in many disciplines and applications such as fuzzy rule based systems and classification systems, pattern recognition, image processing, expert and decision support systems, information retrieval, etc. Furthermore, they have been widely applied in image reduction or decision making problems [18,63]. A further study on penalty functions can be found in [14].

In this section we present a method to construct penalty functions from FSM.

**Definition 4.2** Let $\mathbb{R}$ and $\mathbb{R}^+$ be the set of real numbers and the set of positive real numbers, respectively. For any closed, nonempty interval $I \subseteq \mathbb{R}$, the function

$$P : I^{n+1} \rightarrow \mathbb{R}^+$$

is a penalty function if and only if there exists $c \in \mathbb{R}^+$ such that:

(P1) $P(\bar{x}, y) \geq c$, for all $\bar{x} \in I^n, y \in I$;

(P2) $P(\bar{x}, y) = c$ if and only if $x_i = y$, for all $i = 1, \ldots, n$; and

(P3) For every fixed $\bar{x} \in I^n$, the set of minimizers of $P(\bar{x}, \_)$ is a nonempty interval.

We denote by $\text{Minz}(P(\bar{x}, \_))$ the set of minimizer of $P(\bar{x}, \_)$, that is:

$$\text{Minz}(P(\bar{x}, \_)) = \{ y \in I : P(\bar{x}, y) \leq P(\bar{x}, z), \text{ for each } z \in I \}, \quad (4.5)$$

$^2P(\bar{x}, \_): I \rightarrow \mathbb{R}^+$ is $P(\bar{x}, \_)(y) = P(\bar{x}, y)$. 

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4. A new class of fuzzy subsethood measures

Notation: for each $y \in [0, 1]$, the fuzzy set $A \in F(X)$ such that $A(x_i) = y$ for each $x_i \in X$, will be denoted by $y$.

**Lemma 4.1** Let $M : [0, 1]^n \to [0, 1]$ be a sub-idempotent aggregation function. Then $M$ satisfies

$$M(x_1 \wedge y, \ldots, x_n \wedge y) \leq M(x_1, \ldots, x_n) \wedge y.$$  

**Proof.** Since $M$ is increasing and sub-idempotent we have that for each $(x_1, \ldots, x_n) \in [0, 1]^n$ and $y \in [0, 1]$,

$$M(x_1 \wedge y, \ldots, x_n \wedge y) \leq M(x_1, \ldots, x_n) \text{ and } M(x_1 \wedge y, \ldots, x_n \wedge y) \leq M(y, \ldots, y) \leq y.$$  

So, $M(x_1 \wedge y, \ldots, x_n \wedge y) \leq M(x_1, \ldots, x_n) \wedge y$.  

**Remark 4.2** Note that an aggregation function is sub-idempotent if and only if it is less than or equal to the maximum. In fact, if $M(x_1, \ldots, x_n) \leq \max(x_1, \ldots, x_n)$ then clearly $M(y, \ldots, y) \leq \max(y, \ldots, y) = y$ for every $y \in [0, 1]$. Conversely, if $M(y, \ldots, y) \leq y$ for every $y \in [0, 1]$, then, from the monotonicity of $M$:

$$M(x_1, \ldots, x_n) \leq M(\max(x_1, \ldots, x_n), \ldots, \max(x_1, \ldots, x_n)) \leq \max(x_1, \ldots, x_n).$$  

Taking into account Def. 4.2, the following propositions show how we can construct penalty functions from our new class of FSM.

**Proposition 4.3** Let $M : [0, 1]^n \to [0, 1]$ be a sub-idempotent aggregation function and let $\sigma : F(X) \times F(X) \to [0, 1]$ be a fuzzy subsethood measure on $X$ such that

1. If $B \subseteq A$ then $\sigma(A, B) \leq M(B(x_1), \ldots, B(x_n))$, and
2. $\sigma(1, B) = M(B(x_1), \ldots, B(x_n))$. 

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Then function $P : [0, 1]^{n+1} \rightarrow \mathbb{R}^+$ defined by:

$$P(\bar{z}, y) = 1 - \sigma(A \lor y, A \land y),$$

where $A(x_i) = z_i$ for each $x_i \in X$, is a penalty function. In addition, $\minz(P(\bar{z}, \_))$ is a sub-interval of $[\mathcal{M}(\bar{z}), 1]$.

**Proof.** Property $(P1)$ is satisfied because the co-domain of $\sigma$ is $[0, 1]$. On the other hand, let $\bar{z} \in [0, 1]^n$, $y \in [0, 1]$ and $A \in \mathcal{F}(X)$ such that $A(x_i) = z_i$. Then, by Definition 4.1 (a),

$$P(\bar{z}, y) = 0 \iff \sigma(A \lor y, A \land y) = 1 \iff A \lor y \subseteq A \land y \iff A = y \iff z_i = y \text{ for each } i = 1, \ldots, n.$$ 

Therefore, $(P2)$ holds.

Since $A \land y \subseteq A \lor y$ then by Lemma 4.1 and conditions (c1) and (c2) we have that

$$\sigma(A \lor y, A \land y) \leq \mathcal{M}(A(x_1) \land y, \ldots, A(x_n) \land y)$$

$$\leq \mathcal{M}(A(x_1), \ldots, A(x_n)) \land y \leq \mathcal{M}(A(x_1), \ldots, A(x_n))$$

$$= \sigma(1, A) = \sigma(A \lor 1, A \land 1).$$

Therefore, $1 \in \minz(P(\bar{z}, \_))$ and so $\minz(P(\bar{z}, \_))$ is not empty.

Let $y \in \minz(P(\bar{z}, \_))$. Then $P(\bar{z}, 1) = P(\bar{z}, y)$. So, $\sigma(A \lor 1, A \land 1) = \sigma(A \lor y, A \land y)$.

Thus, by (c2), $\sigma(A \lor y, A \land y) = \mathcal{M}(A(x_1), \ldots, A(x_n))$. Thus, because $A \land y \subseteq A \lor y$ and by (c1) and Lemma 4.1, we have that

$$\mathcal{M}(A(x_1), \ldots, A(x_n)) = \sigma(A \lor y, A \land y)$$

$$\leq \mathcal{M}(A(x_1) \land y, \ldots, A(x_n) \land y) \leq \mathcal{M}(A(x_1), \ldots, A(x_n)) \land y.$$ 

So $\mathcal{M}(A(x_1), \ldots, A(x_n)) \leq y$ and therefore,

$$\mathcal{M}(A(x_1) \land y, \ldots, A(x_n) \land y) = \mathcal{M}(A(x_1), \ldots, A(x_n)). \quad (4.6)$$

Let $y \in \minz(P(\bar{z}, \_))$ and $y' > y$. So, $A \land y' \subseteq A \land y'$. Thus, by Definition 4.1 (c),
condition (c1), Lemma 4.1 and Equation (4.6), we have that

\[
\mathcal{M}(A(x_1), \ldots, A(x_n)) \geq \sigma(A \lor y', A \land y) \geq \sigma(A \lor y', A \land y) \\
\geq \sigma(1, A \land y) = \mathcal{M}(A(x_1) \land y, \ldots, A(x_n) \land y) = \mathcal{M}(A(x_1), \ldots, A(x_n)) .
\]

Therefore, \( \sigma(A \lor y', A \land y) = \mathcal{M}(A(x_1), \ldots, A(x_n)) = \sigma(A \lor y', A \land y) \), i.e. \( y' \in \text{Minz}(\mathcal{P}(\bar{z}, -)) \).

Hence, there is \( y \in [\mathcal{M}(z_1, \ldots, z_n), 1] \) such that \( \text{Minz}(\mathcal{P}(\bar{z}, -)) = (y, 1) \) or \( \text{Minz}(\mathcal{P}(\bar{z}, -)) = [y, 1] \), i.e. \( \text{Minz}(\mathcal{P}(\bar{z}, -)) \) is an interval. Therefore, \( P \) satisfies (P3). \[\blacksquare\]

**Lemma 4.2** Let \( I : [0, 1]^2 \to [0, 1] \) be a function satisfying (I1), (I2), (I6) and (I14), i.e., if \( x < 1 \) then \( I(x, 0) > 0 \). Then \( I \) satisfies (I15).

**Proof.** \((\Rightarrow)\) If \( I(x, y) = 0 \) then by (I2), \( I(x, 0) = 0 \). So, by (I14), \( x = 1 \). On the other hand, if \( I(x, y) = 0 \) then by (I1), \( I(1, y) = 0 \). So, by (I6), \( y = 0 \).

\((\Leftarrow)\) Straightforward from (I6). \[\blacksquare\]

**Proposition 4.4** Let \( \mathcal{M} : [0, 1]^n \to [0, 1] \) be a sub-idempotent aggregation function and let \( I : [0, 1]^2 \to [0, 1] \) be a function that satisfies (I1), (I2), (I6), (I8), (I14) and

\((I16)\) If \( x > y \) then \( I(x, y) \leq y \).

Then, \( P : [0, 1]^{n+1} \to [0, 1] \) defined by \( P(\bar{z}, y) = 1 - \sigma(A \lor y, A \land y) \), where \( A(x_i) = z_i \) for each \( x_i \in X \) is a penalty function for \( \sigma : F(X) \times F(X) \to [0, 1] \) which is defined by

\[
\sigma(A, B) = \mathcal{M}(I(A(x_1), B(x_1)), \ldots, I(A(x_n), B(x_n))).
\]

**Proof.** By Lemma 4.2 and Proposition 4.2 \( \sigma \) is a subsethood fuzzy measure.

On the other hand, let \( A, B \in F(X) \). If \( B \subseteq A \) then \( B(x_i) \leq A(x_i) \) for each \( x_i \in X \). So, by (I16), \( I(A(x_i), B(x_i)) \leq B(x_i) \) for each \( x_i \in X \). Therefore, because \( \mathcal{M} \) is increasing,
\[ \sigma(A, B) = \mathcal{M}(I(A(x_1), B(x_i)), \ldots, I(A(x_i), B(x_i))) \leq \mathcal{M}(B(x_1), \ldots, B(x_n)), \text{ i.e. } \sigma \text{ satisfies (c1)}. \]

By (J6), \( I(1, B(x_i)) = B(x_i) \) for each \( x_i \in X \). Thus,

\[ \sigma(1, B) = \mathcal{M}(I(1, B(x_1)), \ldots, I(1, B(x_n))) = \mathcal{M}(B(x_1), \ldots, B(x_n)). \]

So, \( \sigma \) satisfies (c2). Therefore, by Proposition 4.3, \( P \) is a penalty function. ■

### 4.4 General considerations

In this chapter we provided the definition of the new class of FSM (Definition 4.1). We established the relation between our axiomatization and other FSM and also presented some constructions to obtain \( \sigma \) according to Def. 4.1. The main constructions were presented in Propositions 4.1 and 4.2, but we also built several others to broaden the study on the functions that can be used to construct those FSM and to verify the properties demanded in each particular case. We concluded the chapter defining penalty functions constructed from our new class of FSM.

We carry out the study of such FSM in the next chapter, where we will study distance measures, similarity measures, fuzzy entropy and indexes constructed from \( \sigma \) in the sense of Definition 4.1.
Chapter 5

Constructing measures from the new class of FSM

Distance measure, similarity measure and fuzzy entropy are three basic concepts in fuzzy set theory. They seem to be useful resources whenever we need to deal with uncertain information and they have been applied in many real-life problems involving, for instance, decision making, pattern recognition and medical diagnosis.

In this chapter, we study some measures constructed from FSM in the sense of Definition 4.1.

5.1 Distance measure and FSM

The concept of distance measure describes the difference between fuzzy sets. In 1992, Liu Xuecheng [82] presented the axiomatic definition of distance measure and discussed the relationships between distance measure, fuzzy entropy and similarity measure.

**Definition 5.1** A function \( D : F(X) \times F(X) \to [0, 1] \) is called a distance measure if \( D \) satisfies the following properties:
(D1) \(D(A, B) = D(B, A)\) for all \(A, B \in F(X)\);

(D2) \(D(A, B) = 0\) if and only if \(A = B\);

(D3) \(D(A, B) = 1\) if and only if \(A\) and \(B\) are complementary crisp sets;

(D4) If \(A \leq A' \leq B' \leq B\), then \(D(A, B) \geq D(A', B')\).

Observe that (D4) is equivalent to the following condition: (D4′) If \(A \leq B \leq C\), then \(D(A, B) \leq D(A, C)\) and \(D(B, C) \leq D(A, C)\), for all \(A, B, C \in F(X)\).

Some authors [34] also impose a fifth property on Definition 5.1:

(D5) \(D(A, B) = D(A_N, B_N)\).

Regarding Liu’s definition of distance measure, we can build distance measures between fuzzy sets from the new class of FSM \(\sigma\), given according to Definition 4.1.

**Proposition 5.1** If \(\sigma\) is a fuzzy subsethood measure on \(X\) and \(N\) is a strong negation, then

\[
D(A, B) = N(\sigma(A \lor B, A \land B))
\]

is a distance measure on \(F(X)\) in the sense of Definition 5.1.

**Proof.**

(D1). Evident.

(D2). If \(D(A, B) = 0\), then \(\sigma(A \lor B, A \land B) = 1\). Since \(\sigma\) satisfies (a), we have \(A \lor B \leq A \land B\); therefore \(A \lor B = A \land B\); that is, \(A = B\).

If \(A = B\), then \(A \lor B = A \land B\), therefore \(\sigma(A \lor B, A \land B) = 1\), therefore \(D(A, B) = 0\).

(D3) If \(D(A, B) = 1\), then \(\sigma(A \lor B, A \land B) = 0\). Since \(\sigma\) satisfies (b), we have then \(A \lor B = 1\) and \(A \land B = 0\); that is \(A\) and \(B\) are complementary crisp sets.
If $A$ and $B$ are complementary crisp sets, then $A \lor B = 1$ and $A \land B = 0$. So, bearing in mind that $\sigma$ satisfies (b) we have $D(A, B) = N(\sigma(A \lor B, A \land B)) = 1$.

(D4) If $A \leq A' \leq B' \leq B$, then

$$D(A, B) = N(\sigma(A \lor B, A \land B)) = N(\sigma(B, A))$$

$$D(A', B') = N(\sigma(A' \lor B', A' \land B')) = N(\sigma(B', A'))$$

Since $\sigma$ satisfies (c), then $\sigma(B, A) \leq \sigma(B', A')$. So $D(A, B) \geq D(A', B')$. ■

In particular, we can get distance measures in the sense of Liu (Def. 5.1) from functions $M$ and $I$, and also from automorphisms.

**Corollary 5.1** Let $N$ be a strong negation and let $\sigma : F(X) \times F(X) \to [0, 1]$, be given by Eq. (4.2), for all $A, B \in F(X)$, where $M : [0, 1]^n \to [0, 1]$ is an aggregation function, and $I : [0, 1]^2 \to [0, 1]$ is a function that satisfies (I1), (I2), (I8) and (I15). Then the following items hold.

i) 
$$D(A, B) = N \left( \frac{1}{n} \sum_{i=1}^{n} \left( M(I(A(x_i) \lor B(x_i)), A(x_i) \land B(x_i)) \right) \right)$$

$$= N \left( \frac{1}{n} \sum_{i=1}^{n} \left( M(I(A(x_i), B(x_i)) \land I(B(x_i), A(x_i))) \right) \right)$$

it is a distance measure on $F(X)$;

ii) If $I$ satisfies (I12), then $D$ satisfies (D5);

iii) If $I$ satisfies (I6), (I7), (I12) and (I13), then

$$D(A, B) = \varphi^{-1} \left( 1 - \varphi \left( \frac{1}{n} \sum_{i=1}^{n} \left( M(\varphi^{-1}(1 \land (1 - \varphi(A(x_i) \lor B(x_i)) + \varphi(A(x_i) \land B(x_i)))) \right) \right) \right)$$

is a distance measure on $F(X)$ where $\varphi$ is an automorphism of the unit interval such that $N(x) = \varphi^{-1}(1 - \varphi(x))$, for all $x \in [0, 1]$.
5. Constructing measures from the new class of FSM

5.2 Similarity measure and FSM

The similarity measure of two fuzzy sets indicates the similarity between them. It is clear that distance measure and similarity measure are two dual concepts. In [82], Liu also introduced the axiomatic definition of a similarity measure as follows.

