Analytic Study of Cosmological Perturbations in a Unified Model of Dark Matter and Dark Energy with a Sharp Transition

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We study cosmological perturbations in a model of unified dark matter and dark energy with a sharp transition in the late-time universe. The dark sector is described by a dark fluid which evolves from an early stage at redshifts $z > z_C$ when it behaves as cold dark matter (CDM) to a late time dark energy (DE) phase ($z < z_C$) when the equation of state parameter is $w = -1 + \epsilon$, with a constant $\epsilon$ which must be in the range $0 < \epsilon < 2/3$. We show that fluctuations in the dark energy phase suffer from an exponential instability, the mode functions growing both as a function of comoving momentum $k$ and of conformal time $\eta$. In order that this exponential instability does not lead to distortions of the energy density power spectrum on scales for which we have good observational results, the redshift $z_C$ of transition between the two phases is constrained to be so close to zero that the model is unable to explain the supernova data.

I. INTRODUCTION

Observations of large scale structure [1, 2] are consistent with a cosmological model where cold dark matter (CDM) dominates the evolution of the universe after recombination. CDM produces a decelerating expansion of the universe. However, magnitude-redshift studies of supernovae of type Ia (SNIa) [3, 4] indicate that the universe’s expansion is currently accelerating. This demands the presence of an exotic component dubbed dark energy (DE) which has an equation of state close to that of a cosmological constant and which dominates the energy density at the present time. This component affects the detailed peak structure of the angular power spectrum of cosmic microwave background radiation (CMB) anisotropies (see e.g. [5, 6] and [7, 8] for textbook discussions). The current data (see e.g. [9]) give detailed support of the current picture in which the universe is dominated by CDM until the DE component takes over in the recent past.

The cosmological constant $\Lambda$ is the simplest candidate for dark energy: it is constant in space and time and has a geometrical justification in that it is a term which can be added to the Einstein-Hilbert action of General Relativity, and it fits current observational data available with great accuracy (see e.g. [10]). However, positing a cosmological constant as dark energy leads to a number of conceptual problems (see e.g. [11, 12]): Since the vacuum energy of quantum fields acts as a cosmological constant and has an expected magnitude which can be estimated to be many orders of magnitude larger than the currently observed dark energy density, it leads to the problem of not knowing what causes the cosmological constant to be so small. This is the “old” cosmological constant problem. If we had an explanation for a suppressed value, we would still have to explain the coincidence that the cosmological constant begins to rear its head at the present time. This is the “coincidence” problem. Hence, it is of interest to pursue other explanations for the presence of dark energy.

Alternatives to a cosmological constant as dark energy (the so-called $\Lambda$CDM model) have been explored as possible explanations for the present-day acceleration while preserving the conditions for structure formation in the young universe. A class of such scenarios is based on adding a new component of matter, e.g. quintessence [13–18], k-essence models [19, 21] – both field-theory-based frameworks – or a Chaplygin gas [22, 23] – a single fluid with an exotic equation of state – to name but a few. Constraints to some of these models were presented in [24]. There are

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Note that even if dark energy is a new component of matter, the “old” cosmological constant problem remains.
also attempts to explain dark energy using modified gravity (see e.g. [25–27]). They prefer to alter the geometrical

To explain the observed background cosmology, what one needs is: (1) two dark components which in turn dominate

The purpose of this paper is to study the constraints for our quartessence models which come from demanding

In Ref. [42] Aurich and Lustig worked out the CMB angular power spectrum predicted by the arctan-UDM model

However, an acceptable model for the unified dark sector must also agree with the data on cosmological pertur-

Both in the case of CGC and in the arctan-UDM model mentioned above, the linearized fluctuation equations were

To simplify the analytic analysis of the perturbed equations in a UDM model we will simplify the functional form

with a constant $\epsilon$ in the range

2 Positive values of the parameter $\alpha$ in GCG models lead to spurious oscillations in the small-scales regime of the matter power spectrum also if $|\alpha| < 10^{-5}$.

3 This is the case if one admits a vanishing effective speed of sound $c_{\text{eff}}^2$. 

\[ w(z) = \begin{cases} 
-1 + \epsilon, & 0 < z < z_C \\
0, & z \geq z_C
\end{cases} \] 

\[ 0 < \epsilon \leq 2/3. \]
The value \( z = z_C \) marks the sharp transition between a matter-like behavior \((w = 0)\) at high redshift \((z \geq z_C)\) and a dark-energy-like regime at recent times \((0 \leq z \leq z_C)\). For \( w = -1 + \epsilon \) to be compatible with a DE phase, it should satisfy \( w < -1/3 \), which explains the upper limit for the interval in (4). The value \( w = -1 \) corresponds to a cosmological constant. Note that \( w < -1 \) is the phantom regime. Since we want to avoid the phantom regime and also not describe a pure cosmological constant, we should have \( \epsilon > 0 \) as in the lower limit (4).

Notice that Eq. (3) indeed describes a UDM model because the single dark fluid evolves from a CDM regime to a DE regime, even though this transition is instantaneous. Conversely, in the \( \Lambda \)CDM model, dark matter and dark energy co-exist, even though CDM dominates initially and then DE takes over as the main (but not the only) component in the form of \( \Lambda \). However, to be consistent with observations, our piecewise \( w(z) \) should reproduce the background dynamics of the \( \Lambda \)CDM model today. In particular, this implies that the first line of Eq. (3) should reproduce the effective EoS parameter of that model at the current time:

\[
 w_{\text{eff},0}^{\Lambda\text{CDM}} = \left(\frac{p_{\text{CDM},0} + p_A}{\epsilon_{\text{CDM},0} + \epsilon_A}\right) \frac{-1}{(\Omega_{\text{CDM},0}/\Omega_A) + 1} \simeq -0.72. \tag{5}
\]

\( p \) is the pressure and \( \epsilon \) is the energy density. Hence, we should have \( \epsilon \simeq 0.28 \). Notice that \( \epsilon \) is not negligible, so our model’s dark energy phase is very different from a pure \( \Lambda \) phase.

In spite of its simplicity, the piecewise constant UDM model can be viewed as a limit towards which other UDM models tend to. This is particularly true for the case of the arctan-UDM model of Eq. (1). For this model, the transition redshift \( z_C = \beta/\alpha \) corresponds to \( w(z_C) = -1/2 \) and the \( \alpha \) parameter controls the rate in which the transition occurs. By taking \( (\alpha, \beta) \to \infty \) but keeping \( z_C \) finite we guarantee that the arctan-UDM EoS parameter undergoes a sharp transition just like the one described by our piecewise-UDM equation of state.

The parameters \( (\alpha, \beta) \) in the generalized Chaplygin gas \( w(z) \) – Eq. (2) – can also be tuned to produce a brisk transition from CDM-like behavior to a DE-like pattern, although this choice is more evolved than in the arctan-UDM case. This is because the transition rate \( v_C \) in a GCG model is intertwined with the transition redshift \( z_C \). For instance, when one increases the value of \( \alpha > 0 \) to increase the transition slope \( v_C \) then \( z_C \) approaches zero.