**Definition 5.2** A real function $SM : F(X) \times F(X) \rightarrow \mathbb{R}^+$ is called a similarity measure in Liu’s sense if $SM$ satisfies the following properties:

1. **(SM1)** $SM(A, B) = SM(B, A)$, for all $A, B \in F(X)$;
2. **(SM2')** $SM(A, A_N) = 0$, for all crisp set $A$;
3. **(SM3')** $SM(C, C) = \max_{A,B\in F(X)} SM(A, B)$, for all $C \in F(X)$;
4. **(SM4)** For all $A, B, C, D \in F(X)$, if $A \leq B \leq C \leq D$, then $SM(A, D) \leq SM(B, C)$.

Nevertheless, there are other definitions of similarity measures in the fuzzy literature which do not require simultaneously the four properties given in Definition 5.2 as it is the case of those definitions given by Pappis and Karacapilidis [61] or by Wang, De Baets and Kerre [78].

In 1999, Fan and Xie propose the definition of a proximity measure in [35] as follows.

**Definition 5.3** A similarity measure in Liu’s sense is called a proximity measure, if for all $A, B \in F(X)$, $SM(A, B) = SM(A_N, B_N)$.

Gathering together Def. 5.2, Def. 5.3, the definition of equivalence given by Fodor and Roubens [37] and the four conditions usually demanded from the measures used in image processing for comparing two images (see [8, 24, 23, 28]), they lead to the next definition presented by Bustince et al. in [21]:
**Definition 5.4** A real function $SM : F(X) \times F(X) \to \mathbb{R}^+$ is called a similarity measure on $F(X)$ if $SM$ satisfies the properties: (SM1), (SM4) and

1. (SM2) $SM(A, B) = 0$ if and only if $A$ and $B$ are complementary crisp sets;
2. (SM3) $SM(A, B) = 1$ if and only if $A = B$.

It is easy to see that (SM4) is equivalent to:

If $A \leq B \leq C$, then $SM(C, B) \geq SM(C, A)$ and $SM(A, B) \geq SM(A, C)$.

**Proposition 5.2** If $\sigma$ is a fuzzy subsethood measure on $X$, then

$$SM(A, B) = \sigma(A \vee B, A \wedge B)$$

is a similarity measure on $F(X)$ in the sense of Definition 5.4.

**Proof.**

(SM1) Evident.

(SM2) If $SM(A, B) = 0$, then $\sigma(A \vee B, A \wedge B) = 0$. Since $\sigma$ satisfies (b), we have then $A \vee B = 1$ and $A \wedge B = 0$; that is $A$ and $B$ are complementary crisp sets.

If $A$ and $B$ are complementary crisp sets, then $A \vee B = 1$ and $A \wedge B = 0$. Therefore, bearing in mind that $\sigma$ satisfies (b), we have $SM(A, B) = \sigma(A \vee B, A \wedge B) = 0$.

(SM3) If $SM(A, B) = 1$, then $\sigma(A \vee B, A \wedge B) = 1$. Since $\sigma$ satisfies (a), we have $A \vee B \leq A \wedge B$. So, $A \vee B = A \wedge B$; that is, $A = B$.

If $A = B$, then $A \vee B = A \wedge B$, and therefore $\sigma(A \vee B, A \wedge B) = 1 = SM(A, B)$.

(SM4) If $A \leq A' \leq B' \leq B$, then $SM(A, B) = \sigma(A \vee B, A \wedge B) = \sigma(B, A)$ and $SM(A', B') = \sigma(A' \vee B', A' \wedge B') = \sigma(B', A')$. 
Since \( \sigma \) satisfies (c), then \( \sigma(B, A) \leq \sigma(B', A') \). Thus, \( SM(A, B) \leq SM(A', B') \). ■

It is clear that the similarity measure constructed from \( \sigma \) given in Proposition 5.2 is dual to the distance measure constructed from \( \sigma \) given in Proposition 5.1, that is, if we take \( N(SM(A, B)) \), we have: \( N(\sigma(A \lor B, A \land B)) = D(A, B) \).

### 5.3 Fuzzy entropy and FSM

A fuzzy entropy measure is a way to determine the amount of vagueness, or fuzziness, in a given fuzzy set. In 1972, Deluca and Termini [55] provided a formalization of this notion, which was generalized by Bustince et al. in [19] by using an arbitrary strong negation.

**Definition 5.5** Let \( N \) be a strong negation with equilibrium point \( e \). A function \( E : F(X) \to [0, 1] \) is called an entropy on \( F(X) \), if \( E \) has the following properties:

1. \((E1)\) \( E(A) = 0 \) if and only if \( A \) is crisp;
2. \((E2)\) \( E(A) = 1 \) if and only if \( A = \{ (x, A(x) = e) : x \in X \} \);
3. \((E3)\) \( E(A) \leq E(B) \) if \( A \) refines \( B \), that is, \( A(x) \leq B(x) \) when \( B(x) \leq e \) and \( A(x) \geq B(x) \) when \( B(x) \geq e \);
4. \((E4)\) \( E(A) = E(A_N) \).

The relation between fuzzy entropy and a fuzzy subsethood measure \( \sigma \) is straight, as we can see in the next result.

**Theorem 5.1** If \( \sigma \) is a fuzzy subsethood measure on \( X \), then \( E \) defined by

\[
E(A) = \sigma(A \lor A_N, A \land A_N), \text{ with } A \in F(X),
\]

is a fuzzy entropy on \( X \).
5. Constructing measures from the new class of FSM

Proof. \((E1)\). If \(E(A) = 0\), then \(\sigma(A \lor A_N, N(A \lor A_N)) = 0\). Since \(\sigma\) satisfies \((b)\), we have \(A \lor A_N = 1\), therefore \(A\) is crisp.

If \(A\) is crisp, then \(A \lor A_N = 1\) and \(N(A \lor A_N) = 0\). Since \(\sigma\) satisfies \((b)\), we have \(E(A) = \sigma(A \lor A_N, N(A \lor A_N)) = \sigma(1, 0) = 0\).

\((E2)\). If \(E(A) = 1 = \sigma(A \lor A_N, N(A \lor A_N))\), then since \(\sigma\) satisfies \((a)\), we have that \(A \lor A_N \leq N(A \lor A_N)\); that is \(A(x) \lor N(A(x)) \leq N(A(x) \lor N(A(x)))\).

With respect to \(A(x_i)\) three things can happen:

1) If \(A(x) < e\), then \(A(x) < e < N(A(x))\), therefore

\[
A(x) \lor N(A(x)) = N(A(x)) > e
\]
\[
N(A(x) \lor N(A(x))) = A(x) < e,
\]
then for \(x\) we have: \(A(x) \lor N(A(x)) > e > N(A(x) \lor N(A(x)))\).

2) If \(A(x) > e\), then \(N(A(x)) < e < A(x)\), therefore

\[
A(x) \lor N(A(x)) = A(x) > e
\]
\[
N(A(x) \lor N(A(x))) = N(A(x)) < e,
\]
then for \(x\) we have: \(A(x) \lor N(A(x)) > e > N(A(x) \lor N(A(x)))\).

So, in both cases we have a contradiction with the fact that \(A(x) \lor N(A(x)) \leq N(A(x) \lor N(A(x)))\).

And only the third possibility can happen; that is, \(A(x) = e\).

If \(A = e\); that is, \(A = \{x : A(x) = e : x \in X\}\), then \(A_N = e\), therefore \(A \lor A_N = e = N(A \lor A_N)\). Since \(\sigma\) satisfies \((a)\), we have \(E(A) = \sigma(A \lor A_N, N(A \lor A_N)) = 1\).

\((E3)\). If \(A\) refines \(B\); that is,
1) \( A(x) \leq B(x) \) when \( B(x) \leq e \), then \( A(x) \leq B(x) \leq e \leq N(B(x)) \leq N(A(x)) \), therefore

\[
A(x) \lor N(A(x)) = N(A(x)) \geq N(B(x)) = B(x) \lor N(B(x))
\]

\[
N(A(x) \lor N(A(x))) \leq N(B(x) \lor N(B(x))),
\]

and

2) \( A(x) \geq B(x) \) when \( B(x) \geq e \), then \( N(A(x)) \leq N(B(x)) \leq e \leq B(x) \leq A(x) \), therefore

\[
A(x) \lor N(A(x)) = A(x) \geq B(x) = B(x) \lor N(B(x))
\]

\[
N(A(x) \lor N(A(x))) \leq N(B(x) \lor N(B(x))).
\]

Since \( \sigma \) satisfies (c), we have \( E(A) \leq E(B) \).

(E4). Evident, as \( N \) is involutive because it is a strong negation.

As mentioned before, similarity measure and distance measure constructed from FSM in the sense of Definition 4.1 are dual from each other. It is easy to realize that in Theorem 5.1 if we take \( A_N = B \), then from Proposition 5.2 we have \( SM(A, A_N) = E(A) \).

The following result shows the relations between distance measure, similarity measure and fuzzy entropy constructed from FSM \( \sigma \).

**Theorem 5.2** Let \( N \) be a strong negation and let \( SM, D \) and \( E \) be a similarity measure, a distance measure and a fuzzy entropy, respectively, constructed from \( \sigma \) in the sense of Definition 4.1. Then, the following items hold:

i) \( D(A, B) = N(SM(A, B)) \);

ii) \( E(A) = N(D(A, A_N)) = SM(A, A_N) \).

**Proof.** Straightforward. ■
In the literature, different definitions of entropy can be found, see for instance [29, 48].

In 1983, Ebanks [33] presented the following definition of fuzzy entropy (also generalized in [19] by using an arbitrary strong negation).

**Definition 5.6** Let $N$ be a strong negation with equilibrium point $e$. A real function $E : F(X) \rightarrow \mathbb{R}^+$ is called an entropy on $F(X)$, if $E$ has the following properties:

- $(E1)$ $E(A) = 0$ if and only if $A$ is crisp;
- $(E2)$ $E(A) = 1$ if and only if $A = e$;
- $(E3)$ $E(A) \leq E(B)$ if $A$ refines $B$;
- $(E4)$ $E(A) = E(A_N)$;
- $(E5)$ $E(A \land B) + E(A \lor B) = E(A) + E(B)$, valuation property.

Ebanks provided the following characterization result for a fuzzy entropy function which satisfies $(E1)$-$(E5)$ according to Def. 5.6.

**Theorem 5.3** Let $E : F(X) \rightarrow \mathbb{R}^+$ and $\mu_i = A(x_i)$ for all $i \in \{1, 2, \ldots, n\}$ with $\text{Cardinal}(X) = n$. Then $E$ satisfies $(E1)$-$(E5)$ if and only if $E$ has the form

$$E(A) = \sum_{i=1}^{n} g(\mu_i)$$

for some functions $g : [0, 1] \rightarrow \mathbb{R}^+$ that satisfy:

- $(G1)$ $g(0) = g(1) = 0; g(x) > 0$ for all $x \in (0, 1)$;
- $(G2)$ $g(x) < g(e)$ for all $x \in [0, 1] - \{e\}$;
- $(G3)$ $g$ is non decreasing on $[0, e)$ and non increasing on $(e, 1]$;
- $(G4)$ $g(x) = g(N(x))$ for all $x \in [0, 1]$, where $N$ is a strong negation such that $N(e) = e$. 

Lemma 5.1 Let $N$ be a strong negation and let $I : [0, 1]^2 \to [0, 1]$ be a function that satisfies $(I1), (I2), (I8)$ and $(I15)$. Then $G : [0, 1] \to [0, 1]$ given by:

$$G(x) = \frac{1}{n} I(x \lor N(x), N(x \lor N(x)))$$

where $n$ is a positive integer, satisfies the following properties:

$(G1)$ $G(0) = G(1) = 0$; $G(x) > 0$ for all $x \in (0, 1)$;

$(G2)$ $G(x) < G(e)$ for all $x \in [0, 1] - \{e\}$;

$(G3)$ $G$ is non decreasing on $[0, e)$ and non increasing on $(e, 1]$;

$(G4)$ $G(x) = G(N(x))$ for all $x \in [0, 1]$.

Proof. Analogous to Lemma (2) in [19].  

Theorem 5.4 Let $N$ be a strong negation and let $\sigma : F(X) \times F(X) \to [0, 1]$, be given by Eq. (4.2), for all $A, B \in F(X)$, where $M : [0, 1]^n \to [0, 1]$ and $I : [0, 1]^2 \to [0, 1]$ is a function that satisfies $(I1), (I2), (I8)$ and $(I15)$. Therefore:

$$E(A) = \sigma(A \lor A_N, N(A \lor A_N))$$

is a fuzzy entropy on $X$ that satisfies the valuation property $(E5)$ if and only if $M(x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i$.

Proof. Throughout the proof we have to remind that the maximum entropy for $A = e$ is one, i.e. $E(A) = 1$, as we have proved in Theorem 5.1.

(Necessity) By hypothesis, $E$ is an entropy that satisfies $(E5)$. If we take $G(A(x)) = \frac{1}{n} I(A(x) \lor N(A(x)), N(A(x) \lor N(A(x))))$, which by Lemma 5.1 satisfies $(G1)-(G4)$ and if
we apply Ebanks’ characterization, Theorem 5.3, we have that

\[ E(A) = \sigma(A \lor A_N, N(A \lor A_N)) = \]

\[ \mathcal{M}(I(A(x_i) \lor N(A(x_i))), N(A(x_i) \lor N(A(x_i)))) = \]

\[ \sum_{i=1}^{n} G(A(x_i)) = \sum_{i=1}^{n} \frac{1}{n} I(A(x_i) \lor N(A(x_i))), N(A(x_i) \lor N(A(x_i)))) \]

Let us call \( z_i = I(A(x_i) \lor N(A(x_i))), N(A(x_i) \lor N(A(x_i)))) \) for all \( i \in \{1, \ldots, n\} \), under these conditions we have \( \mathcal{M}(z_1, \ldots, z_n) = \frac{1}{n} \sum_{i=1}^{n} z_i \).

(Sufficiency) Evident by Theorem 5.3 and Lemma 5.1.

5.3.1 An example of fuzzy entropy constructed from FSM in a unified framework for fuzzy thresholding

In this subsection we provide an example where we show how important fuzzy entropy can be in the technique of thresholding [42, 62], which is a gray-level segmentation used in the field of image processing to reduce the complexity of the data and therefore it simplifies the procedures of recognition and classification in many problems. A trivial example of where this technique is used is in a scanner that basically thresholds a document into text and background.

Image segmentation is a difficult research problem that plays a key role in areas such as pattern recognition, computer vision, among others. The segmentation process is performed by dividing an image into disjoint classes so that each of the classes has similar attributes or properties and represents an object of the image. From the different segmentation techniques, one of the most important is the one denoted as thresholding which is a simple but effective method to separate objects from the background.

Let us suppose an image made up of one object. Then, the global thresholding classifies, only considering the level of gray of the pixels into two types: those that belong to the
background and those that belong to the object. In this way, the classification is done setting up a threshold \( t \) such that all the pixels with higher intensities than \( t \) belong to the background (or object) and those with lower intensities than \( t \) belong to the object (or background). Therefore, to determine the threshold \( t \), only the histogram (the number of pixels in an image at each different gray level found in that image) is used, and a binary image is obtained where all the pixels belonging to the background are depicted in black color and all the pixels that belongs to the object are depicted in white color (or vice versa).

In this example, we denote by \((i, j)\) the coordinates of each pixel on an image \( Q \). By \( q(i, j) \) we represent the intensity or level of gray of pixel \((i, j)\). We assume that there exist \( L \) levels of gray, so that \( 0 \leq q(i, j) \leq L - 1 \) for each \((i, j) \in Q\).

First of all, we normalize the image dividing each pixel by \( L - 1 \), therefore \( 0 \leq q(i, j) \leq 1 \). After that, we apply to each pixel \((i, j)\), of an original image \( Q \), the \( E_N \) function

\[
E_N(q(i, j)) = 1 \land (2 \cdot (q(i, j) \land 1 - q(i, j))),
\]

obtaining a new image \( Q_{Ent} \), that is, an image where each pixel \((i, j)\) has associated a gray level given by Eq. (5.1).