A third class of UDM models which tends to the piecewise-UDM model at the background level is the one proposed by Linder and Huterer \[30\] characterized by \( ^4 \):

\[
 w(z) = w_i + \frac{w_f - w_i}{1 + (\frac{z}{z_C})^{1/d}}, \tag{6}
\]

where label \( i \) (\( f \)) stand for the initial (final) values of \( w \), \( z_i = z_C \) is the transition redshift and \( d \) is the fourth parameter in the model. This model is a generalization of the GCG scenario, which can be recovered by letting \( w_i = 0 \), \( w_f = -1 \), \( (1 + z_i) = \beta^d \) and \( d = 1/(1 + \alpha) \). Consequently, a sharp CDM-to-DE transition is not trivially obtained in the Linder-Huterer parameterization for \( w(z) \). It is nonetheless possible to show there is such a transition in this four-parameter model \( ^5 \).

Given these connections with other UDM parameterizations, it is interesting to use the model with the piecewise constant EoS parameter of Eq. (3) to perform the study of cosmological perturbations induced by the unified dark fluid. This is the topic of this paper. In Sect. II we set up the basic equations. The details of the analytic analysis leading to the formulas for density fluctuations for our model are found in Sect. III. The consequences for structure formation in the universe are discussed in Sect. IV. Final comments are given in Sect. V.

Note that we work in natural units in which the speed of light and Planck’s constant are set to 1. We work in the context of an expanding Friedmann-Robertson-Walker-Lemaitre space-time with superimposed linear cosmological perturbations. Since the equations of motion simplify, we use conformal time \( \eta \) instead of the physical time \( t \). The comoving spatial coordinates are denoted by \( x \).

\section{II. COSMOLOGICAL PERTURBATIONS}

We work in longitudinal gauge in which the perturbed line element takes the form \[7, 44]\n
\[
ds^2 = a^2(\eta) \left[ (1 + 2\phi) d\eta^2 - (1 - 2\phi) \delta_{ij} dx^i dx^j \right], \tag{7}
\]

\(^4\)Ref. \[43\] offers a similar parameterization but in terms of the deceleration parameter.

\(^5\)In particular, for \( w_i = 0 \), \( d \) controls the transition rate while both \( d \) and \( w_f \) determine \( z_C \) which is defined as the value setting \( w(z_C) = -1/2 \). Assuming \( z_C \sim 0.5 \), one shows that the abrupt transition occurs for large negative values of \( w_f \) such that \( |w_f| \gg 1/d \).

The price for this choice is to end up with a pronounced phantom regime.
if only linear scalar fluctuations for a perfect fluid are taken into account. The function $a = a(\eta)$ is the scale factor, and the potential $\phi(x, \eta)$ is the generalization of Newtonian gravitational potential.

The equation governing the evolution of scalar perturbations for a barotropic fluid $p = p(\varepsilon)$ is given by [44, 45]:

$$\phi'' + 3(1 + c_s^2) \mathcal{H}\phi' - c_s^2 \nabla^2 \phi + [2\mathcal{H}' + (1 + 3c_s^2) \mathcal{H}^2] \phi = 0,$$

where a prime indicates the derivative with respect to conformal time $\eta$ and $\mathcal{H}$ is Hubble function in conformal time, i.e. $\mathcal{H} = d'/a$. The quantity $c_s^2 \equiv p'/\varepsilon'$ is the square of the (adiabatic) speed of sound.

We consider a generic parametrization of the equation of state

$$p = w(z) \varepsilon,$$

where the cosmological redshift $z$ is given as usual in terms of $a$ as $(1 + z) = a_0/a$, and one usually normalizes the scale factor to be one at the present time $\eta_0$, i.e. $a(\eta_0) = a_0 = 1$. Given the time dependent equation of state of Eq. (9),

$$c_s^2 = w(z) + \frac{1}{3} \frac{(1 + z)}{1 + w(z)} \frac{\partial w}{\partial z}.$$

For a constant $w$ perfect fluid matter we have $c_s^2 = w$. Since in the dark energy phase $w < 0$, this leads to an instability in the fluctuation equations. This is not the case for a scalar field model of dark energy such as quintessence when (for a canonically normalized scalar field) $c_s^2 = 1$.

Taking the Fourier transform of Eq. (8) one obtains:

$$\phi''_{k} + 3[1 + w(z)] \mathcal{H}(z)\phi'_{k} + w(z) k^2 \phi_{k} + \frac{1}{3} \frac{(1 + z)}{[1 + w(z)]} \frac{\partial w}{\partial z} \left[3 \mathcal{H} \phi_{k} + k^2 \phi_{k} + 3 \mathcal{H}^2 (z) \phi_{k}\right] = 0.$$

The expansion rate $\mathcal{H}$ (in conformal time) appearing above can be expressed as

$$\mathcal{H}(z) = \frac{H_0 \Omega_{\text{ref}}^{1/2}}{(1 + z)} \exp \left\{ \frac{3}{2} \int \left[ \frac{1 + w(z')}{(1 + z')} \right] dz' \right\},$$

where $H_0$ is the current expansion rate (in terms of physical time) and $\Omega$ is the ratio of the energy density to the critical energy density for a spatially flat universe, i.e. $\Omega = \varepsilon / \varepsilon_c$ with $\varepsilon_c = 3H_0^2/(8\pi G)$ (tilde signs on top of quantities like $\varepsilon$ indicate they are calculated in the unperturbed space-time). $\Omega_{\text{ref}}$ and $z_{\text{ref}}$ are integration constants to be conveniently chosen. Eq. (11) stems from Eq. (9) and from the background energy conservation equation.

For a general time dependence of the equation of state parameter it is not easy to find solutions of Eq. (10). This difficulty can be overcome by adopting a model with an instantaneous transition between the matter regime ($w = 0$) and the dark energy phase, i.e. by taking an EoS modeled by Eq. (3). We emphasize that $\varepsilon = \text{constant}$ so that even during the DE-phase $\partial w/\partial z$ vanishes along with the second line of the differential Eq. (10). This drastically simplifies the analysis, as will be explored in Sect. III.

Once we have determined the relativistic potential $\phi_k$ by solving the differential equation (10), the result can be used to determine the energy density perturbation $\delta \varepsilon_k$. From the 00-component of the perturbed Einstein equations in momentum space [7, 44] we have

$$\delta_k = \frac{\delta \varepsilon_k}{\varepsilon} = -2 \left[ \frac{1}{3} \left(k\mathcal{H}^{-1}\right)^2 + 1 \right] \phi_k - \frac{2}{\mathcal{H}} \phi_k,$$

where $\delta_k$ is the fractional density contrast.