In this way, in \( Q_{Ent} \) we can consider the following two situations:

- Pixels with intensity values near to 1, \( q_{Ent}(i, j) \approx 1 \) (white color). This implies that these pixels, in the original image, have a lot of uncertainty, \( q(i, j) \approx 0.5 \);

- In contrast, pixels with intensity values near to 0, \( q_{Ent}(i, j) \approx 0 \) (black color). This implies that these pixels, in the original image, have \( q(i, j) \approx 0 \) or \( q(i, j) \approx 1 \).

**Remark.** Notice that a thresholding problem is a classification problem in two classes (black and white pixels). For this reason the maximum entropy is obtained when an image is a binary image (an image only with black and white pixels).
In Table 5.1 we can see that in the first column it is depicted the original images $Q$ with their names on the left, in the second column we have their ideals (the ones segmented by humans), and in the third we find their $Q_{Ent}$ images.

<table>
<thead>
<tr>
<th>Original</th>
<th>Ideal</th>
<th>$Q_{Ent}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="00" alt="Image" /></td>
<td><img src="01" alt="Image" /></td>
<td><img src="02" alt="Image" /></td>
</tr>
<tr>
<td><img src="03" alt="Image" /></td>
<td><img src="04" alt="Image" /></td>
<td><img src="05" alt="Image" /></td>
</tr>
<tr>
<td><img src="06" alt="Image" /></td>
<td><img src="07" alt="Image" /></td>
<td><img src="08" alt="Image" /></td>
</tr>
</tbody>
</table>

Table 5.1: Original images, their ideals and their corresponding $Q_{Ent}$ images

Next, the entropy of $Q$ is calculated in the following way:

$$E(Q) = \sigma(Q \lor Q_N, Q \land Q_N) = \frac{1}{n} \sum_{i,j} E_N(q(i,j)), \text{ with } n = |Q|.$$  \hspace{1cm} (5.2)
After that, for each image \( Q \), we execute two different thresholding methods: Otsu \([59]\) and Pagola et al. \([60]\). The difference between both techniques is that the first represents the images by numbers and the second one by intervals, as briefly described in the following items:

i. Otsu method \([59]\) is based on a very simple idea which consists of finding the threshold that minimizes the weighted within-class variance. In this way, the histogram is divided into two classes (object and background) where the inter-class variance is minimized and therefore it maximizes the between-class variance.

ii. In Pagola et al. method \([60]\), the thresholding algorithm is based on the idea of choosing the correct membership functions that represent the objects of the image. In this way, an expert can select multiple membership functions that can correctly represent the problem. Then, from these membership functions, an interval-valued fuzzy set is constructed and, by minimizing its entropy, a threshold is chosen to threshold the image. In particular, we apply the interval-valued fuzzy set algorithm where the membership functions used are the ones denoted by \( \mu_1, \mu_5 \) and a normal-distribution-based function (see Table I in \([60]\)).

Finally, we compute the percentage of accuracy just comparing the ideal images with the segmented images obtained with Otsu and Pagola et al. methods. Table 5.2 shows the following values:

- First column. The name of the original images.
- Second column. The value of entropy calculated for each image by means of Eq. (5.2) (the table is ordered in increasing order of these values);
- Third and fourth column. The thresholds obtained with Otsu and Pagola et al. methods, respectively;
• Fifth and sixth column. The accuracy obtained comparing the ideal images and the thresholded images with Otsu and Pagola at al. methods, respectively. The best results are stressed in bold-face.

As we can observe, it is important to highlight the following facts:

• If $E(Q)$ values are less than 0.5, the accuracy results given by Otsu method are better or equal than the ones given by Pagola et al. This occurs in 5 out of 6 images: (00), (01), (02), (03), (04) and (06);

• If $E(Q)$ values are greater or equal than 0.5, the best accuracy results are obtained when using interval-valued fuzzy sets, see values of images (05) and (07).

We consider that the entropy calculated considering fuzzy subsethood measures, can be used as a guide to perform a thresholding method or another, depending on the uncertainty associated to the original image. In particular, between methods that use numbers or intervals to represent the entropy computation.

<table>
<thead>
<tr>
<th>Image</th>
<th>$E(Q)$</th>
<th>Otsu threshold</th>
<th>Interval threshold</th>
<th>% Accuracy Otsu</th>
<th>% Accuracy interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>original</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(02)</td>
<td>0.2060</td>
<td>104</td>
<td>67</td>
<td>0.49</td>
<td>0.49</td>
</tr>
<tr>
<td>(00)</td>
<td>0.2660</td>
<td>79</td>
<td>50</td>
<td>0.63</td>
<td>0.62</td>
</tr>
<tr>
<td>(06)</td>
<td>0.2797</td>
<td>72</td>
<td>49</td>
<td>0.63</td>
<td>0.62</td>
</tr>
<tr>
<td>(01)</td>
<td>0.2812</td>
<td>74</td>
<td>40</td>
<td>0.63</td>
<td>0.60</td>
</tr>
<tr>
<td>(04)</td>
<td>0.3247</td>
<td>127</td>
<td>146</td>
<td>0.61</td>
<td>0.62</td>
</tr>
<tr>
<td>(03)</td>
<td>0.4868</td>
<td>137</td>
<td>135</td>
<td>0.34</td>
<td>0.33</td>
</tr>
<tr>
<td>(07)</td>
<td>0.5647</td>
<td>123</td>
<td>121</td>
<td>0.36</td>
<td>0.36</td>
</tr>
<tr>
<td>(05)</td>
<td>0.6927</td>
<td>135</td>
<td>153</td>
<td>0.81</td>
<td>0.83</td>
</tr>
</tbody>
</table>

Table 5.2: $E(Q)$ entropy values for the original images shown in Fig. 5.1

5.4 General considerations

In this chapter we applied our proposal of a new class of FSM to construct distance measure ($D$), similarity measure ($SM$) and fuzzy entropy ($E$). We also showed through Theorem
the relations between \( D, SM \) and \( E \) constructed from FSM \( \sigma \) in the sense of Definition 4.1. In subsection 5.3.1 we provided an example where the fuzzy entropy calculated from FSM can be applied. We showed through a simple example, how our proposal can be useful whenever it is necessary to choose between two thresholding techniques, Otsu [59] or Pagola et al. [60].
A study on indexes generated from FSM

It is common sense that fuzzy set theory is a useful tool to deal with imprecise information in fields such as image processing, especially in comparison of images. Some methods focus on obtaining the similarity between images through the analysis of gray levels of the pixels that compose those images.

The study of fuzzy indexes seems relevant to our work as they evaluate to which extent a set is fuzzy, or in other words, they attempt to quantify how fuzzy a fuzzy set is and, in the context of image processing, several studies have been making use of such indexes, as in many image definitions there are concepts which are vague in nature such as edges, boundaries, etc.

In this chapter, we have an initial study of some fuzzy indexes obtained from our new class of FSM in the sense of Definition 4.1. We start recalling some relevant definitions that will be considered in our study.

6.1 Equality indexes

The two most common definitions of the concept of equality index have been given by Klir and Yuan [48], and by D. Dubois and H. Prade [31]. Other definitions can be found in
Definition 6.1 An equality index in the sense of Klir and Yuan [48] is a mapping $EQ_{KY} : F(X) \times F(X) \rightarrow [0, 1]$, such that

1. $EQ_{KY}(A, B) = 1$ if and only if $A = B$;
2. If $A \leq B \leq C$, then $EQ_{KY}(C, B) \geq EQ_{KY}(C, A)$ and $EQ_{KY}(A, B) \geq EQ_{KY}(A, C)$.

Definition 6.2 An equality index in the sense of Dubois and Prade [31] is a mapping $EQ_{DP} : F(X) \times F(X) \rightarrow [0, 1]$, such that

1. $EQ_{DP}(A, B) = 1$ if and only if $A = B$;
2. $EQ_{DP}(A, B) = 0$ if $\text{supp}(A) \cap \text{supp}(B) = \emptyset$ and $\text{supp}(A) \cup \text{supp}(B) = X$;
3. $EQ_{DP}(A, B) = EQ_{DP}(B, A)$.

From the analysis of the definitions of equality indexes that exist in the fuzzy literature one can deduce that the property: $EQ(A, B) = 1$ if and only if $A = B$ is always maintained, whereas the property: $EQ(A, B) = 0$ if $A \land B = 0$ is not always considered.

Next, we show two expressions constructed from our new class of FSM that satisfy Definitions 6.1 and 6.2.

Theorem 6.1 If $\sigma$ is a fuzzy subsethood measure, then

$$EQ(A, B) = \begin{cases} 
0, & \text{if } \text{supp}(A) \cap \text{supp}(B) = \emptyset \text{ and } \\
\sigma(A \lor B, A \land B), & \text{otherwise.}
\end{cases}$$

is an equality index in the sense of Definitions 6.1 and 6.2.
6. A study on indexes generated from FSM

**Proof.** (EQ1<sub>DP</sub>) (Necessity) If $EQ(A, B) = 1$, then $\sigma(A \lor B, A \land B) = 1$. Since $\sigma$ satisfies item (a) of Definition 4.1 we have $A \lor B \leq A \land B$; that is, $A \lor B = A \land B$, then $A = B$.

(Sufficiency) If $A = B$, two things can happen:

1) $A = B = 0$, then $EQ(A, B) = \sigma(0, 0) = 1$.

2) $A = B$ and $A \neq 0$, then $A \land B = A \land A = A \neq 0$, then $EQ(A, B) = \sigma(A, A) = 1$.

(EQ2<sub>DP</sub>) If $supp(A) \cap supp(B) = \emptyset$ and $supp(A) \cup supp(B) = X$, by definition we have $EQ(A, B) = 0$.

(EQ3<sub>DP</sub>) By definition $EQ(A, B) = EQ(B, A)$.

(EQ2<sub>KY</sub>) If $A \leq B \leq C$. Then $supp(A) \subseteq supp(B)$ and $supp(B) \subseteq supp(C)$.

If $A = 0$ and $C = X$, then $EQ(C, B) \geq 0 = EQ(C, A)$.

Otherwise, $EQ(C, B) = \sigma(C \lor B, C \land B) = \sigma(C, B) \geq \sigma(C, A) = \sigma(C \lor A, C \land A) = EQ(C, A)$.

The second inequality is shown in a similar way. □

**Theorem 6.2** If $\sigma$ is a fuzzy subsethood measure, then

$$EQ_\land(A, B) = \begin{cases} 0, & \text{if } supp(A) \cap supp(B) = \emptyset \text{ and } \\ \sigma(A, B) \land \sigma(B, A), & \text{otherwise.} \end{cases}$$

is an equality index in the sense of Def. 6.1 and in the sense of Def. 6.2.

**Proof.** (EQ1) (Necessity) If $EQ_\land(A, B) = 1$, then $\sigma(A, B) \land \sigma(B, A) = 1$, therefore $\sigma(A, B) = 1$ and $\sigma(B, A) = 1$, then $A \leq B$ and $B \leq A$; that is, $A = B$.

(Sufficiency) If $A = B$, then $\sigma(A, B) = \sigma(B, A) = 1$, therefore
6. A study on indexes generated from FSM

\[ EQ_\land(A, B) = \sigma(A, B) \land \sigma(A, B) = 1. \]

(EQ2) If \( \text{supp}(A) \cap \text{supp}(B) = \emptyset \) and \( \text{supp}(A) \cup \text{supp}(B) = X \), then by definition \( EQ_\land(A, B) = 0. \)

(EQ3) Obvious.

(EQ2KY) If \( A \leq B \leq C \). Then \( \text{supp}(A) \subseteq \text{supp}(B) \) and \( \text{supp}(B) \subseteq \text{supp}(C) \).

If \( A = 0 \) and \( C = X \), then \( EQ_\land(C, B) \geq 0 = EQ_\land(C, A) \).

Otherwise, \( EQ_\land(C, B) = \sigma(C \lor B, C \land B) = \sigma(C, B) \geq \sigma(C, A) = \sigma(C \lor A, C \land A) = EQ_\land(C, A) \).

The second inequality is shown in a similar way. ■

Corollary 6.1 For all \( A, B \in F(X) \), the following items hold:

i) \( EQ(A, B) \leq EQ_\land(A, B) \).

ii) \( EQ(A, A_N) = E(A) \).

Proof. i) We only need to bear in mind item \( c \) of Definition 4.1

ii) We only need to recall that if \( A \land A_N = 0 \), then \( A \) is crisp and if \( A = 0 \), then \( A_N = 1 \) and vice versa. So by Theorem 5.1, it is evident. ■

In the following theorem we show two expressions that are equality indexes in the sense of Klir and Yuan (Def. 6.1), but they are not in the sense of Dubois and Prade (Def. 6.2).

Theorem 6.3 If \( \sigma \) is a fuzzy subsethood measure, then

\[ SE_\sigma(A, B) = \sigma(A \lor B, A \land B) \]

and

\[ se_\sigma(A, B) = \sigma(A, B) \land \sigma(B, A) \]

are equality indexes in the sense of Def. 6.1 and are not so in the sense of Def. 6.2.
Proof. For $\sigma(A \lor B, A \land B)$.

\((EQ1_{KY})\) $\sigma(A \lor B, A \land B) = 1$ if and only if $A \lor B \leq A \land B$; that is, $A = B$.

\((EQ2_{KY})\) If $A \leq B \leq C$, then

$\sigma(A \lor B, A \land B) = \sigma(B, A) \geq \sigma(C, A) = \sigma(C \lor A, C \land A)$.

$\sigma(C \lor B, C \land B) = \sigma(C, B) \geq \sigma(C, A) = \sigma(C \lor A, C \land C \land B)$.

Obviously, $\sigma(A \lor B, A \land B) = \sigma(B \lor A, B \land A)$.

If $supp(A) \cap supp(B) = \emptyset$ and $supp(A) \cup supp(B) = X$, we have $\sigma(A \lor B, 0)$ of which nothing can be said.

In a similar way, it can be proven for $\sigma(A, B) \land \sigma(B, A)$. ■

Table 6.1 shows the equality indexes constructed from FSM $\sigma$ and their relation with the equality index in the sense of Klir and Yuan [48], Def. 6.1, and with the equality index in the sense of Dubois and Prade [31], Def. 6.2.

<table>
<thead>
<tr>
<th>Equality indexes constructed from $\sigma$</th>
<th>Equality index in the sense of Def. 6.1</th>
<th>Equality index in the sense of Def. 6.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>EQ(A,B)</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>$EQ_{\land}(A, B)$</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>SE$_{\sigma}(A,B)$</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>se$_{\sigma}(A,B)$</td>
<td>yes</td>
<td>no</td>
</tr>
</tbody>
</table>

Table 6.1: Equality indexes constructed from FSM $\sigma$.

6.2 Semi-equality index

In this section we discuss the possible relationships between the concepts of FSM and semi-equality index. We start with the following definition:

Definition 6.3 A function $SE_{\sigma} : F(X) \times F(X) \rightarrow [0, 1]$ is called semi-equality index if it satisfies the following conditions:
6. A study on indexes generated from FSM

(SE1) $SE_\sigma(A, B) = SE_\sigma(B, A)$ for all $A, B \in F(X)$;

(SE2) $SE_\sigma(A, B) = 1$ if and only if $A = B$;

(SE3) $SE_\sigma(A, B) = 0$ if and only if $A$ and $B$ are complementary crisp sets;

(SE4) If $A \leq A' \leq B' \leq B$, then $SE_\sigma(A, B) \leq SE_\sigma(A', B')$.

It is easy to see that (SE4) is equivalent to the following property:

If $A \leq B \leq C$, then $SE_\sigma(C, B) \geq SE_\sigma(C, A)$ and $SE_\sigma(A, B) \geq SE_\sigma(A, C)$.

**Proposition 6.1** The following item holds:

i) Every semi-equality index is an equality index in the sense of Def. 6.1

**Proof.** $(EQ_{1KY}) = (SE2)$ and $(EQ_{2KY}) = (SE3)$. ■

**Proposition 6.2** If $\sigma$ is a fuzzy subsethood measure on $X$, then $SE_\sigma$ is a semi-equality index.

**Proof.** (SE1). Evident. (SE2). If $SE_\sigma(A, B) = 1$, then $\sigma(A \lor B, A \land B) = 1$, since $\sigma$ satisfies (a) we have $A \lor B \leq A \land B$, therefore $A \lor B = A \land B$; that is, $A = B$.