The power spectrum $P_k$ is calculated with $\delta_k$ and then confronted with observations. In order to obtain $\delta_k$ and $\phi_k$ before it, we first need to determine the background evolution, i.e. the functions $\mathcal{H}(\eta)$ and $a(\eta)$ for our piecewise-UDM model this is done with Eq. (11) and the constraint $w = \text{const}$. It yields

$$\mathcal{H}^2 = H_0^2 \frac{\Omega_0}{(1 + z_{\text{ref}})^2} \left( \frac{1 + z}{1 + z_{\text{ref}}} \right)^{(1 + 3w)} \left( w = \text{const} \right).$$

The scale factor $a(\eta)$ is then obtained by using the definition of $\mathcal{H}$ and $z$ in terms of $a$

$$a(\eta) = a_{\text{ref}} \left[ 1 + \frac{(1 + 3w)}{2} H_0 a_{\text{ref}} \Omega_{\text{ref}}^{1/2} (\eta - \eta_{\text{ref}}) \right]^{2 / (1 + 3w)} \left( w = \text{const} \right).$$
Substituting this result into (13), we have, up to integration constants:

$$H(\eta) = \frac{2}{(1 + 3w) \eta} \quad (w = \text{const}),$$

so that Eq. (10) for $\phi_k$ leads to:

$$\phi''_k + \frac{6(1 + w)}{(1 + 3w) \eta} \phi'_k + w k^2 \phi_k = 0 \quad (w = \text{const}).$$

(16)

In the following we will study the solutions of this equation. The perturbations in our model first evolve like in a dust-dominated universe, and at some critical time $\eta_C$ transit to unstable solutions in the dark energy phase. Note that subtleties concerning the integration constants missing in Eqs. (15) and (16) are discussed in the Appendix, where the background cosmology results for our piecewise-UDM model are presented in detail.

III. PERTURBATIONS IN THE UNIFIED DARK FLUID MODEL

We will study the fluctuations in the two regimes of our model specified in Eq. 3. The fluctuations generated in some primordial phase first evolve in the CDM phase. There is then an abrupt transition to the dark energy phase. We will discuss these phases in turn, as well as the matching of the fluctuation modes at the transition between the phases.

A. Perturbations in the CDM Phase

Dust-like matter is characterized by $w = 0$ in which case Eq. (16) reduces to:

$$\phi''_k + \frac{6 \phi'_k}{\eta} = 0,$$

(17)

whose solutions are

$$\phi_k = \bar{C}_{k,1} + \frac{\bar{C}_{k,2}}{\eta^5},$$

(18)

where $\bar{C}_{k,1}$ and $\bar{C}_{k,2}$ are integrations constants determined from early-universe physics (inflation or alternative scenarios). We will be normalizing them by fitting to the matter power spectrum at some pivot scale – see Sect. IV.

Eq. (18) exhibits a constant mode and a decaying mode (scaling with $\eta^{-5}$) for the gravitational potential. The density contrast $\delta_k$ is calculated by using (18) and inserting $H$ (Eq. (15)) into (12), resulting in

$$\delta_k = -2\bar{C}_{k,1} \left[ \frac{1}{2} \left( \frac{(k\eta)^2}{6} + 1 \right) - 3\bar{C}_{k,2} \left[ \frac{1}{3} \frac{(k\eta)^2}{6} - 1 \right] \right] \frac{1}{\eta^5}.$$

(19)

Fluctuations in the energy density are coordinate-dependent, but the gauge dependence is not important on sub-Hubble scales, i.e. on scales where the physical wavelength of the fluctuation mode is smaller than the Hubble radius $d_H = H^{-1}$. Scales of observational interest today are initially super-Hubble. They cross the Hubble radius when $\lambda = d_H$, i.e. $k = H$. Since in the sub-Hubble regime $k\eta \gg 1$, the matter density contrast (19) in this regime can be approximated by

$$\delta_k = -\frac{(k\eta)^2}{6} \left( \bar{C}_{k,1} + \frac{\bar{C}_{k,2}}{\eta^5} \right) \quad (k\eta \gg 1).$$

(20)

This is the equation we will use (together with (18)) when matching the fluctuations to those in the DE phase.

B. Perturbations for the DE Phase

Inserting the DE sector EoS parameter $w = -1 + \epsilon$ into the differential equation (16) for $\phi_k$ we obtain

$$\phi''_k + \frac{6\epsilon}{(-2 + 3\epsilon) \eta} \phi'_k + (-1 + \epsilon) k^2 \phi_k = 0.$$

(21)
We see that there is an instability for $\epsilon < 1$. This instability affects shorter wavelengths more than longer wavelength ones. This leads, in addition to an amplification of the overall amplitude of the spectrum (which could be compensated by a decrease in the amplitude of the primordial spectrum) to a change in the spectral shape. It is this change in shape which will lead to the strong constraints on the model.

Note that in the case of DE described by a cosmological constant ($\epsilon = 0$) it would appear at first sight that there also is an instability leading to $\phi_k = D_1 e^{\epsilon k} + D_2 e^{-\epsilon k}$. However, the 0-component of the Einstein equations for perturbations demands $D_1 = D_2 = 0$, meaning that $\Lambda$ produces no $\phi_k$, therefore no $\delta_k$, i.e. $\Lambda$ does not agglomerate. In the case of scalar field dark energy, even though $w < 0$ the speed of sound is positive, and hence there is no instability.

The general solution of Eq. (21) can be written in terms of Bessel functions [46]:

\[
\phi_k (\sqrt{w}k\eta) = (\sqrt{w}k\eta)^{-\nu} [C_{k,1}J_\nu (\sqrt{w}k\eta) + C_{k,2}Y_\nu (\sqrt{w}k\eta)] ,
\]

where

\[
\nu = -\frac{1}{2} \frac{5 + 3w}{1 + 3w} = -\frac{1}{2} \left( \frac{2 + 3\epsilon}{2 - 3\epsilon} \right) < 0 .
\]

The solution for $\phi_k$ in terms of Bessel functions can be simplified to elementary functions if we consider perturbation which are sub-Hubble in the DE regime, i.e. $k\eta \gg 1$. Then, the argument $\zeta = \sqrt{w}k\eta$ of the Bessel functions in $\phi_k$ is large and we can use the asymptotic forms [46]:

\[
J_{\pm|\nu|} (\zeta) \simeq \frac{2}{\pi \zeta} \cos \left( \frac{\zeta}{2} \mp \frac{|\nu| + \frac{1}{2}}{4} \right) \left[ \Gamma \left( |\nu| + \frac{3}{2} \right) \sin \left( \frac{\zeta}{2} \left( |\nu| - \frac{1}{2} \right) \right) \right],
\]

\[
Y_{\pm|\nu|} (\zeta) \simeq \frac{2}{\pi \zeta} \sin \left( \frac{\zeta}{2} \mp \frac{|\nu| + \frac{1}{2}}{4} \right) \left[ \Gamma \left( |\nu| + \frac{3}{2} \right) \cos \left( \frac{\zeta}{2} \left( |\nu| - \frac{1}{2} \right) \right) \right].
\]

Because $\nu < 0$ and $w < 0$, we obtain hyperbolic functions. In fact, using the properties of the Gamma functions (see e.g. Ref. [46]), $\phi_k$ is cast into the form:

\[
\phi_k (\eta) = (i\omega_k \eta)^{-\frac{3\epsilon}{2(3\epsilon - 2)}} \times
\]

\[
\left\{ \left[ \left( C_{k,1} + \frac{C_{k,2}}{\omega_k \eta} Q \right) X - \left( C_{k,2} - \frac{C_{k,1}}{\omega_k \eta} Q \right) Y \right] \cosh [\omega_k (\eta - \eta_C)] 
\right.
\]

\[
\left. + \left[ \left( C_{k,1} + \frac{C_{k,2}}{\omega_k \eta} Q \right) Z - \left( C_{k,2} - \frac{C_{k,1}}{\omega_k \eta} Q \right) W \right] \sinh [\omega_k (\eta - \eta_C)] \right\} \quad (k\eta \gg 1),
\]

where

\[
\omega_k^2 = (1 - \epsilon)k^2 ,
\]

and $\eta_C$ is the time of transition between the CDM and the DE phases. Quantities $\{Q, X, Y, Z, W\}$ are constants which depend on $\epsilon$ and $\eta_C$:

\[
U = \frac{\pi}{2} \frac{3\epsilon}{(3\epsilon - 2)} ,
\]

\[
X = \cos U \cosh (\omega_k \eta_C) + i \sin U \sinh (\omega_k \eta_C) ,
\]

\[
Y = \sin U \cosh (\omega_k \eta_C) - i \cos U \sinh (\omega_k \eta_C) ,
\]

\[
Z = \cos U \sinh (\omega_k \eta_C) + i \sin U \cosh (\omega_k \eta_C) ,
\]

\[
W = \sin U \sinh (\omega_k \eta_C) - i \cos U \cosh (\omega_k \eta_C) ,
\]

\[
Q = \frac{1}{i} \frac{3\epsilon}{(3\epsilon - 2)^2} .
\]

With this, the potential $\phi_k$ in the DE phase is determined up to the integration constants $C_{k,1}$ and $C_{k,2}$. These are related to the constant $C_{k,1}$ and $C_{k,2}$ in the matter phase solution by general relativistic matching conditions. This is the subject of the next subsection.
C. Matching Conditions

In our model there is an instantaneous transition in the equation of state at redshift $z_C$. On either side of the matching surface the Einstein equations are satisfied. Across the surface, the metric must obey matching conditions which are a generalization of the Israel matching conditions [47] and which were derived in Ref. [48] (see also [49]).

Let us use the index ($-$) as a label for quantities in the CDM regime and ($+$) for quantities in the DE regime. The CDM fluctuations $\phi_k-$ and $\delta_k-$ are then connected to the DE ones, $\phi_k+$ and $\delta_k+$, through the matching conditions

$$\phi_k-(\eta_C) = \phi_k+(\eta_C) \quad \text{and} \quad \frac{\delta_k-(\eta_C)}{1+w_-} = \frac{\delta_k+(\eta_C)}{1+w_+}. \quad (26)$$

Inserting the sub-Hubble solutions in the CDM and DE periods derived in the previous subsections we obtain:\footnote{Making use of $w_- = w_{\text{CDM}} = 0$ and $w_+ = w_{\text{DE}} = (-1 + \epsilon)$.}

$$\phi_k-(\eta_C) = \frac{1}{(i\omega_k \eta_C)^{\frac{1}{2} \kappa^2}} \left[ \left( C_{k,1} + C_{k,2} Q \frac{\omega_k}{\eta C} \right) X - \left( C_{k,1} - C_{k,2} \frac{\omega_k}{\eta C} \right) Y \right];$$

$$\phi_k-(\eta_C) \times \left\{ \frac{1}{6} \left( k \eta_C \right)^2 \left[ (-1 + \epsilon) (3 \epsilon - 2) \right] - \frac{3 \epsilon}{(3 \epsilon - 2)} \right\}$$

$$= \frac{(\omega_k \eta_C)}{(i\omega_k \eta_C)^{\frac{1}{2} \kappa^2}} \left[ \frac{Q}{(\omega_k \eta_C)^2} \left( C_{k,2} X + C_{k,1} Y \right) - \left( C_{k,1} + C_{k,2} \frac{\omega_k}{\eta C} \right) Z + \left( C_{k,2} - C_{k,1} \frac{\omega_k}{\eta C} \right) W \right], \quad (27)$$

where

$$\phi_k-(\eta_C) = \tilde{C}_{k,1} + \frac{C_{k,2}}{(3\epsilon - 2) \eta C}. \quad (28)$$

The factor $(3\epsilon - 2)$ in Eq. (28) comes from the integration constants we neglected when writing Eq. (15) for $H \propto \eta^{-1}$. The details about these constants can be found in the Appendix.

The system in Eq. (27) can be solved to find the constants $\left\{ C_{k,1}, C_{k,2} \right\}$ appearing in $\phi_k-$ in terms of the constants $\left\{ \tilde{C}_{k,1}, \tilde{C}_{k,2} \right\}$ of the CDM phase. They, in turn, are determined in terms of the primordial spectrum of fluctuations and the evolution until the time of equal matter and radiation. Note that even if $\tilde{C}_{k,2}$ is of the same magnitude of $C_{k,1}$ at the end of the phase of very early universe cosmology when the fluctuations are generated, it is the coefficient of the decaying mode of $\phi_k$ which can be neglected.

In solving for $\left\{ C_{k,1}, C_{k,2} \right\}$ we keep only terms scaling with the highest power of $(\omega_k \eta_C)$ (this is consistent in the sub-Hubble case). The constants in Eq. (24) are then determined to be:

$$C_{k,1} = \frac{(i\omega_k \eta_C)^{\frac{1}{2} \kappa^2}}{\left| XW - YZ \right|} \left\{ \frac{1}{6} \left( 3 \epsilon - 2 \right) \left( \omega_k \eta_C \right) Y + QX \right\}$$

$$C_{k,2} = \frac{(i\omega_k \eta_C)^{\frac{1}{2} \kappa^2}}{\left| XW - YZ \right|} \left\{ \frac{1}{6} \left( 3 \epsilon - 2 \right) \left( \omega_k \eta_C \right) X + QY \right\} \quad (29)$$

Then, by using the identities

$$[X^2 + Y^2] = 1, \quad [XZ + YW] = 0, \quad [WX - YZ] = -i,$$

we, at last, get $\phi_k+$ in its final form:

$$\phi_k+(\eta) = \phi_k-(\eta_C) \left( \frac{\eta_C}{\eta} \right)^{\frac{1}{2} \kappa^2} \left\{ \left[ 1 - \frac{1}{6} \left( 3 \epsilon - 2 \right) \left( 1 - \frac{\eta C}{\eta} \right) \right] \cosh \left[ \omega_k (\eta - \eta_C) \right] \right.$$ \n
$$\left. + \left[ -\frac{1}{6} \left( 3 \epsilon - 2 \right) \left( \omega_k \eta_C \right) \right] \sinh \left[ \omega_k (\eta - \eta_C) \right] \right\}. \quad (30)$$

Notice that the expression for the DE-regime $\phi_k+$ reduces to the result in the CDM-phase $\phi_k-$ at $\eta = \eta_C$ as it must according to the matching conditions.
Note that the \( k \)-dependence of \( \phi_k^+ \) appears both in the argument of the hyperbolic functions and in the coefficient of \( \sinh \) through \( \omega_k = \sqrt{1 - \epsilon k} \). Eq. (30) for \( \phi_k^+ \) can be substituted into Eq. (12) in order to obtain \( \delta_k^+ \). The expression is long. Keeping only the dominant terms in powers of \((k\eta)\), it reduces to:

\[
\delta_k^+(\eta) \approx \delta_k^-(\eta_C) \left( \frac{\eta_C}{\eta} \right)^{\frac{(3\epsilon - 4)}{2\epsilon}} \left\{ \left[ \epsilon \left( \frac{\eta_C}{\eta} \right) + \left[ 1 - \frac{1}{6} (3\epsilon - 2) \right] \left( 1 - \frac{\eta_C}{\eta} \right) \right] \cosh \left[ \omega_k (\eta - \eta_C) \right] \right. \\
+ \left[ \frac{1}{6} (3\epsilon - 2) (\omega_k \eta_C) \right] \sinh \left[ \omega_k (\eta - \eta_C) \right] \right\} \tag{31}
\]

where

\[
\delta_k^-(\eta_C) \approx \left[ -\frac{1}{6} (3\epsilon - 2)^2 (k\eta)^2 \right] \phi_k^-(\eta_C). \tag{32}
\]

Notice that Eq. (31) guarantees that

\[
\delta_k^+(\eta_C) = \epsilon \delta_k^-(\eta_C) \tag{33}
\]

as required by Eq. (26).