If $A = B$, then $A \lor B = A \land B$, therefore $\sigma(A \lor B, A \land B) = 1 = SE_\sigma(A, B)$.

(SE3) If $SE_\sigma(A, B) = 0$, then $\sigma(A \lor B, A \land B) = 0$, since $\sigma$ satisfies (b) we have then $A \lor B = 1$ and $A \land B = 0$; that is $A$ and $B$ are complementary crisp sets.

If $A$ and $B$ are complementary crisp sets, then $A \lor B = 1$ and $A \land B = 0$. Then bearing in mind that $\sigma$ satisfies (b) we have $SE_\sigma(A, B) = \sigma(A \lor B, A \land B) = 0$.

(SE4) If $A \leq A' \leq B' \leq B$, then

$$SE_\sigma(A, B) = \sigma(A \lor B, A \land B) = \sigma(B, A)$$

$$SE_\sigma(A', B') = \sigma(A' \lor B', A' \land B') = \sigma(B', A')$$
6. A study on indexes generated from FSM

Since \( \sigma \) satisfies \((c)\), then \( \sigma(B, A) \leq \sigma(B', A') \), therefore \( SE_\sigma(A, B) \leq SE_\sigma(A', B') \). ■

Let \( \sigma \) be a fuzzy subsethood measure. If we take \( se_\sigma \) by Theorem 6.3, we have that both \( se_\sigma \) and \( SE_\sigma \) are equality indexes in the sense of Klir and Yuan (Def. 6.1) whereas they are not so in the sense of Dubois and Prade (Def. 6.2).

However, we have seen in Theorem 6.2 that from \( se_\sigma \) we can construct \( EQ \wedge \) that is an equality index in the sense of Dubois and Prade. (That is, it happens the same as with \( SE_\sigma \) and \( EQ \) according to Theorem 6.1.)

Due to all these considerations and bearing in mind that \( SE_\sigma(A, B) \leq se_\sigma(A, B) \), (Corollary 6.1), it seems logic to wonder if \( se_\sigma \) is a semi-equality index. The answer is no. It does not fulfill property \((SE3)\) of the Definition 6.3

In the following proposition we show the most important properties fulfilled by \( se_\sigma \).

**Proposition 6.3** Let \( \sigma \) be a fuzzy subsethood measures and \( N \) be a strong fuzzy negation. Then the following items hold for \( se_\sigma \):

i) \( se_\sigma(A, B) = se_\sigma(B, A) \);

ii) \( se_\sigma(A, B) = 1 \) if and only if \( A = B = 1 \);

iii) \( se_\sigma(A, B) = 0 \) if and only if \( (A = 1 \text{ and } B = 0) \text{ or } (A = 0 \text{ and } B = 1) \);

iv) If \( A \leq B \leq C \), then \( se_\sigma(A, B) \geq se_\sigma(A, C) \text{ and } se_\sigma(C, B) \geq se_\sigma(C, A) \);

v) \( SE_\sigma(A, B) \leq se_\sigma(A, B) \);

vi) If \( \sigma \) satisfies \( \sigma(A, B) = \sigma(B_N, A_N) \), then \( se_\sigma(B_N, A_N) = se_\sigma(A, B) \).

**Proof.** i) Evident by definition. ii) \( se_\sigma(A, B) = 1 \) if and only if \( \sigma(A, B) = \sigma(B, A) = 1 \) if and only if \( A \leq B \) and \( A \geq B \); that is \( A = B \).

iii) (Necessity) If \( se_\sigma(A, B) = 0 \), then two things can happen:
6. A study on indexes generated from FSM

1) \( \sigma(A, B) = 0 \) if and only if \( A = 1 \) and \( B = 0 \);

or

2) \( \sigma(B, A) = 0 \) if and only if \( B = 1 \) and \( A = 0 \).

(Sufficiency) If \( A = 1 \) and \( B = 0 \), then \( \sigma(A, B) = 0 \) or if \( A = 0 \) and \( B = 1 \), then \( \sigma(B, A) = 0 \). Therefore \( se_\sigma(A, B) = \sigma(A, B) \land \sigma(B, A) = 0 \).

iv) If \( A \leq B \leq C \), then we have that

\[
\begin{align*}
se_\sigma(C, B) &= \sigma(C, B) \land \sigma(B, C) = \sigma(C, B) \\
se_\sigma(C, A) &= \sigma(C, A) \land \sigma(A, C) = \sigma(C, A)
\end{align*}
\]

by item (c) of Definition 4.1 we have \( \sigma(C, A) \leq \sigma(C, B) \), then

\[
se_\sigma(C, A) \leq se_\sigma(C, B).
\]

\[
\begin{align*}
se_\sigma(A, B) &= \sigma(A, B) \land \sigma(B, A) = \sigma(B, A) \\
se_\sigma(A, C) &= \sigma(A, C) \land \sigma(C, A) = \sigma(C, A)
\end{align*}
\]

since \( \sigma(B, A) \geq \sigma(C, A) \), then \( se_\sigma(A, B) \geq se_\sigma(A, C) \).

v) We know that \( SE_\sigma(A, B) = \sigma(A \lor B, A \land B) \leq \sigma(A, A \land B) \leq \sigma(A, B) \) and \( SE_\sigma(A, B) = \sigma(A \lor B, a \land B) \leq \sigma(B, A \land B) \leq \sigma(B, A) \). Therefore \( SE_\sigma(A, B) \leq \sigma(A, B) \land \sigma(B, A) = se_\sigma(A, B) \).

vi) If \( \sigma \) satisfies \( \sigma(B_N, A_N) = \sigma(A, B) \) we have \( se_\sigma(B_N, A_N) = \sigma(B_N, A_N) \land \sigma(A_N, B_N) = \sigma(A, B) \land \sigma(B, A) = se_\sigma(A, B) \).

\[\blacksquare\]

It is not hard to see that \( se_\sigma \) is not a semi-equality index as every equality index in the sense of Dubois and Prade satisfies the first three conditions of the measures of equivalence of Liu. Please note that it satisfies the rest of the conditions demanded in Definition 6.3.

In the following corollary we show the way of constructing semi-equality indexes by
aggregating implication operators. In the last item we construct these indexes using automorphisms of the unit interval.

**Corollary 6.2** Let $N$ be a strong fuzzy negation and let $\sigma : F(X) \times F(X) \to [0, 1]$, be given by Eq. \ref{eq:4.2} for all $A, B \in F(X)$, where $\mathcal{M} : [0, 1]^n \to [0, 1]$ is a function that satisfies (A1), (A2), (A3), and $I : [0, 1]^2 \to [0, 1]$ is a function that satisfies (I1), (I2), (I8) and (I15).

Then the following items hold.

i) $SE_{\sigma}(A, B) = \mathcal{M}(I(A(x_i) \lor B(x_i)), A(x_i) \land B(x_i)))$

is a semi-equality index.

ii) $SE_{\sigma}(A, B) \leq \sigma(A, B) \land \sigma(B, A)$, being $\sigma$ the fuzzy subsethood measure on $X$ generated from Proposition \ref{prop:4.2}.

iii) If $I$ satisfies (I12), then $SE_{\sigma}(A_N, B_N) = SE_{\sigma}(A, B)$.

iv) If $I$ satisfies (I6), (I7), (I12) and (I13), then

$$SE_{\sigma}(A, B) = \mathcal{M}(\varphi^{-1}(1 - \varphi(A(x_i) \lor B(x_i))) + \varphi(A(x_i) \land B(x_i)))$$

is a semi-equality index, $\varphi$ being an automorphism of the unit interval such that $N(x) = \varphi^{-1}(1 - \varphi(x))$ for all $x \in [0, 1]$.

**Proof.** i) We only need to bear in mind Prop. \ref{prop:2.2} the fact that $I$ satisfies (I8) and Propositions \ref{prop:4.2} and \ref{prop:6.2}.

ii) Since $\mathcal{M}$ satisfies (A3) we have
\[ SE_\sigma(A, B) = \sum_{i=1}^{n} \lambda(I(A(x_i), B(x_i)) \land I(B(x_i), A(x_i))) \leq \sum_{i=1}^{n} \lambda(I(A(x_i), B(x_i))) = \sigma(A, B) \]

\[ SE_\sigma(A, B) = \sum_{i=1}^{n} \lambda(I(A(x_i), B(x_i)) \land I(B(x_i), A(x_i))) \leq \sum_{i=1}^{n} \lambda(I(B(x_i), A(x_i))) = \sigma(B, A), \]

therefore \( SE_\sigma(A, B) \leq \sigma(A, B) \land \sigma(B, A). \)

iii) \( I \) satisfies \( (I12) \) and \( (\lor, \land, N) \) is a De Morgan triple, therefore

\[ SE_\sigma(A_N, B_N) = \sum_{i=1}^{n} \lambda(I(N(A(x_i))) \lor N(B(x_i)), N(A(x_i)) \land N(B(x_i))) \]
\[ = \sum_{i=1}^{n} \lambda(I(N(A(x_i))) \land N(B(x_i)), N(N(A(x_i))) \lor N(B(x_i)))) \]
\[ = \sum_{i=1}^{n} \lambda(I(A(x_i)) \lor B(x_i), A(x_i) \land B(x_i))) \]
\[ = SE_\sigma(A, B). \]

iv) We only need to bear in mind Theorem 4.5 and Proposition 6.2. ■

### 6.2.1 Relationships between semi-equality index, fuzzy entropy, fuzzy distance and proximity measure

**Theorem 6.4** Let \( N \) be a strong negation. If \( SE_\sigma \) is a semi-equality index, then the following items hold.

i) \( E(A) = SE_\sigma(A, A_N); \)

ii) \( D(A, B) = N(SE_\sigma(A, B)). \)

**Proof.** i) \((E1)\). If \( E(A) = 0 \), then \( SE_\sigma(A, A_N) = 0 \), since \( SE_\sigma \) satisfies \( (SE3) \) we have that \( A \) and \( A_N \) are complementary crisp sets.

If \( A \) is a crisp set, then so is \( A_N \) and besides, they are complementary. Bearing in mind \( (SE3) \) we have \( E(A) = SE_\sigma(A, A_N) = 0. \)
(E2). If $E(A) = 1 = SE_\sigma(A, A_N)$, then by (SE2) we have $A = A_N$; that is, $A(x_i) = N(A(x_i))$ for all $i \in \{1, \ldots, n\}$. Since $N$ is a strong negation, it has a single equilibrium point $e$, therefore $A(x_i) = e$ for all $i \in \{1, \ldots, n\}$.

If $A(x_i) = e$ for all $i \in \{1, \ldots, n\}$, then $N(A(x_i)) = e$, therefore $A = A_N = e$. Since $SE_\sigma$ satisfies (SE2) we have $E(A) = SE_\sigma(A, A_N) = 1$.

(E3). If $A(x_i) \leq B(x_i) \leq e$, then

$$A(x_i) \leq B(x_i) \leq e \leq N(B(x_i)) \leq N(A(x_i)).$$

If $e \leq B(x_i) \leq A(x_i)$, then

$$N(A(x_i)) \leq N(B(x_i)) \leq e \leq B(x_i) \leq A(x_i).$$

Since $SE_\sigma$ satisfies (SE1) and (SE4) we have $E(A) = SE_\sigma(A, A_N) \leq SE_\sigma(B, B_N) = E(B)$.

(E4). Bearing in mind that $N$ is a strong negation and that $SE_\sigma$ satisfies (SE1) we have $E(A_N) = SE_\sigma(A_N, N(A_N)) = SE_\sigma(A_N, A) = SE_\sigma(A, A_N) = E(A)$.

ii) Evident, we only need to recall Def. 6.3 and the fact that $N$ is a strong negation. ■

**Corollary 6.3** Let $N$ be a strong negation. If $D$ is a distance measure on $F(X)$, then

$$E(A) = N(D(A, A_N))$$

is a fuzzy entropy.

**Proof.** Similar to the one done in Theorem 6.4 ■

In 1999, J. Fan and W. Xie [34] defined proximity measures in the following way:
Definition 6.4 A similarity measure in the sense of Liu is called a proximity measure, if

\[ SM(A, B) = SM(A_N, B_N), \]

for all \( A, B \in F(X). \)

In Theorem 6.1 we have proved that every semi-equality index is a similarity measure in the sense of Liu. From this result we have the following corollary.

Corollary 6.4 Under the conditions of Proposition 6.2 if \( \sigma \) satisfies \( \sigma(B_N, A_N) = \sigma(A, B) \), then the following items hold.

i) \( SE_\sigma \) is a proximity measure in the sense of Def. 6.4.

ii) \( SE_\sigma \) is a correlation.

Proof. Since \((\wedge, \vee, N)\) is a De Morgan triple and \( \sigma \) satisfies \( \sigma(B_N, A_N) = \sigma(A, B) \) we have:

\[
SE_\sigma(A_N, B_N) = \sigma(A_N \vee B_N, A_N \wedge B_N) = \sigma(N(A_N \wedge B_N), N(A_N \vee B_N)) = \sigma(A \vee B, A \wedge B) = SE_\sigma(A, B).
\]

Therefore, it is also a proximity measure. Obviously it is also a correlation.

A correlation between two fuzzy sets \( A \) and \( B \), according to \([15, 32, 40, 44, 45]\), are functions that fulfill the four conditions \((ME_{1 FR})-(ME_{4 FR})\) given by Fodor and Roubens in \([37]\) and called a measure of equivalence, together with \((ME_{5 FR})\), as seen in the following definition.

Definition 6.5 A function \( ME_{FR} : F(X) \times F(X) \to [0, 1] \) is called a correlation if for all \( A, B \in F(X) \) it satisfies:
6. A study on indexes generated from FSM

\( ME_{FR}(A, B) = ME_{FR}(B, A) \): 

\( ME_{FR}(A, A) = 1 \);

\( ME_{FR}(1, 0) = ME_{FR}(0, 1) = 0 \);

\( ME_{FR}(A, B) \leq ME_{FR}(A', B') \);

\( ME_{FR}(A, B) = ME_{FR}(A_N, B_N) \).

6.3 Inclusion index of Kosko

We know that from Zadeh’s\[84\] definition of set inclusion (subsethood measure), \( A \subseteq B \) if \( A(x) \leq B(x) \) for all \( x \in X \). Kosko\[51\] states that if this inequality holds for almost all \( x \) except a few, one can still consider \( A \) to be a subset of \( B \) to some degree. He then generalizes Zadeh’s definition using the following subsethood measure:

\[
\sigma_{K_k}(A, B) = \begin{cases} 
\frac{\sum_{i=1}^{n} A(x_i) \land B(x_i)}{\sum_{i=1}^{n} A(x_i)} = \frac{|A \land B|}{|A|}, & \text{if } A \neq 0, \\
1, & \text{if } A = 0.
\end{cases}
\]

So, we can say that \( A \) is a subset of \( B \) to a degree \( \sigma_S(A, B) \). In other words, this function \( \sigma_S \) measures how well \( A \) and \( B \) satisfy the inequality \( A \leq B \) with respect to the size of \( A \).

From the analysis of the expression shown by Kosko we deduce that it satisfies the following properties:

1. \( A \leq B \) if and only if \( \sigma_{K_k}(A, B) = 1 \);

2. \( \sigma_{K_k}(A, B) = 0 \) if and only if \( supp(A) \cap supp(B) = \emptyset \), with \( A \neq 0 \);

3. If \( A \leq B \), then \( \sigma_{K_k}(C, A) \leq \sigma_{K_k}(C, B) \); that is, it is increasing in the second argument.
It is clear that $\sigma_{Kk}$ is not a fuzzy subsethood measure in the sense of Definition 4.1 because it does not satisfy item (b) and nor it is decreasing in the first component.

The third property suggested Virginia Young to impose a fourth condition:

4. If $A \leq B \leq C$, then $\sigma_{Kk}(C, A) \leq \sigma_{Kk}(C, B)$

in her definition [83].

Next we show an expression, denoted by $RIC_\sigma$ (that will be referred to as the $RIC_\sigma$ index) constructed from the new class of FSM according to Def. 4.1 that satisfy all of the four conditions. Later we will see the conditions under which our expression coincides with Kosko’s inclusion index.