We will apply these matching conditions in two different cases. First, we shall consider modes which enter the Hubble radius after the time of equal matter and radiation. We call these the “large-scale” fluctuations (Sect. III C1). Then, in Sect. III C2, we take consider “small-scale” perturbations which enter the Hubble radius in the radiation phase\(^7\). In both cases we will be matching sub-Hubble modes at the transition surface \( z = z_C \). The difference between these two cases is the evolution of the fluctuations once they enter the Hubble radius. For large-scale fluctuations the primordial spectrum is unchanged, for small-scale fluctuations the primordial spectrum is processed since the fluctuation modes grow only logarithmically between when they enter the Hubble radius and the time of equal matter and radiation.

1. Large-scale fluctuations

Large scales enter the Hubble radius when the universe is already dominated by matter (but before \( z_C \)). Therefore, the coefficients in Eqs. (18) and (20) for \( \phi_k^- \) and \( \delta_k^- \) are determined directly by the primordial spectrum which needs to be nearly scale-invariant to agree with observations. Accordingly, we introduce the notation \( \phi_k^- = \hat{C}_{k,1} = \phi_{kL} \), where label “L” stands for large-scale and “p” for primordial.

Note that for a scale-invariant primordial spectrum of curvature fluctuations such as emerges from an early phase of inflation [50] (or alternatives such as the “matter bounce” scenario [51] or “string gas cosmology” [52, 53])

\[
\phi_k^- \sim k^{-3/2}, \tag{34}
\]

and hence

\[
\delta_k^-(\eta) \sim k^{1/2}. \tag{35}
\]

Then, Eq. (24) for \( \phi_{kL} \) can be used to calculate \( \delta_k^+ \) through the “Poisson” Eq. (12).

2. Small-scale fluctuations

Small-scale modes enter the Hubble radius when the universe is radiation dominated. The radiation pressure will only allow the density fluctuation modes to grow logarithmically in time between when they enter the Hubble radius and the time of equal matter and radiation. This is the Meszaros effect [54]. This means that \( \delta_k \) will evolve as

\[
\delta_k \sim \ln(k\eta) \tag{36}
\]

\(^7\) Note that we consider only fluctuations which are sub-horizon at the transition redshift \( z_C \).
after the mode enter the horizon (see e.g. [8]). After the time of equal matter and radiation regime the fractional density contrast will then grow linearly in the scale factor, i.e. proportional to $a/a_{\text{eq}}$ \(^8\). Hence
\[
\delta_{k^-}(\eta) = K_k \left( \frac{a}{a_{\text{eq}}} + \frac{2}{3} \right) \approx K_k \left[ \frac{1}{4} H_0^2 \Omega_{\text{DE},0} \left( \frac{a_C}{a_{\text{eq}}} \right) \left( \frac{a_0}{a_C} \right)^{3\epsilon - 2} \right] (\eta + 3w_+ \eta_C)^2 ,
\]
if $a \gg a_{\text{eq}}$ where $w_+ = -1 + \epsilon$ and
\[
K_k = \frac{3}{2} (A \phi^S_{kp}) \ln \left[ \frac{4B}{e^\epsilon \sqrt{2} k/k_{\text{eq}}} \right] .
\]

Here, the constant $\phi^S_{kp}$ (label “S” means “small scales”) comes from the primordial spectrum, $A = 9.0$ and $B = 0.62$ are numbers given by Dodelson in [8] (and are not affected by the DE phase of our model), $\Omega_{\text{DE},0}$ is the density parameter of the dark fluid today, and $a_C = a(\eta_C)$ is the scale factor at CDM-DE transition from which we build $z_C = (a_0/a_C) - 1$. Finally, by following the procedure explained in Ref. [8], the scale $k_{\text{eq}}$ (the comoving wavenumber entering the Hubble radius at $t_{\text{eq}}$) is calculated to be:
\[
k_{\text{eq}} = \sqrt{2} H_0 \Omega_{\text{DE},0}^{\frac{1}{2}} (1 + z_C)^{3w_+} .
\]

This equation contains $\Omega_{\gamma,0}$, the present-day value of radiation density parameter. Eq. (37) uses the background solution (14) with the integration constants obtained from continuity of $a(\eta)$ and $H(\eta)$ at $\eta = \eta_{\text{eq}}$. The interested reader is referred to the Appendix where the details are worked out.

By reverse engineering involving the “Poisson” Eq. (12) and Eq. (37), we determine:
\[
\phi_{k^-} \approx -K_k \left[ \frac{3}{2} H_0^2 \Omega_{\text{DE},0} \left( \frac{a_C}{a_{\text{eq}}} \right) \left( \frac{a_0}{a_C} \right)^{3\epsilon - 2} \right] \left[ 1/k^2 \right] .
\]

With the small-scale $\phi_{k^-}(\eta)$ and $\delta_{k^-}(\eta)$ at hand – Eqs. (37) and (40), the next step is to obtain $\phi_{k^+}(\eta)$ and $\delta_{k^+}(\eta)$ good for small scales. As small-scale DE fluctuations also evolve in a DE phase for the background, all the reasoning involving $\phi_{k^+}$ in Sect. III B – previous to the imposition of the matching conditions in Sect. III C 1 – still holds; i.e. Eq. (24) remains unchanged and also the related $\delta_{k^+}$. This $\phi_{k^+}$ is then glued to $\delta_-$, Eq. (37), according to the matching conditions. Also, $\phi_{k^+}$ as given by Eq. (24) is set equal to $\phi_{k^-}$, Eq. (40), at $\eta = \eta_C$. This way, constants $\{C_{k,1}, C_{k,2}\}$ for small scales are calculated.