**Theorem 6.5** If $\sigma$ is a fuzzy subsethood measure on $X$, then $RIC_\sigma : F(X) \times F(X) \to [0, 1]$ given by

$$RIC_\sigma(A, B) = \begin{cases} \frac{\sigma(1, A \wedge B)}{\sigma(1, A)}, & \text{if } A \neq 0. \\ 1, & \text{if } A = 0. \end{cases}$$

satisfies the following items.

i) If $A \leq B$, then $RIC_\sigma(A, B) = 1$;

ii) $RIC_\sigma(A, B) = 0$ if and only if $\text{supp}(A) \cap \text{supp}(B) = \emptyset$, with $A \neq 0$;

iii) If $A \leq B$, then $RIC_\sigma(C, A) \leq RIC_\sigma(C, B)$;

**Proof.** i) If $A \leq B$, then $A \wedge B = A$, therefore $\sigma(1, A \wedge B) = \sigma(1, A)$, then $RIC_\sigma(A, B) = 1$.

ii) If $RIC_\sigma(A, B) = 0$ (obviously $A \neq 0$), then $\sigma(1, A \wedge B) = 0$. Since $\sigma$ is a fuzzy subsethood measure on $X$, it satisfies condition (b) of Definition 6.1 that is, $A \wedge B = 0$, then $\text{supp}(A) \cap \text{supp}(B) = \emptyset$. 
If \( \text{supp}(A) \cap \text{supp}(B) = \emptyset \) (\( A \neq 0 \)), then \( A \wedge B = 0 \), therefore \( \sigma(1, A \wedge B) = \sigma(1, 0) \). By condition (b) of Def. 6.1 we have \( \sigma(1, 0) = 0 \), therefore \( RIC_\sigma(A, B) = 0 \).

iii) If \( A \leq B \), two things can happen:

1) If \( C = 0 \), then \( RIC_\sigma(C, A) = 1 = RIC_\sigma(C, B) \).

2) In any other case, since \( A \leq B \), then \( C \wedge A \leq C \wedge B \), therefore since \( \sigma \) satisfies (c) of Definition 4.1 we have \( \sigma(1, C \wedge A) \leq \sigma(1, C \wedge B) \), then \( RIC_\sigma(C, A) \leq RIC_\sigma(C, B) \). ■

Note that item i) is the necessary condition of property (a) of Definition 4.1. And it must be said that there are authors, such as E. Tsiporkova et al. [76], who only demand item i) instead of (a) from the inclusion indexes. In this sense we have the following result.

**Corollary 6.5** Under the conditions of Theorem 6.5, \( RIC_\sigma \) is a fuzzy subsethood measure in the sense of J. Fan.

**Proof.** Evidently, \( RIC_\sigma \) satisfies i) of Theorem 6.5 then it satisfies the necessary condition of Definition 4.1. On the other hand, it satisfies that \( RIC_\sigma(1, 0) = 0 \) (by condition (b) of Def. 4.1). It also satisfies the fourth condition of Theorem 6.5 and since \( RIC_\sigma \) is increasing in the second argument, it satisfies the condition of J. Fan:

If \( A \leq B \leq C \), then \( RIC_\sigma(A, A) \leq RIC_\sigma(C, B) \). ■

Please note that items ii) and iii) of Theorem 6.5 are two of the conditions that are demanded from the overlap index [29, 85]. However, \( RIC_\sigma \) does not fulfill the symmetry or the fact of being one if and only if \( A \subseteq B \).

In the following corollary we show a construction of \( RIC_\sigma \) from an aggregation function \( \mathcal{M} \) and an implication operator \( I \).

**Corollary 6.6** Let \( \mathcal{M} : [0, 1]^n \rightarrow [0, 1] \) be such that it satisfies (A1), (A2), (A3) and \( I : [0, 1]^2 \rightarrow [0, 1] \) is a function that satisfies (I1), (I2), (I8) and (I15).
Under these conditions

\[
RIC_\sigma(A, B) = \begin{cases} 
\frac{\sum_{i=1}^{n} (1 \land (1 - A(x_i) + B(x_i)))}{\sum_{i=1}^{n} (1 \land (1 - A(x_i) + B(x_i)))} = \frac{\sum_{i=1}^{n} (1 \land (1 - A(x_i) + B(x_i)))}{\sum_{i=1}^{n} (1 \land (1 - A(x_i) + B(x_i)))}, & \text{if } A \neq 0. \\
1, & \text{if } A = 0.
\end{cases}
\]

satisfies the following items

(i) If \( A \leq B \), then \( RIC_\sigma(A, B) = 1 \);

(ii) \( RIC_\sigma(A, B) = 0 \) if and only if \( \text{supp}(A) \cap \text{supp}(B) = \emptyset \), with \( A \neq 0 \);

(iii) If \( A \leq B \), then \( RIC_\sigma(C, A) \leq RIC_\sigma(C, B) \);

(iv) If \( A \leq B \leq C \), then \( RIC_\sigma(C, A) \leq RIC_\sigma(B, A) \).

**Proof.** We only need to bear in mind Proposition 4.2 and Theorem 6.5. By means of Proposition 4.2 we can construct fuzzy subsethood measures on \( X \) from \( M \) fulfilling (A1), (A2), (A3) and \( I \) fulfilling (I1), (I2), (I8) and (I15), by means of Eq. (4.2) for all \( A, B \in F(X) \). With this construction and Theorem 6.5 we can assure that \( RIC_\sigma(A, B) \) satisfies items i)-iv).  

**Example 6.1** Let us take \( \sigma \) in the following example,

\[
\sigma(A, B) = \frac{\bigvee_{i=1}^{n} (1 \land (1 - A(x_i) + B(x_i)))}{\bigvee_{i=1}^{n} (1 \land (1 - A(x_i) + B(x_i))) + \bigvee_{i=1}^{n} (0 \lor (A(x_i) - B(x_i)))}
\]

then, we have

\[
RIC_\sigma(A, B) = \begin{cases} 
\frac{\bigvee_{i=1}^{n} (A(x_i) \land B(x_i)) \cdot \left( \bigvee_{i=1}^{n} (A(x_i)) + \bigvee_{i=1}^{n} (1 - A(x_i)) \right)}{\bigvee_{i=1}^{n} (A(x_i)) \cdot \left( \bigvee_{i=1}^{n} (A(x_i) \land B(x_i)) + \bigvee_{i=1}^{n} (1 - A(x_i) \land B(x_i)) \right)}, & \text{if } A \neq 0. \\
1, & \text{if } A = 0.
\end{cases}
\]
In the following theorem we show a way of constructing inclusion indexes that satisfy the same four conditions as Kosko’s index satisfies. In particular, we also get this index. In this construction we will use the result of Proposition 4.2 imposing on \( M \) some additional conditions.

**Theorem 6.6** Let \( M : [0, 1]^n \rightarrow [0, 1] \) be an operator of Kolmogorov [49] and Nagumo [58] generated by the continuous and strictly increasing function \( f : [0, 1] \rightarrow \mathbb{R} \), and let \( I : [0, 1]^2 \rightarrow [0, 1] \) be a function that satisfies (I1), (I2), (I6), (I8) and (I15). Then:

\[
RIC_\sigma(A, B) = \begin{cases} 
\frac{\sum_{i=1}^{n} (I_{M}(1, A(x_i)))}{\sum_{i=1}^{n} (I_{M}(1, A(x_i)))} = f^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} f(A(x_i) \land B(x_i)) \right), & \text{if } A \neq 0. \\
1, & \text{if } A = 0.
\end{cases}
\]

satisfies the following items

i) \( RIC_\sigma(A, B) = 1 \) if and only if \( A \leq B \);

ii) \( RIC_\sigma(A, B) = 0 \) if and only if \( \text{supp}(A) \cap \text{supp}(B) = \emptyset \), with \( A \neq 0 \);

iii) If \( A \leq B \), then \( RIC_\sigma(C, A) \leq RIC_\sigma(C, B) \);

iv) If \( A \leq B \leq C \), then \( RIC_\sigma(C, A) \leq RIC_\sigma(B, A) \).

**Proof.** Items ii)-iii) hold by Corollary 6.6. We only need to bear in mind that

\[
M(x_1, \ldots, x_n) = f^{-1} \left( \sum_{i=1}^{n} f(x_i) \right)
\]

(where \( f : [0, 1] \rightarrow \mathbb{R} \) is continuous and strictly increasing), satisfies (A1), (A2), (A3S), it is idempotent, continuous and decomposable by Theorem 5 in [20]. Moreover, \( I \) satisfies the
same conditions that are demanded in Corollary \[ \text{Corollary 6.6} \] therefore

\[
RIC_{\sigma}(A, B) = \begin{cases} \left( \frac{\sum_{i=1}^{n}(A(x_i) \wedge B(x_i))}{\sum_{i=1}^{n}A(x_i)} \right)^{\frac{1}{\lambda}}, & \text{if } A \neq 0. \\ 1, & \text{if } A = 0. \end{cases}
\]

satisfies $\text{ii) } - \text{iv}$ and the necessary condition of $\text{i) }$. Let us see the sufficient condition.

If $RIC_{\sigma}(A, B) = 1$, then $A = 0$ in such case $A \leq B$ or

\[
\frac{f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}f(A(x_i) \wedge B(x_i))\right)}{f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}f(A(x_i))\right)} = 1.
\]

Therefore, regarding that $f^{-1}$ is injective, \[ \sum_{i=1}^{n}f(A(x_i) \wedge B(x_i)) = \sum_{i=1}^{n}f(A(x_i)) \]; that is,

\[
\sum_{i=1}^{n}f(A(x_i)) - f(A(x_i) \wedge B(x_i)) = 0,
\]

then since $f$ is strictly increasing, we have $f(A(x_i)) = f(A(x_i) \wedge B(x_i))$ for all $i \in \{1, \ldots, n\}$, therefore $A(x_i) = A(x_i) \wedge B(x_i)$; that is, $A(x_i) \leq B(x_i)$. $\blacksquare$

**Example 6.2** Some examples can be seen as follows.

1) Let $f$ be given by: $f(x) = x^\lambda$, where $\lambda \geq 1$. Then, the resulting $RIC_{\sigma}$ is:

\[
RIC_{\sigma}(A, B) = \begin{cases} \left( \frac{\sum_{i=1}^{n}(A(x_i) \wedge B(x_i))}{\sum_{i=1}^{n}A(x_i)} \right)^{\frac{1}{\lambda}}, & \text{if } A \neq 0. \\ 1, & \text{if } A = 0. \end{cases}
\]

2) Let $f$ be given by: $f(x) = e^x$. Then, the resulting $RIC_{\sigma}$ is:
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$RIC_\sigma(A, B) = \begin{cases} 
\frac{\ln \left( \frac{\sum_{i=1}^{n} e^{(A(x_i) \land B(x_i))}}{\sum_{i=1}^{n} e^{A(x_i)}} \right)}{\ln \left( \frac{\sum_{i=1}^{n} e^{A(x_i)}}{\sum_{i=1}^{n} e^{B(x_i)}} \right)}, & \text{if } A \neq 0. \\
1, & \text{if } A = 0.
\end{cases}$

**Remarks.** Concerning Theorem 6.6 we have three remarks: A), B) and C).

A) Obviously, if in Theorem 6.6 $f$ is the identity function, then the fuzzy subsethood measure that we obtain is Kosko’s, that is, $RIC_\sigma(A, B) = \sigma_{Kk}(A, B)$.

B) Please note that in Theorem 4.5 we have studied $FSM$ on $X$ that fulfill the property $\sigma(1, A) = \mathcal{M}(A)$. Therefore, if this property holds, we have that under the conditions of the aforementioned theorem:

$RIC_\sigma(A, B) = \begin{cases} 
\sigma_{(1, A \land B)} / \sigma_{(1, A)}), & \text{if } A \neq 0. \\
1, & \text{if } A = 0.
\end{cases} = \begin{cases} 
\frac{\mathcal{M}(A(x) \land B(x))}{\mathcal{M}(A(x))}, & \text{if } A \neq 0. \\
1, & \text{if } A = 0.
\end{cases}$

satisfies the following items

i) If $A \leq B$, then $RIC_\sigma(A, B) = 1$;

ii) $RIC_\sigma(A, B) = 0$ if and only if $supp(A) \cap supp(B) = \emptyset$, with $A \neq 0$;

iii) If $A \leq B$, then $RIC_\sigma(C, A) \leq RIC_\sigma(C, B)$;

iv) If $A \leq B \leq C$, then $RIC_\sigma(C, A) \leq RIC_\sigma(B, A)$.

In order to obtain from Theorem 4.5 the fulfillment of the necessary condition of i), it will be enough to take $\mathcal{M}$ generated by the same functions as those shown in Theorem 6.6.

C) Obviously, the result of Theorem 6.6 can be obtained without considering the conditions of Theorems 4.3 or 4.5, that is, without bearing in mind all the constructions of $FSM$ $\sigma$ developed in this section (constructions obtained by applying $\mathcal{M}$ to functions $I$). We only need to take an adequate $\mathcal{M}$ and apply it to the minimum of sets $A$ and $B$. Nevertheless, we showed here the way of getting Kosko’s index from our constructions.
6.4 Equality indexes REC, S and L constructed from FSM

As already said before, one of the fields where similarity measures have been successfully applied is in image processing, in which such measures are used to compare images. There are different approaches to perform that comparison and generally they are divided into two groups: the one based on obtaining the similarity between two images, starting from the shape, size, boundaries, etc. of the objects that compose the images; and the other which is based on identifying how similar two images are by considering the intensities of gray (or gray levels) of the pixels that make up the images \[21\]. Nevertheless, it has been observed that, among all known similarity measures, the only ones that produce good results in image processing are those that fulfill a special set of properties \[28\]. In \[21\], Bustince et al. presented six properties that were proposed as minimal for the global comparison of two images. This kind of comparison globally studies the distribution of intensities of gray, that is, it directly applies the similarity measures to two images.

Three indexes commonly used to compare objects are: the REC index (a similarity measure based on the operations of union and intersection), the S index (based on the difference and the sum of degrees of membership) and the L index (based on the maximum difference). In the next three subsections we are going to build FSM in the sense of Definition 4.1 from indexes: REC, S and L.

6.4.1 REC index based on the operations of union and intersection

The equality index \( REC(A, B) \) of fuzzy sets \( A \) and \( B \), is defined by:

\[
REC(A, B) = \begin{cases} 
1, & \text{if } A = B = 0, \\
\frac{\sum_{i=1}^{n} A(x_i) \land B(x_i)}{\sum_{i=1}^{n} A(x_i) \lor B(x_i)}, & \text{otherwise.}
\end{cases}
\]

This index is widely used in different applications. First Pappis and Karacapilidis \[61\]
and later Wang, Baets and Kerre [78] studied its main properties. Among the most important properties, we have the following:

1. $REC(A, B) = 1$ if and only if $A = B$;
2. $REC(A, B) = REC(B, A)$;
3. $REC(A, B) = 0$ if and only if $supp(A) \cap supp(B) = \emptyset$ and $supp(A) \cup supp(B) = X$.

The $REC$ index also satisfies the following property:

4. If $A \leq B \leq C$, then $REC(C, B) \geq REC(C, A)$ and $REC(A, B) \geq REC(A, C)$.

**Corollary 6.7** The following items hold:

i) $REC$ is an equality index in the sense of Klir and Yuan and in the sense of Dubois and Prade;

ii) $REC$ is a similarity measure in the sense of Liu.

**Proof.** Evident. □

Observe that $REC$ also satisfies the following property:

5. $REC(A \lor A_N, A \land A_N)$ is a fuzzy entropy;

And Wang, De Baets and Kerre also proved [78] that:

6. If $A \sim_{\alpha} B$, then $A \lor C \sim_{\alpha} B \lor C$. 
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Where \( A \sim_{\alpha} B \) means that \( A \) is approximately equal \( B \) with grade \( \alpha \), formally defined in subsection 6.5.

It is important to note that, by fulfilling (2) and (3), this index also fulfills two of the four properties demanded by Dubois and Prade [29] from the overlap indexes (see definition in subsection 6.6). Among them, property (3) is the one that characterizes these indexes.

In this subsection, we study the way of getting the \( REC \) index from \( FSM \) on \( X \). So, we start showing a similar expression to the one given for \( REC \) but now with functions \( \sigma \).