3. Summary of the Results

The result for the small-scale $\phi_{k^+}$ is formally the same as the large-scales $\phi_{k^+}$, but because the functional form of $\phi_{k^-}$ changes, the $k$-dependence is different in each case. In fact,
\[
\phi_{k^+}^{L1}(\eta) = \phi_{kp}^{L1} \left( \frac{\eta_C}{\eta} \right)^{\frac{1}{2} \epsilon - 1 \epsilon} \left\{ \left[ 1 - \frac{1}{6} \frac{3\epsilon}{3\epsilon - 2} \left( 1 - \frac{\eta_C}{\eta} \right) \right] \cosh \left[ \sqrt{1 - \epsilon} (k\eta - k\eta_C) \right] \\
+ \left[ -\frac{1}{6} \frac{3\epsilon - 2}{3\epsilon - 2} \sqrt{1 - \epsilon} (k\eta_C) \right] \sinh \left[ \sqrt{1 - \epsilon} (k\eta - k\eta_C) \right] \right\} ,
\]

(the superscript “L” standing for large-scale) while
\[
\phi_{kp}^{L1} \left( \frac{\eta_C}{\eta} \right)^{\frac{1}{2} \epsilon - 1 \epsilon} \left\{ \left[ 1 - \frac{1}{6} \frac{3\epsilon}{3\epsilon - 2} \left( 1 - \frac{\eta_C}{\eta} \right) \right] \cosh \left[ \sqrt{1 - \epsilon} (k\eta - k\eta_C) \right] \\
+ \left[ -\frac{1}{6} \frac{3\epsilon - 2}{3\epsilon - 2} \sqrt{1 - \epsilon} (k\eta_C) \right] \sinh \left[ \sqrt{1 - \epsilon} (k\eta - k\eta_C) \right] \right\} .
\]

---

\(^8\) Quantities with subscript $\text{eq}$ are calculated at the time $\eta_{\text{eq}}$ of equal matter and radiation energy density.
This has a direct impact on the power spectrum [55] predicted by these two regimes, as we shall see in the next section.

IV. UNIFIED DARK FLUID POWER SPECTRUM

The current power spectrum of density fluctuations can be defined in terms of $\delta_k(\eta)$ as:

$$P_k(\eta) = 2\pi^2 |\delta_k(\eta)|^2$$

where $P_k$ must be calculated today ($\eta = \eta_0$) for comparison with observations [56, 57]. The observational power spectrum is a rough measure of the distribution of matter. In our unified model, cold dark matter evolves into a dark energy fluid. Hence, it is the current dark energy power spectrum that would have to coincide with the observed matter power spectrum today.

In order to agree with observations, we need to obtain the following asymptotic limits on large and small scales, respectively [8]:

$$P_k^L(\eta_0) \sim k \quad (k \ll k_{eq})$$

and

$$P_k^S(\eta_0) \sim k^{-3} \ln k \quad (k \gg k_{eq}).$$

Our model must give a power spectrum which agrees with $P_k$ from (44) and (45) within the observational error bars to be viable. However, we have seen that a factor of $\cosh[\omega_k(\eta - \eta_C)]$ appears in $\phi_k^+$ for both large and small scales – Eqs. (41) and (42). This factor introduces instabilities which grow both in $\eta$ and in $k$, and these instabilities propagate to $\delta_k^+$ and consequently to $P_k$.

In fact, by using the background evolution $a(\eta)$, it is possible to express $P_k(\eta_0)$ for the dark fluid in the general form:

$$P_k(\eta_0) \sim (\phi_k^p)^2 \times \{\text{usual } k\text{-behavior}\} \times \{\text{Enhancement Factor } F(k, z_C, \epsilon, \Omega_{DE,0})\}$$

which is valid both for large scales and small scales. Here, the first factor on the right hand side, $\phi_k^p \propto k^{-3/2}$, is the primeval spectrum of the potential [8] (assuming an initial scale-invariant spectrum). It is related to the curvature perturbation $R$ on super-Hubble scales via [45]:

$$R = \left[1 + \frac{2}{3} \frac{1}{(1 + w)}\right] \phi \ .$$

Large-scale fluctuations enter the horizon in a matter dominated universe, when $w = 0$. Conversely, small-scale perturbations make the transition from super-horizon to sub-horizon in a radiation dominated universe with $w = 1/3$. Therefore, Eq. (47) gives:

$$R = \begin{cases} 
\frac{5 \phi_p^L}{2} & \ (w = 0) \\
\frac{\phi_p^S}{2} & \ (w = 1/3) 
\end{cases} \ .$$

The second factor on the right hand side of (46) is the usual transfer function of density fluctuations which takes into account the different evolution of large and small scale density fluctuations (discussed at the end of the previous section), and the final factor is the extra growth factor in our model compared to what is obtained in the $\Lambda$CDM scenario.

The specific functional form of Eq. (46) on large scales is

$$P_k^L(\eta_0) \simeq (\Phi_{kp}^L)^2 \left\{ \frac{c}{H_0} k \right\} F(k, z_C, \epsilon, \Omega_{DE,0}) ,$$

with

$$\Phi_{kp}^L \equiv \sqrt{2\pi^2} \left( -\frac{2}{3} \phi_{kp}^L \right) \left( \frac{c}{H_0} \right)^2$$

(50)
and the enhancement factor
\[
F = F(k, z_C, \epsilon, \Omega_{DE,0}) \equiv \left\{ \frac{(1 + z_C)^{3w_+ + 2}}{\Omega_{DE,0}} \left[ (\epsilon + \zeta_C) \cosh \frac{k \tilde{\eta}_C}{\sqrt{\Omega_{DE,0}}} - \frac{1}{3} \frac{e^{1 - \epsilon k}}{H_0 \sqrt{\Omega_{DE,0}}} \sinh \frac{k \tilde{\eta}_C}{\sqrt{\Omega_{DE,0}}} \right] \right\}^2,
\]
where we have defined
\[
\tilde{\eta}_C = \tilde{\eta}_C(z_C, \epsilon) \equiv \left( \frac{e}{H_0} \right)^2 \frac{1}{2} \sqrt{1 - \epsilon} \frac{2}{(3 \epsilon - 2)} \left[ 1 - (1 + z_C)^{\frac{3(\epsilon - 2)}{2}} \right]
\]
and
\[
\zeta_C = \zeta_C(z_C, \epsilon) \equiv \left[ 1 - \frac{1}{2} \frac{\epsilon}{(3 \epsilon - 2)} \right] \left[ (1 + z_C)^{-\frac{3(\epsilon - 2)}{2}} - 1 \right].
\]
Note that the enhancement factor introduces a departure from the shape of the initial spectrum which grows exponentially in $k$. Moreover,
\[
\lim_{z_C \to 0} \tilde{\eta}_C = \lim_{z_C \to 0} \zeta_C = 0.
\]

Eq. (49) for $P^S_k(\eta_0)$ holds as long as $k \ll k_{eq}$.

The small scale (S) power spectrum for the dark fluid today is:
\[
P^S_k(\eta_0) \approx \left( \Phi^S_p \right)^2 \left\{ \frac{k_{eq}}{\sqrt{2k}} \right\}^3 \ln^2 \left[ \frac{4B \sqrt{2k}}{e^\epsilon k_{eq}} \right] \mathcal{G}(k, z_C, \epsilon, \Omega_{DE,0}),
\]
with
\[
\Phi^S_p \equiv \sqrt{2 \pi^2} \left( \frac{3}{2} \frac{\Lambda \phi_0^3}{H_0^2} \right) \left( \frac{e}{H_0} \right)^2,
\]
and
\[
\mathcal{G}(k, z_C, \epsilon, \Omega_{DE,0}) \equiv \frac{\Omega_{DE,0}}{\sqrt{\Omega_{DE,0}}} (1 + z_C)^{3w_+} F(k, z_C, \epsilon, \Omega_{DE,0}),
\]
given in terms of the enhancement factor $F$ in Eq. (51). Eq. (54) for $P^S_k(\eta_0)$ holds as long as $k \gg k_{eq}$. The enhancement factor once again introduces an exponential divergence from the shape of the initial spectrum.