**Theorem 6.7** If \( \sigma \) is a fuzzy subsethood measure on \( X \), then \( REC_\sigma : F(X) \times F(X) \to [0, 1] \) given by

\[
REC_\sigma(A, B) = \begin{cases} 
1, & \text{if } A = B = 0, \\
\frac{\sigma(1, A \wedge B)}{\sigma(1, A \vee B)}, & \text{otherwise.}
\end{cases}
\]

satisfies the following items.

i) If \( A = B \), then \( REC_\sigma(A, B) = 1 \).

ii) \( REC_\sigma(A, B) = 0 \) if and only if \( supp(A) \cap supp(B) = \emptyset \).

iii) \( REC_\sigma(A, B) = REC_\sigma(B, A) \).

iv) If \( A \leq A' \leq B' \leq B \), then \( REC_\sigma(A, B) \leq REC_\sigma(A', B') \).

v) \( REC_\sigma(A, B) \leq \frac{\sigma(1, A) \wedge \sigma(1, B)}{\sigma(1, A) \vee \sigma(1, B)} \).

**Proof.**

i) If \( A = B \), it can happen that \( A = B = 0 \), and in such case \( REC_\sigma(A, B) = 1 \). Otherwise, we have that \( \sigma(1, A \wedge B) = \sigma(1, A) = \sigma(1, A \vee B) \), therefore \( REC_\sigma(A, B) = 1 \).

ii) If \( REC_\sigma(A, B) = 0 \), then \( \sigma(1, A \wedge B) = 0 \), therefore \( A \wedge B = 0 \).

If \( A \wedge B = 0 \), then \( \sigma(1, A \wedge B) = 0 \), therefore \( REC_\sigma(A, B) = 0 \).

iii) Obvious by the commutative property of \( \vee \) and \( \wedge \).
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iv) If \( A \leq A' \leq B' \leq B \), then

\[
A \land B = A \leq A' = A' \land B' \\
A \lor B = B \geq B' = A' \lor B'
\]

therefore

\[
\sigma(1, A \land B) \leq \sigma(1, A' \land B') \\
\sigma(1, A \lor B) \geq \sigma(1, A' \lor B')
\]

then

\[
REC_\sigma(A, B) = \frac{\sigma(1, A \land B)}{\sigma(1, A \lor B)} \leq \frac{\sigma(1, A' \land B')}{\sigma(1, A' \lor B')} = REC_\sigma(A', B').
\]

v) We only need to bear in mind Propositions 1 and 2 in [19] by which \( \sigma \) fulfills condition (c) of Definition 4.1 and then we have

\[
\sigma(1, A \land B) \leq \sigma(1, A) \land \sigma(1, B) \\
\sigma(1, A \lor B) \geq \sigma(1, A) \lor \sigma(1, B).
\]

In the following corollary we show the way of constructing \( REC_\sigma \) from FSM on \( X \) constructed by means of functions \( \mathcal{M} \) and \( I \).

**Corollary 6.8** Let \( \mathcal{M} : [0, 1]^n \to [0, 1] \) be such that it satisfies (A1), (A2), (A3) and \( I : [0, 1]^2 \to [0, 1] \) is a function that satisfies (I1), (I2), (I8) and (I15).

Under these conditions

\[
REC_\sigma(A, B) = \begin{cases} 
\frac{\sigma(1, A \land B)}{\sigma(1, A \lor B)}, & \text{if } A \neq 0 \text{ and } B \neq 0. \\
1, & \text{if } A = B = 0.
\end{cases}
\]

\[
= \begin{cases} 
\frac{\sum_{i=1}^{n} (\mathcal{M}(1(1, A(x_i)) \land I(1, B(x_i))))}{\mathcal{M}(1(1, A(x_i)) \lor I(1, B(x_i)))}, & \text{if } A \neq 0 \text{ and } B \neq 0. \\
1, & \text{if } A = B = 0.
\end{cases}
\]

satisfies the following items
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\[i) \text{ If } A = B, \text{ then } \text{REC}_\sigma(A, B) = 1.\]

\[ii) \text{ REC}_\sigma(A, B) = 0 \text{ if and only if } \text{supp}(A) \cap \text{supp}(B) = \emptyset.\]

\[iii) \text{ REC}_\sigma(A, B) = \text{REC}_\sigma(B, A).\]

\[iv) \text{ If } A \leq A' \leq B' \leq B, \text{ then } \text{REC}_\sigma(A, B) \leq \text{REC}_\sigma(A', B').\]

\[v) \text{ REC}_\sigma(A, B) \leq \frac{\sigma(1A) \land \sigma(1B)}{\sigma(1A) \lor \sigma(1B)}.\]

Proof. We only need to bear in mind Proposition 2.2, the construction of \(\sigma\) shown in Prop. 4.2 and Theorem 6.7.

Example 6.3 If we take \(\mathcal{M}(x_1, \ldots, x_n) = \frac{\bigvee_{i=1}^n x_i}{\bigvee_{i=1}^n (1-x_i)}\) and \(I(x, y) = 1 \land (1 - x + y)\), we have

\[
\text{REC}_\sigma(A, B) = \begin{cases} 
\frac{\bigvee_{i=1}^n (A(x_i) \land B(x_i)) - \left(\bigvee_{i=1}^n (A(x_i) \lor B(x_i))^n + \bigvee_{i=1}^n (1-A(x_i) \lor B(x_i))\right)}{\bigvee_{i=1}^n (A(x_i) \lor B(x_i)) - \left(\bigvee_{i=1}^n (A(x_i) \land B(x_i))^n + \bigvee_{i=1}^n (1-A(x_i) \land B(x_i))\right)}, & \text{if } A \neq 0 \text{ and } B \neq 0. \\
1, & \text{if } A = B = 0.
\end{cases}
\]

It is clear that we are interested in constructing \(\text{REC}_\sigma\) from FSM on \(X\) that satisfy not only the following property:

If \(A = B\), then \(\text{REC}_\sigma(A, B) = 1\)

but also

\[A = B \text{ if and only if } \text{REC}_\sigma(A, B) = 1.\]

Thus, we are going to take functions \(\mathcal{M}\) constructed from the theorem of Kolmogorov and Nagumo and obviously they fulfill the conditions demanded in Corollary 6.8; that is, (A1), (A2) and (A3).

Theorem 6.8 Let \(\mathcal{M} : [0, 1]^n \to [0, 1]\) be an operator of Kolmogorov [49] and Nagumo [58] generated by a continuous and strictly increasing function \(f : [0, 1] \to \mathbb{R}\), and let \(I : [0, 1]^2 \to [0, 1]\) be a function that satisfies (I1), (I2), (I6), (I8) and (I15).
Then

\[
REC_\sigma(A, B) = \begin{cases} 
\frac{\sigma(1A \land B)}{\sigma(1A \lor B)}, & \text{if } A \lor B \neq 0, \\
1, & \text{if } A = B = 0.
\end{cases}
\]

satisfies the following items

i) \( REC_\sigma(A, B) = 1 \) if and only if \( A = B \).

ii) \( REC_\sigma(A, B) = 0 \) if and only if \( \text{supp}(A) \cap \text{supp}(B) = \emptyset \).

iii) \( REC_\sigma(A, B) = REC_\sigma(B, A) \).

iv) If \( A \leq A' \leq B' \leq B \), then \( REC_\sigma(A, B) \leq REC_\sigma(A', B') \).

v) \( REC_\sigma(A, B) \leq \frac{\sigma(1A \land \sigma(1B))}{\sigma(1A \lor \sigma(1B))} \).

**Proof.** Theorem 5 in [20] states that: a continuous, symmetric \((A4)\), monotone operator \( \mathcal{M} \) is idempotent, decomposable and satisfies \((A1)\), \((A2)\) and \((A3S)\) if and only if

\[
\mathcal{M}(x_1, \ldots, x_n) = f^{-1}\left(\frac{1}{n} \sum_{i=1}^{n} f(x_i)\right)
\]

where \( f : [0, 1] \rightarrow \mathbb{R} \) is continuous and strictly increasing.

Therefore, \( \mathcal{M} \) satisfies the conditions demanded in Corollary 6.8. Function \( I \) also satisfies the conditions previously demanded, and besides we demand \((I6)\) to be satisfied; that is, \( I(1, x) = x \). Therefore, the proof of items \( ii) - v) \) is the same as the one done previously. Let us see that if \( REC_\sigma(A, B) = 1 \), then \( A = B \) holds.
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Obviously, if \( A = B = 0 \), then \( \text{REC}_\sigma(A, B) = 1 \). Otherwise we have

\[
f^{-1}\left( \frac{1}{n} \sum_{i=1}^{n} f(A(x_i) \wedge B(x_i)) \right) = f^{-1}\left( \frac{1}{n} \sum_{i=1}^{n} f(A(x_i) \vee B(x_i)) \right);
\]

that is,

\[
\sum_{i=1}^{n} f(A(x_i) \wedge B(x_i)) = \sum_{i=1}^{n} f(A(x_i) \vee B(x_i)),
\]

then

\[
\sum_{i=1}^{n} f(A(x_i) \vee B(x_i)) - f(A(x_i) \wedge B(x_i)) = 0
\]

since \( f \) is strictly increasing and for all \( i \in \{1, \ldots, n\}, A(x_i) \vee B(x_i) \geq A(x_i) \wedge B(x_i) \) holds. So we have that all addends are positive, therefore

\[
f(A(x_i) \vee B(x_i)) - f(A(x_i) \wedge B(x_i)) = 0;
\]

that is,

\[
f(A(x_i) \vee B(x_i)) = f(A(x_i) \wedge B(x_i)),
\]

therefore \( A(x_i) \vee B(x_i) = A(x_i) \wedge B(x_i) \), for all \( i \in \{1, \ldots, n\} \), then \( A = B \).  

**Example 6.4** Some examples can be seen as follows.

1) Let \( f \) be given by: \( f(x) = x^\lambda \), where \( \lambda \geq 1 \).

Then, the resulting \( \text{REC}_\sigma \) is:

\[
\text{REC}_\sigma(A, B) = \begin{cases} 
\left( \frac{\sum_{i=1}^{n}(A(x_i)\wedge B(x_i))^\lambda}{\sum_{i=1}^{n}(A(x_i)\vee B(x_i))^\lambda} \right)^{\frac{1}{\lambda}}, & \text{if } A \vee B \neq 0. \\
1, & \text{if } A = B = 0.
\end{cases}
\]

2) Let \( f \) be given by: \( f(x) = e^x \).

Then, the resulting \( \text{REC}_\sigma \) is:
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\[ REC_\sigma (A, B) = \begin{cases} 
\frac{\ln\left(\frac{\sum_{i=1}^{n} \sigma(A(x_i) \land B(x_i))}{\sum_{i=1}^{n} \sigma(A(x_i) \lor B(x_i))}\right)}{\ln\left(\frac{\sum_{i=1}^{n} \sigma(A(x_i) \land B(x_i))}{\sum_{i=1}^{n} \sigma(A(x_i) \lor B(x_i))}\right)}, & \text{if } A \lor B \neq 0. \\
1, & \text{if } A = B = 0.
\end{cases} \]

Remarks

1) Obviously if \( f \) is the identity function, then we obtain the \( REC \) index commonly used in the literature.

2) Note that the \( REC_\sigma \) indexes constructed from Theorem \( \text{6.8} \) are equality indexes in the sense of Klir and Yuan (Def. \( \text{6.1} \)) and in the sense of Dubois and Prade (Def. \( \text{6.2} \)) by Corollary \( \text{6.8} \) and they are also similarity measures in the sense of Liu (Def. \( \text{5.2} \)).

6.4.2 \( S \) equality index based on the difference and the sum of degrees of membership

The equality index \( S(A, B) \) of fuzzy sets \( A \) and \( B \), is defined by:

\[ S(A, B) = \begin{cases} 
1, & \text{if } A = B = 0. \\
1 - \frac{\sum_{i=1}^{n} |A(x_i) - B(x_i)|}{\sum_{i=1}^{n} [A(x_i) + B(x_i)]}, & \text{otherwise.}
\end{cases} \]

The most general properties of the equality index \( S \) are the following:

1) \( S(A, B) = 1 \) if and only if \( A = B \);

2) \( S(A, B) = S(B, A) \);

3) \( S(A, B) = 0 \) if and only if \( \text{supp}(A) \cap \text{supp}(B) = \emptyset \) and \( \text{supp}(A) \cup \text{supp}(B) = X \);

4) If \( A \leq B \leq C \), then \( S(C, A) \leq S(C, B) \);
Wang, De Baets and Kerre [78] also proved the following property:

(5) If $A \sim_\alpha B$, then $A \lor C \sim_\alpha B \lor C$.

**Corollary 6.9** $S$ is an equality index in the sense of Definition 6.2.

**Proof.** Obviously, $(EQ1)_{DP} = (1)$, $(EQ3)_{DP} = (2)$ and if (3) is satisfied then $(EQ2_{DP})$ holds. ■

It is important to note that the index $S$ does not fulfill that if $A \leq B \leq C$, then $S(A, B) \geq S(A, C)$. Therefore, it is not an equality index in the sense of Klir and Yuan, or a similarity measure in the sense of Liu Xuecheng or a semi-equality index.

This expression has been quite studied by C.P. Pappis and N.I. Karacapilidis [61] and Wang, De Baets and Kerre [78].

Our intent in this subsection is to obtain indexes from FSM on $X$. First, the expression above has suggested us to take $S(A, B)$ in the following way:

$$S(A, B) = \begin{cases} 1, & \text{if } A = B = 0. \\ 1 - \frac{\sigma(1, A \lor B - A \land B)}{\sigma(1, A) + \sigma(1, B)}, & \text{otherwise.} \end{cases}$$

Obviously, $S(A, B) \leq 1$ for all $A, B \in F(X)$. However, it does not always happen that $S(A, B) \geq 0$. For example, if we take

$$\mathcal{M}(x_1, \ldots, x_n) = \left(\frac{1}{n} \sum_{i=1}^n x_i^2\right)^2;$$

that is, the aggregation of Kolmogorov and Nagumo generated by $f : [0, 1] \to R$, such that $f(x) = x^2$. We find that $\mathcal{M}$ is idempotent and satisfies $(A1)$, $(A2)$, $(A3S)$. In addition, if we take $I$ so that it satisfies $(I1)$, $(I2)$, $(I6)$, $(I8)$ and $(I15)$, we have by Proposition 4.2 that the function:

$$\sigma(A, B) = \prod_{i=1}^n (I(A(x_i), B(x_i))),$$
is a fuzzy subsethood measure on $X$. Under these conditions, if we take the sets:

$$
A = \{ \langle x_1, 0 \rangle, \langle x_2, 0.1 \rangle \}
$$

$$
B = \{ \langle x_1, 0.9 \rangle, \langle x_2, 0 \rangle \}
$$

we have $S(A, B) < 0$.

Due to this, the expression that we are going to study is the one we propose below.

**Proposition 6.4** If $\sigma$ is a fuzzy subsethood measure on $X$, then $S_{\sigma} : F(X) \times F(X) \rightarrow [0, 1]$ given by

$$
S_{\sigma}(A, B) = \begin{cases} 
1, & \text{if } A = B = 0. \\
1 - \left(1 \wedge \frac{\sigma(1, A \lor B - A \land B)}{\sigma(1, A) + \sigma(1, B)} \right), & \text{otherwise.}
\end{cases} \tag{6.1}
$$

satisfies the following items.

i) $S_{\sigma}(A, B) = 1$ if and only if $A = B$.

ii) $S_{\sigma}(A, B) = S_{\sigma}(B, A)$.

**Proof.** i) Obviously, if $A = B = 0$ holds. If $S_{\sigma}(A, B) = 1$, then

$$
\left(1 \wedge \frac{\sigma(1, A \lor B - A \land B)}{\sigma(1, A) + \sigma(1, B)} \right) = 0,
$$

therefore $\sigma(1, A \lor B - A \land B) = 0$. Since $\sigma$ satisfies (b) we have $A \lor B - A \land B = 0$; that is, $A = B$.