We perform a numerical fit of the power spectrum to the data which indicates that exponential divergence $P_k(\eta_0)$ (see (49) and (54)) caused by the enhancement factor (51) can only be reconciled with the data if the transition redshift $z_C$ is extremely close to $z_C = 0$—see the plots in Figs. 1 and 2. On large scales, the observational data come from CMB anisotropies (see the points labelled CMB in the plots) and from the power spectrum of galaxies on large scales (see the points labelled 2dFGRS). On small scales the tightest observational constraints come from Lyman alpha observations (the Lyman-$\alpha$ points on the graphs) 9.

In these figures, we have independently fixed the normalization of the primordial power spectrum to either obtain agreement with the data in the $k \to 0$ limit (in the case of large scales) or at $k \sim 1 \times h\text{Mpc}^{-1}$ (in the case of small scales). In fact, we should use the small-scale normalization also on larger scales. We see that the tightest constraints on the redshift $z_C$ come from the smallest scales which are in the linear regime. With values of $z_C$ for which the small-scale power spectrum is consistent with the data, the large-scale spectrum is also consistent.

The only hope for our unified model is to take the transition time $\eta_C$ to be very close to the present-day time $\eta_0$. In this limit $\cosh [\omega_k (\eta - \eta_C)] \to 1$ and $\sinh [\omega_k (\eta - \eta_C)] \to 0$.

9 Note that there are a number of references that use the Lyman-alpha Forest in the linear regime, see e.g. [58–60].
Figure 1: Observational and predicted values of $P_k$ for our unified dark matter model with piecewise constant equation of state. Both large-scale fit (upper plot) and small-scale fit use $\epsilon = 0.28$ and $\Omega_{DE,0} = 1$. The justification for this particular value of $\epsilon$ is given in Sect. I. We take the present-day value of the dark component to be one because we neglect the 4% contribution coming from baryonic matter to simplify the treatment.

V. DISCUSSION

We studied cosmological perturbations in a unified dark fluid model in which perfect fluid matter with an initial cold dark matter equation of state $w = 0$ evolves to a DE regime characterized by a constant EoS parameter of the type $w = -1 + \epsilon$ (cf. Sect. I) and the transition is assumed to be instantaneous. In this simple model the equations
Figure 2: $P_k(\eta_0)$ for the unified dark sector with a sharp transition. Large-scale curves are valid for $k \ll k_{eq}$ when the data points come from CMB and large-scale structure data. The small-scale curves are compared to Lyman-Alpha Forest data for $k \gg k_{eq}$. Small values of $z_C$ which are consistent with large-scale points already causes a strong divergence at small scales.

for the relativistic potential $\phi_k$ can be solved analytically (Sects. II and III). The solutions on either side of the transition hypersurface must be connected using the general relativistic matching conditions. The power spectrum (PS) of density fluctuations was computed both in the limit of large scales and small scales (Sect. IV). Due to the negative speed of sound in the dark energy phase which is a consequence of the perfect fluid description of our matter, there is an exponential instability both in time and in comoving wavenumber $k$ which leads both to an exponential growth of the amplitude of the power spectrum and to a shape distortion. Demanding agreement with the observed spectrum requires setting the transition redshift between the two phases extremely close to zero, $z_C = 0$. If we do this, however, the model is unable to match the background cosmology data \cite{9, 36, 38, 61}. This rules out a unified dark sector model given by perfect fluid matter with a piecewise constant equation of state. Our analytical approach reveals the origin of the instabilities faced by some unified dark sector models. Our analytical approach also makes it clear that the problems cannot be solved by smoothing the transition.

We have seen in Sect. I that our model can be obtained as particular limits of other UDM models at the background level. The general results of our analysis may be applicable to these other models. In particular, this is the case for GCG in the range $\alpha < 0$ - see Eq. (2). Ref. \cite{62} showed that pronounced divergences in the GCG matter power spectrum appear for $|\alpha| > 10^{-5}$. Our analysis enables us to pinpoint the origin of these divergences in the power spectrum: they come about early in the DE-phase due to the exponential increase (both in time and in $k$) of the relativistic potential.

Another way to construct unified models of dark matter and dark energy is to make use of scalar fields instead of perfect fluids. In this case $c_s^2 \neq w$ (see e.g. the review in \cite{21}). In particular, $c_s^2$ can be positive semi-definite, and hence the instability of fluctuations which we have encountered in our dark fluid model does not arise. For canonical scalar fields $c_s^2 = 1$. This then gives rise to an opposite problem: at the level of linear cosmological perturbations it becomes impossible to explain the observed gravitational clustering of dark matter. This problem can be avoided by ceasing to regard the matter field to be coherent \footnote{\label{footnote1}Note that the constraint on the transition redshift $z_C$ would be even tighter had we taken into account the nonlinear effects which boost the power spectrum on small scales, in particular on the QSO scales. We have provided a conservative bound, but this bound already rules out the model.}. For recent examples of unified scalar field models of dark matter\footnote{\label{footnote2}This is how axion dark matter works.}.
and dark energy see [63–65].

In spite of the limits in which our unified model with piecewise constant equation of state can be obtained from other unified models (such as GCG, the arctan-UDM and the Linder-Huterer model) the fact that matter power spectrum has instabilities in our model does not imply automatically that such problems will always be present in these other UDM framework. Even in the case of GCG the divergences can be cured by making sure that the effective speed of sound squared is not negative, as shown e.g. in [66] and commented on in [40]. Such a construction could also work in an extended model of our type, as is already indicated in Ref. [42] in the case of the arctan-UDM model.

One merit of the present work is to show that it is possible to learn about general features of unified dark sector models from analytical calculations. In our view, the physical intuition gained in the process is an important result of our work.

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Appendix: Background Results

Inserting Eq. (14) into (13) leads to:

$$H(\eta) = \frac{2}{(1 + 3w)} \left[ \frac{2}{(1 + 3w)} \frac{1}{H_{0}a_{\text{ref}}^{\Omega_{\text{ref}}} + (\eta - \eta_{\text{ref}})} \right]. \quad (A.1)$$

Eqs. (14) and (A.1) for $a(\eta)$ and $H(\eta)$ depend on three constants: $a_{\text{ref}}$, $\eta_{\text{ref}}$ and $\Omega_{\text{ref}}$. Two of them, say $a_{\text{ref}}$ and $\eta_{\text{ref}}$, are determined by imposing that the background evolution is continuous for all values of time. The remaining constant $\Omega_{\text{ref}}$ is fixed from physically meaningful boundary conditions. In fact, the continuity equation for the background,

$$\bar{\varepsilon} + 3H(\bar{\varepsilon} + \bar{p}) = 0, \quad (A.2)$$

with a barotropic EoS of constant $w$ reads:

$$\Omega(a) = \Omega_{\text{ref}} \left( \frac{a_{\text{ref}}}{a} \right)^{3(1+w)} \quad (w = \text{constant}). \quad (A.3)$$

The matching condition at the CDM-to-DE transition is that the CDM density parameter $\Omega_{\text{CDM}}(\eta_{C})$ calculated at the transition time $\eta_{C}$ should be the same as DE density parameter $\Omega_{\text{DE}}(\eta_{C})$. This is true because in the unified matter approach there exists a single fluid that transforms itself from matter to dark energy, preserving the same energy budget in both phases. Let

$$\Omega_{C} \equiv \Omega_{\text{CDM}}(\eta_{C}) = \Omega_{\text{DE}}(\eta_{C}). \quad (A.4)$$

This is the natural choice for energy density reference value $\Omega_{\text{ref}}$:

$$\Omega_{\text{ref}} = \Omega_{C} = \Omega_{\text{DE,0}}(1 + z_{C})^{3\bar{\varepsilon}} \quad \text{(CDM-to-DE transition).} \quad (A.5)$$

where

$$\Omega_{\text{DE}}(\eta_{0}) \equiv \Omega_{\text{DE,0}}. \quad (A.6)$$

The last equality in Eq. (A.5) follows from Eq. (A.3) applied to the DE phase ($\eta_{C} \leq \eta \leq \eta_{0}$), where the EoS parameter is $w = (-1 + \epsilon)$. Let $a_{\text{CDM}}(\eta)$ [$a_{\text{DE}}(\eta)$] be the scale factor during CDM [DE] regime; and let us use a similar notation for $H(\eta)$. By imposing

$$a_{\text{CDM}}(\eta_{C}) = a_{\text{DE}}(\eta_{C}). \quad (A.7)$$
and

\[ \mathcal{H}_{\text{CDM}}(\eta_C) = \mathcal{H}_{\text{DE}}(\eta_C), \]  
(A.8)

then one gets the remaining constants \( a_{\text{ref}} \) and \( \eta_{\text{ref}} \). They become

\[ \Omega_{\text{ref}} = \Omega_C, \quad \eta_{\text{ref}} = \eta_C, \quad a_{\text{ref}} = a_C \quad \text{(CDM regime)}, \]  
(A.9)

(which is very natural and consistent, by the way) under the assumption

\[ \Omega_{\text{ref}} = \Omega_{\text{DE,0}}, \quad \eta_{\text{ref}} = \eta_0, \quad a_{\text{ref}} = a_0 = 1 \quad \text{(DE regime)} . \]  
(A.10)

Similar considerations can be made at the radiation-to-matter transition. The whole picture is the set of equations below which summarize piecewise-UDM background cosmology:

\[
\begin{align*}
    a(\eta) &= \begin{cases} 
        a_{\text{eq}} \left[ 1 - \left( \frac{a_0}{a_{\text{eq}}} \right)^\frac{3(\epsilon - 2)}{2} \frac{H_0 a_0 (\Omega_{\text{DE,0}})^{1/2} (\eta_{\text{eq}} - \eta)}{2} \right], & \eta \leq \eta_{\text{eq}} \\
        a_C \left[ 1 - \left( \frac{a_0}{a_C} \right)^\frac{3(\epsilon - 2)}{2} \frac{H_0 a_0 (\Omega_{\text{DE,0}})^{1/2} (\eta_C - \eta)}{2} \right]^2, & \eta_{\text{eq}} < \eta \leq \eta_C \\
        a_0 \left[ 1 - \left( \frac{a_0}{a_C} \right)^\frac{3(\epsilon - 2)}{2} \frac{H_0 a_0 (\Omega_{\text{DE,0}})^{1/2} (\eta_0 - \eta)}{2} \right]^\frac{3(\epsilon - 2)}{2}, & \eta_C < \eta \leq \eta_0 
    \end{cases} 
\end{align*}
\]  
(A.11)

with

\[ a_C = a_0 \left[ 1 - \left( \frac{3\epsilon - 2}{2} \frac{H_0 a_0 (\Omega_{\text{DE,0}})^{1/2} (\eta_0 - \eta)}{a_0} \right)^\frac{3(\epsilon - 2)}{2} \right]^{\frac{3(\epsilon - 2)}{2}} \]  
(A.12)

and

\[ a_{\text{eq}} = a_C \left[ 1 - \left( \frac{a_0}{a_C} \right)^\frac{3(\epsilon - 2)}{2} \frac{H_0 a_0 (\Omega_{\text{DE,0}})^{1/2} (\eta_0 - \eta)}{2} \right]^2 \]  
(A.13)

such that \( a(\eta) \) is indeed continuous throughout the cosmic history. Also:

\[ \mathcal{H}(\eta) = \frac{2}{(1 + 3w)} \frac{1}{\bar{\eta}} \]  
(A.14)

where

\[
\bar{\eta}(\eta) = \begin{cases} 
    \eta_{\text{eq}}(\eta) = \frac{1}{2} \bar{\eta}_{\text{CDM}}(\eta_{\text{eq}}) - (\eta_{\text{eq}} - \eta), & \eta \leq \eta_{\text{eq}} \\
    \bar{\eta}_{\text{CDM}}(\eta) = (3\epsilon - 2) \bar{\eta}_{\text{DE}}(\eta_C) - (\eta_C - \eta), & \eta_{\text{eq}} \leq \eta \leq \eta_C \\
    \bar{\eta}_{\text{DE}}(\eta) = \frac{1}{2} \frac{3(\epsilon - 2)}{H_0 a_0 (\Omega_{\text{DE,0}})^{1/2}} - (\eta_0 - \eta), & \eta_C \leq \eta \leq \eta_0 
\end{cases}
\]  
(A.15)

with

\[ \bar{\eta}_{\text{CDM}}(\eta) = \bar{\eta}_{\text{DE}}(\eta) + 3 \left( -1 + \epsilon \right) \bar{\eta}_{\text{DE}}(\eta_C) \]  
(A.16)

and

\[ \bar{\eta}_{\text{DE}}(\eta) = \bar{\eta}_{\text{DE}}(\eta) + 3 \left( -1 + \epsilon \right) \bar{\eta}_{\text{DE}}(\eta_C) . \]  
(A.17)

Note how \( \mathcal{H}(\eta) \) is also continuous at \( \eta_{\text{eq}} \) and \( \eta_C \).

The first line in Eq. (A.11) is precisely the expression needed for writing the last equality in Eq. (37).

Eqs. (A.16) and (A.17) show that \( \bar{\eta} \) is discontinuous both at \( \eta = \eta_{\text{eq}} \) and \( \eta = \eta_C \). This happens in order to compensate for the discontinuity introduced by our piecewise \( w \) — see Eq. (3) — and makes it possible that \( a(\eta) \) and \( \mathcal{H}(\eta) \) are continuous. The discontinuity in \( \bar{\eta} \) is not a problem because this quantity is defined only for convenience; it does not have an intrinsic physical meaning. The convenience is that the perturbation equation for \( \phi_k \) assumes the form (16) in terms of \( \bar{\eta} \); and this form is free from subtleties regarding integration constants from the physics of the
background cosmology. Although these subtleties are not important for the technical task of obtaining the solution of (16), they are relevant when one performs the matching of different phases according to matching conditions. This explains the reason for the factor $(3\epsilon - 2)\eta_C$ in the expression for $\phi_k(\eta_C)$, Eq. (28).