If $A = B$, then $\sigma(1, A \lor B - A \land B) = \sigma(1, 0) = 0$. Therefore $S_{\sigma}(A, B) = 1$.

ii) Evident. $\blacksquare$

From Eq. (6.1) shown in Proposition 6.4 we cannot say anything when $S_{\sigma}(A, B)$ is 0. Therefore, it only satisfies two of the three conditions demanded from the equality indexes by D. Dubois and H. Prade.
Next we will study the conditions that have to be satisfied by functions \( \mathcal{M} \) and \( I \) so that functions \( S \), defined from FSM \( \sigma \) constructed with \( \mathcal{M} \) and \( I \) and obtained according to Proposition 6.4, satisfy the three conditions of D. Dubois and H. Prade.

**Theorem 6.9** Let \( \mathcal{M} : [0, 1]^n \rightarrow [0, 1] \) be an operator of Kolmogorov [49] and Nagumo [58] generated by continuous and strictly increasing \( f : [0, 1] \rightarrow \mathbb{R}^+ \) such that \( f^{-1}(a + b) = f^{-1}(a) + f^{-1}(b) \) for all \( a, b \in \mathbb{R}^+ \), and let \( I : [0, 1]^2 \rightarrow [0, 1] \) be a function that satisfies (I1), (I2), (I6), (I8) and (I15).

Then

\[ S_\sigma(A, B) = \begin{cases} 
1, & \text{if } A = B = 0. \\
1 - \frac{\sigma(\sum_{i=1}^n f(A(x_i)) + \sum_{i=1}^n f(B(x_i)))}{\sigma(\sum_{i=1}^n f(A(x_i)) + \sum_{i=1}^n f(B(x_i)))}, & \text{otherwise.}
\end{cases} \]

satisfies the following items

i) \( S_\sigma(A, B) = 1 \) if and only if \( A = B \);

ii) \( S_\sigma(A, B) = 0 \) if and only if \( \text{supp}(A) \cap \text{supp}(B) = \emptyset \) and \( \text{supp}(A) \cup \text{supp}(B) = X \);

iii) \( S_\sigma(A, B) = S_\sigma(B, A) \).

**Proof.** i) and iii) are evident by Proposition 6.4.

ii) If \( S_\sigma(A, B) = 0 \), then bearing in mind that \( f : [0, 1] \rightarrow \mathbb{R}^+ \) is continuous and strictly increasing such that \( f^{-1}(a + b) = f^{-1}(a) + f^{-1}(b) \) for all \( a, b \in \mathbb{R}^+ \) we have

\[

t^{-1} \left( \frac{1}{n} \sum_{i=1}^n f(A(x_i)) + \frac{1}{n} \sum_{i=1}^n f(B(x_i)) \right) = t^{-1} \left( \frac{1}{n} \sum_{i=1}^n f(A(x_i)) \right) + t^{-1} \left( \frac{1}{n} \sum_{i=1}^n f(B(x_i)) \right) = t^{-1} \left( \frac{1}{n} \sum_{i=1}^n f(A(x_i)) + f(B(x_i)) \right),
\]
therefore
\[ \sum_{i=1}^{n} f(A(x_i)) + f(B(x_i)) - f(A(x_i) \lor B(x_i) - A(x_i) \land B(x_i)) = 0. \]

Since \( f \) is strictly increasing we have two possibilities:

1) If \( A(x_i) \geq B(x_i) \), then since \( A(x_i) \geq A(x_i) - B(x_i) \) we have \( f(A(x_i)) + f(B(x_i)) - f(A(x_i) - B(x_i)) \geq 0 \)

2) If \( A(x_i) \leq B(x_i) \), then \( B(x_i) \geq B(x_i) - A(x_i) \) we have \( f(A(x_i)) + f(B(x_i)) - f(B(x_i) - A(x_i)) \geq 0. \)

Therefore for all \( i \in \{1, \ldots, n\} \) we have \( f(A(x_i)) + f(B(x_i)) - f(A(x_i) \lor B(x_i) - A(x_i) \land B(x_i)) = 0. \)

Under these conditions two things can happen:

1) If \( A(x_i) \geq B(x_i) \), then \( f(B(x_i)) = 0. \) Since \( f^{-1} \) satisfies: \( f^{-1}(a+b) = f^{-1}(a) + f^{-1}(b) \), we have \( f^{-1}(a+0) = f^{-1}(a) + f^{-1}(0) = f^{-1}(a) \), then \( f^{-1}(0) = 0; \) that is, \( f(0) = 0. \) Therefore \( B(x_i) = 0. \)

2) If \( A(x_i) \leq B(x_i) \), then \( f(A(x_i)) = 0, \) therefore \( A(x_i) = 0. \)

Therefore \( A \land B = 0. \)

If \( A \land B = 0, \) then
\[
\begin{align*}
&f^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} f(A(x_i)) \right) + f^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} f(B(x_i)) \right) = \\
&f^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} f(A(x_i) + f(B(x_i)) \right) \geq \\
&f^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} f(A(x_i) \lor B(x_i)) \right); \\
\end{align*}
\]

that is,
\[
\frac{1}{n} \sum_{i=1}^{n} f(A(x_i)) + f(B(x_i)) - f(A(x_i) \lor B(x_i)) \geq 0.
\]
Evidently, if $A = B = 0$, then

$$\frac{1}{n} \sum_{i=1}^{n} f(A(x_i)) + f(B(x_i)) - f(A(x_i) \lor B(x_i)) = 0.$$ 

If $A \land B = 0$, then $A(x_i) = 0$ or $B(x_i) = 0$.

If $A(x_i) = 0$, then $f(A(x_i)) + f(B(x_i)) - f(A(x_i) \lor B(x_i)) = 0 + f(B(x_i)) - f(B(x_i)) = 0$.

If $B(x_i) = 0$, then $f(A(x_i)) + f(B(x_i)) - f(A(x_i) \lor B(x_i)) = f(A(x_i)) + 0 - f(A(x_i)) = 0$.

Therefore it always happens:

$$\frac{1}{n} \sum_{i=1}^{n} f(A(x_i)) + f(B(x_i)) - f(A(x_i) \lor B(x_i)) = 0.$$ 

Then

$$f^{-1}\left(\frac{1}{n} \sum_{i=1}^{n} f(A(x_i))\right) + f^{-1}\left(\frac{1}{n} \sum_{i=1}^{n} f(B(x_i))\right) =$$

$$f^{-1}\left(\frac{1}{n} \sum_{i=1}^{n} f(A(x_i)) + f(B(x_i))\right) =$$

$$f^{-1}\left(\frac{1}{n} \sum_{i=1}^{n} f(A(x_i) \lor B(x_i) - A(x_i) \land B(x_i))\right);$$

that is, \( S_\sigma(A, B) = 1 - 1 = 0 \). 

**Corollary 6.10** Under the conditions of Theorem 6.9, \( S_\sigma \) is an equality index in the sense of Definition 6.2.

**Proof.** We only need to bear in mind that if item ii) of the Theorem 6.9 is satisfied, then (EQ2DP) holds.

**Remarks.** We conclude this subsection with remarks 1) and 2) as follows.

1) We know that if \( f \) is a continuous function, then \( f(a + b) = f(a) + f(b) \) for all \( a, b \) if and only if there exists a number \( k \) such that \( f(a) = ka \) for all \( a \). In our case, we are taking
functions \( f : [0, 1] \to \mathbb{R}^+ \), (and considering the inverse functions), therefore \( k > 0 \). Then, the only functions that satisfy the hypothesis of the theorem above are of the form \( f(x) = kx \) with \( k > 0 \). Obviously, from the theorem of Kolmogorov and Nagumo we always obtain the arithmetic mean with these functions; that is,

\[
\mathcal{M}(x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i.
\]

Therefore, under these conditions the expression that we obtain for the functions \( S_{\sigma} \) is the one presented by C.P. Pappis and N.I. Karacapilidis [61]

\[
S_{\sigma}(A, B) = \begin{cases} 
1, & \text{if } A = B = 0. \\
1 - \frac{\sum_{i=1}^{n} |A(x_i) - B(x_i)|}{\sum_{i=1}^{n} (A(x_i) + B(x_i))}, & \text{otherwise.}
\end{cases}
\]

By Theorem [6.9] it is clear that such expression can be obtained from FSM, as we managed to prove.

2) If \( A \leq A' \leq B' \leq B \), then in general \( S_{\sigma}(A, B) \leq S_{\sigma}(A', B') \) does not hold.

### 6.4.3 L equality index based on the maximum difference

The expression:

\[
L(A, B) = 1 - \sqrt[n]{\sum_{i=1}^{n} |A(x_i) - B(x_i)|},
\]

satisfies the following properties:

1. \( L(A, B) = 1 \) if and only if \( A = B \);
2. \( L(A, B) = L(B, A) \);
3. If \( A \leq B \leq C \), then \( L(C, B) \geq L(C, A) \) and \( L(A, B) \geq L(A, C) \);
4. If \( A \sim_{\alpha} B \), then \( A \lor C \sim_{\alpha} B \lor C \) and \( A \land C \sim_{\alpha} B \land C \) and \( A_N \sim_{\alpha} B_N \).
Corollary 6.11  

$L$ is an equality index in the sense of Definition 6.1.

Proof. We only need to bear in mind that $(EQ_{1KY}) = (1)$ and $(EQ_{2KY}) = (3). \square$

Remarks

1) Please note that $L$ is not an equality index in the sense of Dubois and Prade (Def. 6.2), because it does not fulfill that if $A \land B = 0$, then $L(A, B) = 0$.

2) It is important to note that the aggregation $M = \vee$ does not fulfill condition (A1), therefore, we cannot generate $L$ from fuzzy subsethood measures constructed with $M$ and $I$ according to Proposition 4.2.

3) Despite of the considerations above, the expression of the $L$ index given in Eq. (6.2) has led us to present a new index which we will denote by $L_{\sigma}$. Note that it is not either an equality index in the sense of D. Dubois and H. Prade (as it does not fulfill that if $A \land B = 0$, then $L_{\sigma}(A, B) = 0$). Besides, as we will see in the following theorem, it is a semi-equality index and it is also an equality index in the sense of Klir and Yuan (Def. 6.1).

Theorem 6.10  

Let $N$ be a strong negation and let $\sigma$ be a fuzzy subsethood measure on $X$. Then $L_{\sigma} : F(X) \times F(X) \to [0, 1]$ given by

$$L_{\sigma}(A, B) = N(\sigma(1, A \lor B - A \land B))$$

satisfies the following items.

i) $L_{\sigma}(A, B) = 1$ if and only if $A = B$;

ii) $L_{\sigma}(A, B) = 0$ if and only if $A$ and $B$ are complementary crisp sets;

iii) $L_{\sigma}(A, B) = L_{\sigma}(B, A)$;

iv) If $A \leq A' \leq B' \leq B$, then $L_{\sigma}(A, B) \leq L_{\sigma}(A', B')$. 

6.5 Semi-equality index and concept of approximately equal with grade $\alpha$

Wang, Baets and Kerre [78] from the concept of approximately equal with grade $\alpha$ of Pappis and Karacapilidis, give the next definition.

**Definition 6.6** [78] Let $A, B \in F(X)$. $A$ and $B$ are said to be approximately equal with grade $\alpha$ (denoted by $A \sim_\alpha B$) if and only if $W(A, B) \geq \alpha$, where $\alpha \in [0, 1]$ and $W = REC$ or $W = S$ or $W = L$. 

---

**Proof.** i) If $L_\alpha(A, B) = 1$, then $\sigma(1, A \lor B - A \land B) = 0$. Bearing in mind that $\sigma$ satisfies (b) in Def. 4.1 we have $A \lor B - A \land B = 0$; that is, $A = B$.

If $A = B$, then $\sigma(1, A \lor B - A \land B) = \sigma(1, 0) = 0$. Therefore $L_\alpha(A, B) = N(0) = 1$.

ii) If $L_\alpha(A, B) = 0$, then $\sigma(1, A \lor B - A \land B) = 1$. Since $\sigma$ satisfies (a) we have $A \lor B - A \land B = 1$; that is, $A \lor B = 1$ and $A \land B = 0$, therefore $A$ and $B$ are complementary crisp sets.

If $A$ and $B$ are complementary crisp sets we have that $A \lor B = 1$ and $A \land B = 0$, therefore $A \lor B - A \land B = 1$, then $\sigma(1, A \lor B - A \land B) = \sigma(1, 1) = 1$, thus $L_\alpha(A, B) = N(1) = 0$.

iii) Evident.

iv) If $A \leq A' \leq B' \leq B$, then

\[
A \lor B = B \geq B' = A' \lor B' \\
A \land B = A \leq A' = A' \land B',
\]

then $A \lor B - A \land B \geq A' \lor B' - A' \land B'$. Since $\sigma$ satisfies (c) we have $\sigma(1, A \lor B - A \land B) \geq \sigma(1, A' \lor B' - A' \land B')$. Therefore, $L_\alpha(A, B) = N(\sigma(1, A \lor B - A \land B)) \leq N(\sigma(1, A' \lor B' - A' \land B')) = L_\alpha(A', B')$.  

\[\blacksquare\]
Note that $W$ is a measure, defined by Van der Wekens (see [28] for details) and used for the global comparison of two images. $S$ and $L$ refer to equality indexes seen in previous subsections 6.4.2 and 6.4.3 respectively.

In this section we are going to use the definition above taking $W = SE$.

**Proposition 6.5** Under the conditions of Proposition 4.2. Let $A$, $B$ and $C$ be three fuzzy sets in a universe $X$ and $\alpha \in [0, 1]$, then the following properties hold:

i) If $A \sim_{\alpha} B$, then $A \lor C \sim_{\alpha} B \lor C$;

ii) If $A \sim_{\alpha} B$, then $A \land C \sim_{\alpha} B \land C$;

iii) If $I$ satisfies $(I12)$, then

\[ A \sim_{\alpha} B \text{ if and only if } A_N \sim_{\alpha} B_N. \]

**Proof.** We only need to bear in mind the constructions presented in Chapter 4.

If in Definition 6.6 instead of taking $W$ as the semi-equality index, we take $W = se$, it is proven similarly to Proposition 6.5 that under those conditions the three properties below are satisfied:

1) If $A \sim_{\alpha} B$, then $A \lor C \sim_{\alpha} B \lor C$;

2) If $A \sim_{\alpha} B$, then $A \land C \sim_{\alpha} B \land C$;

3) If $I$ satisfies $(I12)$, then

\[ A \sim_{\alpha} B \text{ if and only if } A_N \sim_{\alpha} B_N. \]

It is important to keep in mind that $se$ is not a semi-equality index (as it only satisfies three of the four conditions demanded in Definition 6.3).
6. A study on indexes generated from FSM

6.6 Overlap index

Given two fuzzy sets $A$ and $B \in F(X)$, if we are willing to quantify up to what extent there is an intersection between $A$ and $B$, then we need a degree of intersection (overlap). If the sets were crisp, we could denote such degree as $O(A, B) = 1$ if and only if $A \cap B \neq \emptyset$, but for fuzzy sets that task is not so trivial. In the fuzzy literature, the classic overlap index ($ROC$) is given by the following expression:

$$ROC(A, B) = \frac{\sum_{i=1}^{n} (A(x_i) \land B(x_i))}{\left(\sum_{i=1}^{n} A(x_i)\right) \cdot \left(\sum_{i=1}^{n} B(x_i)\right)}.$$  \hspace{1cm} (6.3)

with $A$ and $B$ different from zero.

This overlap index satisfies the following properties:

1. $ROC(A, B) = 0$ if and only if $A$ and $B$ have disjoint support, that is, $supp(A) \cap supp(B) = \emptyset$;

2. $ROC(A, B) = ROC(B, A)$.

We know that in 1982 Dubois and Prade \cite{31} wrote that the overlap indexes are functions $O : F(X) \times F(X) \rightarrow [0, 1]$, such that $O(A, B)$ fulfills the four conditions below:

$(O1)$ $O(A, B) = 0$ if and only if $supp(A) \cap supp(B) = \emptyset$;

$(O2)$ $O(A, B) = 1$, whenever $A \subseteq B$ or $B \subseteq A$;

$(O3)$ $O(A, B) = O(B, A)$;

$(O4)$ If $B \leq C$, then $O(A, B) \leq O(A, C)$.

Of course, for Dubois and Prade, $ROC$ is not an overlap index, for we have said before $ROC(A, B)$ can be greater than one.
With respect to condition \((O2)\), it has the advantage that if \(A\) is crisp, then \(O(A, A) = 1\). However, D. Dubois, W. Ostasiewicz and H. Prade in \([29]\) reveal that it has the following disadvantages:

1) For subnormal fuzzy sets, \((O2)\) can be violated.

2) \(ROC\) index does not satisfy this property.

It is also interesting to note the following:

3) If \(A = 0\), then \(A \land A = 0\) and by \((O1)\) it results \(O(A, A) = 0\). If we impose at the same time \((O2)\), then \(O(A, A) = 1\). That is, we have a contradiction. Due to this, we rewrite \((O1)\) in the following way:

\[(O1) \ O(A, B) = 0 \text{ if and only if } supp(A) \cap supp(B) = \emptyset \text{ and } supp(A) \cup supp(B) = X.\]

We present in the following theorem a way of constructing FSM from the \(ROC\) index which fulfill at least \((O1)\) and \((O3)\).

**Theorem 6.11** Let \(\sigma\) be a fuzzy subsethood measure on \(X\). Then \(ROC_\sigma : F(X) \times F(X) \to [0, 1]\) given by

\[ROC_\sigma(A, B) = \sigma(1, A \land B)\]  

(6.4)

satisfies the following items.

i) \(ROC_\sigma(A, B) = 0 \text{ if and only if } supp(A) \cap supp(B) = \emptyset;\)

ii) \(ROC_\sigma(A, B) = 1 \text{ if and only if } A = 1 \text{ and } B = 1;\)

iii) \(ROC_\sigma(A, B) = ROC_\sigma(B, A);\)

iv) If \(B \leq C\), then \(ROC_\sigma(A, B) \leq ROC_\sigma(A, C).\)
Proof. i) Since $\sigma$ satisfies (b) we have $ROC_\sigma(A, B) = 0 = \sigma(1, A \land B)$ if and only if $A \land B = 0$.

ii) $ROC_\sigma(A, B) = 1 = \sigma(1, A \land B)$ since $\sigma$ satisfies (a) we have that $\sigma(1, A \land B) = 1$ if and only if $A \land B = 1$; that is, $A = 1$ and $B = 1$.

iii) Evident.

iv) We know that if $B \leq C$, then $A \land B \leq A \land C$. Bearing in mind that $\sigma$ is decreasing in the second argument we have $ROC_\sigma(A, B) = \sigma(1, A \land B) \leq \sigma(1, A \land C) = ROC_\sigma(A, C)$.

Remarks.

1) Both $ROC_\sigma$ and $ROC$ satisfy the two properties (1) = $(O1) = i)$ and (2) = $(O3) = iii)$.

2) The expression given in Eq. (6.4) shown in Theorem 6.11 satisfies not only $(O1)$ in the sense of Dubois and Prade and $(O3)$, but also satisfies $(O4)$. However, it does not satisfy $(O2)$, once item $ii)$ is much more restrictive than the condition demanded by D. Dubois and H. Prade.

3) $ROC_\sigma(A, B) \leq \sigma(1, A) \land \sigma(1, B)$.

4) If $A \leq B$, then $ROC_\sigma(A, B) = \sigma(1, A)$. If $B \leq A$, then $ROC_\sigma(A, B) = \sigma(1, B)$.

In the following corollary we show the way of obtaining $ROC_\sigma$ from an aggregation function $M$ and an implication operator $I$.

**Corollary 6.12** Let $\mathcal{M} : [0, 1]^n \rightarrow [0, 1]$ be such that it satisfies $(A1)$, $(A2)$, $(A3)$ and let $I : [0, 1]^2 \rightarrow [0, 1]$ be a function that satisfies $(I1)$, $(I2)$, $(I8)$ and $(I15)$.

Under these conditions

$$ROC_\sigma(A, B) = \frac{n}{\sum_{i=1}^{n} I(1, A(x_i) \land B(x_i))}$$

satisfies the following items
6. A study on indexes generated from FSM

i) \( ROC_\sigma(A, B) = 0 \) if and only if \( \text{supp}(A) \cap \text{supp}(B) = \emptyset \);

ii) \( ROC_\sigma(A, B) = 1 \) if and only if \( A = 1 \) and \( B = 1 \);

iii) \( ROC_\sigma(A, B) = ROC_\sigma(B, A) \);

iv) If \( B \leq C \), then \( ROC_\sigma(A, B) \leq ROC_\sigma(A, C) \).

Proof. We only need to build \( \text{FSM} \sigma \) from \( M \) and \( I \) according to Proposition 4.2 and apply Theorem 6.11. ■

Note that the classic overlap index given in Eq. (6.3) can be constructed from \( \text{FSM} \) and Corollary 6.12. We only need to take \( M \) equal to the arithmetic mean and take any implication operator \( I \) that satisfies the conditions demanded in Corollary 6.12 and property (I6); that is, \( I(1, x) = x \) for all \( x \in [0, 1] \), in such a way that:

\[
ROC(A, B) = \frac{ROC_\sigma(A, B)}{ROC_\sigma(1, A) \cdot ROC_\sigma(1, B)} = \frac{\sigma(1, A \land B)}{\sigma(1, A) \cdot \sigma(1, B)} = \frac{\sum_{i=1}^{n} A(x_i) \land B(x_i)}{\left(\sum_{i=1}^{n} A(x_i)\right) \cdot \left(\sum_{i=1}^{n} B(x_i)\right)}
\]

However, we still maintain the fact that \( ROC(A, B) > 1 \) can happen.

6.7 Relationships between indexes generated from FSM

We conclude this chapter comparing some of the indexes and measures presented previously. Note that the indexes and measures used here are always generated from the new class of \( \text{FSM} \sigma \).
Corollary 6.13 If $\sigma$ is a fuzzy subsethood measure, then

$$RIC_{\sigma}(A, B) \geq REC_{\sigma}(A, B) \geq ROC_{\sigma}(A, B), \quad \text{for all } A, B \in F(X).$$

**Proof.** By Theorems 6.5, 6.7 and 6.11 we have:

$$RIC_{\sigma}(A, B) = \begin{cases} \frac{\sigma(1, A \land B)}{\sigma(1, A)}, & \text{if } A \neq 0. \\ 1, & \text{if } A = 0. \end{cases}$$

$$REC_{\sigma}(A, B) = \begin{cases} \frac{\sigma(1, A \land B)}{\sigma(1, A \lor B)}, & \text{if } A \lor B \neq 0. \\ 1, & \text{if } A = B = 0. \end{cases}$$

$$ROC_{\sigma}(A, B) = \sigma(1, A \land B).$$

Obviously, if $A = 0$, then $RIC_{\sigma}(A, B) = 1 \geq REC_{\sigma}(A, B)$. If $A = B = 0$, then $RIC_{\sigma}(A, B) = 1 = REC_{\sigma}(A, B)$. Otherwise we have $A \leq A \lor B$, then $\sigma(1, A) \leq \sigma(1, A \lor B)$, therefore

$$\frac{\sigma(1, A \land B)}{\sigma(1, A)} \geq \frac{\sigma(1, A \land B)}{\sigma(1, A \lor B)},$$

then $RIC_{\sigma}(A, B) \geq REC_{\sigma}(A, B)$.

Since $\frac{\sigma(1, A \land B)}{\sigma(1, A \lor B)} \geq \sigma(1, A \land B)$ we have $REC_{\sigma}(A, B) \geq ROC_{\sigma}(A, B)$. ■

Obviously,

$$RIC_{\sigma}(A, B) = \begin{cases} \frac{ROC_{\sigma}(A, B)}{ROC_{\sigma}(1, A)}, & \text{if } A \neq 0. \\ 1, & \text{if } A = 0. \end{cases}$$

$$REC_{\sigma}(A, B) = \begin{cases} \frac{ROC_{\sigma}(A, B)}{ROC_{\sigma}(1, A \lor B)}, & \text{if } A \lor B \neq 0. \\ 1, & \text{if } A = B = 0. \end{cases}$$

In the following corollary we relate $ROC_{\sigma}$ with semi-equality index and fuzzy entropy.
Corollary 6.14 For all $A, B \in F(X)$ if $\sigma$ is a fuzzy subethood measure, then the following items hold.

i) $ROC_\sigma(A, B) = SE_\sigma(1, A \land B) = se_\sigma(1, A \land B)$;

ii) $se_\sigma(A, B) \geq SE_\sigma(A, B) \geq ROC_\sigma(A, B)$.

iii) $E(A) \geq ROC_\sigma(A, A_N)$.

Proof. i) $ROC_\sigma(A, B) = \sigma(1, A \land B) = \sigma(1 \lor (A \land B), 1 \land (A \land B)) = SE_\sigma(1, A \land B)$. On the other hand, $se_\sigma(1, A \land B) = \sigma(1, A \land B) \land \sigma(A \land B, 1) = \sigma(1, A \land B) = ROC_\sigma(A, B)$.

ii) We know by the item v) of Proposition 6.3 that $se_\sigma(A, B) \geq SE_\sigma(A, B)$. Since $A \lor B \leq 1$ we have $SE_\sigma(A, B) = \sigma(A \lor B, A \land B) \geq \sigma(1, A \land B) = ROC_\sigma(A, B)$.

iii) $E(A) = \sigma(A \lor A_N, A \land A_N) \geq \sigma(1, A \land A_N) = ROC_\sigma(A, A_N)$. ■

By item i) of Corollary 6.14 we have

$$
RIC_\sigma(A, B) = \begin{cases} 
\frac{SE_\sigma(1A \land B)}{SE_\sigma(1A)}, & \text{if } A \neq 0. \\
1, & \text{if } A = 0. 
\end{cases}
$$

$$
REC_\sigma(A, B) = \begin{cases} 
\frac{SE_\sigma(1A \lor B)}{SE_\sigma(1A \lor B)}, & \text{if } A \lor B \neq 0. \\
1, & \text{if } A = B = 0. 
\end{cases}
$$

Of course, bearing in mind Corollaries 6.12 and 6.13 we have:

$$
RIC_\sigma(A, B) \geq REC_\sigma(A, B) \geq ROC_\sigma(A, B) = SE_\sigma(1, A \land B) = se_\sigma(1, A \land B).
$$
Considering the previous corollaries and the definitions of \(se_\sigma, SE_\sigma, RIC_\sigma, REC_\sigma\) and \(ROC_\sigma\) constructed from FSM \(\sigma\), it is easily proven that if \(A \neq 0\), then

\[
\frac{se_\sigma(A, B)}{se_\sigma(1, A)} \geq \frac{SE_\sigma(A, B)}{SE_\sigma(1, A)} \geq RIC_\sigma(A, B) \geq REC_\sigma(A, B) \geq ROC_\sigma(A, B) = SE_\sigma(1, A \land B).
\]

With respect to \(S_\sigma\) and \(L_\sigma\) we can say:

\[
S_\sigma(A, B) = \begin{cases} 
1, & \text{if } A = B = 0, \\
1 - \left(1 \land \frac{\sigma(1, A \lor B - A \land B)}{\sigma(1, A) + \sigma(1, B)}\right), & \text{otherwise}.
\end{cases}
\]

\[
L_\sigma(A, B) = \begin{cases} 
1, & \text{if } A = B = 0, \\
1 - \left(1 \land \frac{N(L_\sigma(A, B))}{N(L_\sigma(1, 1 - A)) + N(L_\sigma(1, 1 - B))}\right), & \text{otherwise}.
\end{cases}
\]

### 6.8 General considerations

In this chapter we studied some indexes, namely: equality and semi-equality indexes, the inclusion index of Kosko, equality indexes: \(REC, L\) and \(S\), and also the overlap index. We provided some examples of indexes constructed from FSM in the sense of Definition 4.1 and presented some relationships between the indexes and measures investigated in this chapter.

Note that the construction of indexes from the new class of FSM \(\sigma\) can be useful for the general comparison of any two fuzzy sets. They have common properties but they may also differ from each other. So it is crucial to keep those differences in mind whenever considering the use of those indexes, for instance, in problems where it is required to select the appropriate measure in different applications, for example in fuzzy control, neural networks, etc.
Table 6.2 summarizes some of the properties related to the indexes \(REC, S, L\) and \(ROC\) constructed from FSM \(\sigma\).

<table>
<thead>
<tr>
<th>Indexes constructed from FSM (\sigma)</th>
<th>(REC)</th>
<th>(S)</th>
<th>(L)</th>
<th>(ROC)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equality index in the sense of Klir and Yuan (Def. 6.1)</td>
<td>yes</td>
<td>no</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>Equality index in the sense of Dubois and Prade (Def. 6.2)</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>Similarity measure in the sense of Liu (Def. 5.2)</td>
<td>yes</td>
<td>no</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>Fuzzy entropy</td>
<td>yes</td>
<td>no</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>Semi-equality index (Def. 6.3)</td>
<td>yes</td>
<td>no</td>
<td>yes</td>
<td>yes</td>
</tr>
</tbody>
</table>

Table 6.2: Summary of properties related to the indexes constructed from FSM \(\sigma\).
Chapter 7

Concluding remarks

In this work, our main goal was to introduce a new class of subsethood measures between fuzzy sets. In order to achieve our proposal, we studied the most relevant axiomatizations found in the literature to establish the desirable properties our proposal should have. In this way, we could observe that:

- Kitainik’s FSM demanded only four requirements but they were enough to capture almost the entire essence of Sinha and Dougherty’s approach, which presented twelve axioms;

- Some of the axiomatizations require properties that allow to connect FSM with fuzzy entropy;

- Building subsethood grades by means of aggregating implication operators, such as Eq. [1.1] seems suitable for many applications.

Thus, we proposed a new axiomatic approach showing how we can use aggregation functions and implication operators. In other words, we presented methods for constructing FSM in the sense of Definition [4.1].

It is clear that we managed to introduce a simple axiomatization, by requiring only three axioms, without losing its strength. For example, Goguen’s measure (Eq. [1.2]) proved to be
a specific example of our proposal as it satisfies the properties demanded in Definition 4.1 and therefore our new class of FSM is useful in every application where Goguen’s subsethood grade can be applied.

Besides, the fact that we can use different aggregation functions and implication operators gives us more flexibility to construct, for instance, comparison measures that can be applied in image processing [21].

Other contributions of our work include:

* The introduction of different methods to build our new class of FSM $\sigma$, as seen in Propositions 4.1 and 4.2.

* Analysis of the properties of those FSM, depending on the aggregation functions considered and also depending on the properties demanded for functions $I$, such as Corollaries 4.5, 4.6, 4.7 and Theorems 4.2, 4.3, 4.4 and 4.5.

* Construction of measures from our FSM $\sigma$:
  - Penalty functions, given in Propositions 4.3 and 4.4.
  - Distance measures, given in Proposition 5.1 and Corollary 5.1.
  - Similarity measure, given in Proposition 5.2.
  - Fuzzy entropy, seen in Theorems 5.1 and 5.4.

* Construction of indexes from our FSM $\sigma$:
  - Equality indexes, seen for instance in Theorems 6.1, 6.2 and 6.3.
  - Semi-equality indexes, seen in section 6.2.
  - Inclusion index of Kosko, seen in section 6.3.
  - Indexes: $REC$, $L$ and $S$, seen in section 6.4.
  - Semi-equality index and concept of approximately equal with grade $\alpha$, seen in section 6.5.
In chapters 5 and 6 we concentrated the study on measures and indexes generated from our new class of FSM and also the relationships between them. In this way, an interesting area where our new class of FSM can be applied is in the field of image processing problems. For example, we could observe in the fifth chapter an example that used fuzzy entropy constructed from FSM \( \sigma \) and applied in the technique of thresholding (subsection 5.3.1).

A considerable part of this thesis (up to the fifth chapter, has been submitted to an important journal within the area. The paper is entitled “Fuzzy Subsethood Measures and their relations with Similarity Measures, Distance Measures, Fuzzy Entropy and Penalty Functions” and its current status is under review.

Our future research includes the use of our developments in image processing to build tools such as the SUSAN features detector [73] or to define erosion and dilation in fuzzy mathematical morphology, following the ideas of De Baets et al. [3]. We could also study the use of our constructions in other extensions of fuzzy sets such as Atanassov intuitionistic fuzzy sets, hesitant fuzzy sets and interval-valued hesitant fuzzy sets, for instance in the sense of [12, 25, 26, 53, 54, 66, 70].
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