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Observer-Based Output Feedback Control Using Invariant Polyhedral Sets for Fuzzy T-S Models Under Constraints

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Humanity's deepest desire for
knowledge is justification enough for
our continuing quest.
Stephen Hawking

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Abstract

In this work, is proposed a numerical method for the computation of state observer-based output feedback controllers for fuzzy Takagi-Sugeno (T-S) systems subject to constraints based on invariant set theory. In this regard, Positively Invariant (PI) polyhedral sets are used to ensure that state and input control constraints are satisfied at all times. Sufficient conditions are established for a polyhedron defined in the augmented state-space (state + estimation error) to be PI. The problem of tracking a constant reference signal is also considered, for which the concept of robust positive invariance is used along with a stabilizing Integral-Proportional (I-P) controller. Sufficient conditions are established for a polyhedron defined in the augmented state-space (state + estimation error + tracking error integral state) to be PI in the presence of a constant reference signal, which can be interpreted as a bounded disturbance. From the invariance conditions, a bilinear optimization problem is formulated to simultaneously compute the controller's gains and the positively invariant polyhedron, guaranteeing the satisfaction of the constraints. The two types of observer found in the literature of fuzzy T-S systems are considered: the first considers the membership functions dependent only on the system output; the second, in turn, refers to the general case, where these functions can be associated with any state variables. In the simplest case, although the membership functions depend only on the output, the estimated state feedback results, in general, in controllers with better performance and with larger sets of admissible states associated with them than the output static feedback control. For the general case, as membership functions depend on non-accessible states, an estimation mechanism is needed to calculate these variables. In both cases, this role is played by the fuzzy T-S observer.

Keywords: Fuzzy T-S models, output feedback control, invariant sets, reference tracking, bilinear programming.

Resumo

Neste trabalho, é proposto um método numérico para o cálculo de controladores por realimentação de saída baseado no observador de estado para sistemas fuzzy Takagi-Sugeno (T-S) sujeitos a restrições com base na teoria de conjuntos invariantes. A esse propósito, conjuntos poliédricos Positivamente Invariantes (PI) são usados para garantir que as restrições de estado e controle sejam satisfeitas a todo tempo. Condições suficientes são estabelecidas para que um poliedro definido no espaço de estados aumentado (estado + erro de estimação) seja PI. O problema de rastreamento de um sinal de referência constante também é considerado, para o qual o conceito de invariância positiva robusta é usado em conjunto à um controlador Integral-Proporcional (I-P) estabilizador. São estabelecidas condições suficientes para que um poliedro definido no espaço de estados aumentado (estado + erro de estimação + estado integral do erro de rastreamento) seja PI na presença de um sinal de referência constante, que pode ser interpretado como uma perturbação limitada. A partir das condições de invariância, um problema de otimização bilinear é formulado para calcular simultaneamente os ganhos do controlador e o poliedro positivamente invariante, garantindo a satisfação das restrições. São considerados os dois tipos de observador encontrados na literatura de sistemas fuzzy T-S: o primeiro considera as funções de pertinência dependentes apenas da saída do sistema; o segundo, por sua vez, refere-se ao caso geral, onde estas funções podem estar associadas a quaisquer variáveis de estado. No caso mais simples, embora as funções de pertinência dependam apenas da saída, a realimentação do estado estimado resulta, em geral, em controladores com melhor desempenho e com maiores conjuntos de estados admissíveis associados a eles do que o controle por realimentação estática de saída. Para o caso geral, como as funções de pertinência dependem de estados não acessíveis, é necessário um mecanismo de estimação para o cálculo destas variáveis. Em ambos os casos, este papel é desempenhado pelo observador fuzzy T-S.

Palavras-chave: Modelos fuzzy T-S, controle por realimentação de saída, conjuntos invariantes, rastreamento de referência, programação bilinear.

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:	such that
$\forall$	for all
$\exists$	there exist a least one
$<$	is less than
$>$	is greater than
$\leq$	is less than or equal to
$\geq$	is greater than or equal to
$\neq$	is not equal to
w_i	denotes the i^{th} component of $x \in \mathbb{R}^n$
W_i	denotes the i^{th} row of matrix W
W_{ij}	denotes the $(i,j)^{th}$ element of matrix W
$W \geq 0$	means a non-negative matrix, that is, $W_{ij} \geq 0, \forall i,j$
W^{T}	transpose of the matrix W
$\|\|W\|\|_\infty$	denotes the induced matrix infinity norm of $W \in \mathbb{R}^{m\times n}$: $\|\|W\|\|_\infty = \max_i \sum_{j=1}^{n} \|W_{ij}\|$
1	vector of appropriate dimension whose components are all equal to 1
I	identity matrix of appropriate dimension
$\mathbb{N}$	set of non-negative integers $\mathbb{N} = \{0,1,2,\cdots\}$
$\mathbb{N}^*$	set of positive integers $\mathbb{N} = \{1,2,\cdots\}$
$\mathbb{R}$	set of real numbers
$\mathbb{R}^n$	real space of dimension n, i.e., set of all column vector of n real components
$\mathbb{R}^{m\times n}$	set of all matrices of dimension $m \times n$

$x \in X$	x is a member of X
$X \subset Z$	X is a proper subset of Z
$X \subseteq Z$	X is a subset of Z
$X \cap Z$	set intersection. That is, the set $\{\omega : \omega \in X \text{ and } \omega \in Z\}$
$X \oplus Z$	Minkowski sum operator $X \oplus Z = \{x+z : x \in X, z \in Z\}$, for two given set $X \subset \mathbb{R}^n$ and $Z \subset \mathbb{R}^n$
λ	contraction factor
γ	scalar weighting factor
Φ, $\mathcal{F}$	generic objective functions
$\alpha_i(\cdot)$	membership functions
Δ	A standard simplex $\Delta \in \mathbb{R}^r$, where $r \in \mathbb{N}^*$
$R[Y,\rho]$	A polyhedral set in $\mathbb{R}^n$ defined by a system of linear inequalities $R[Y,\rho] = \{x : Yx \leq \rho\}$. If $\Omega = R[Y,\rho]$, then $\beta\Omega = R[Y,\beta\rho] = \{x : Yx \leq \beta\rho\}$, for any $\beta > 0$
$\mathcal{S}[Y_s,\rho]$	A symmetric polyhedral set in $\mathbb{R}^n$ defined by a system of linear inequalities $\mathcal{S}[Y_s,\rho] = \{x : Y_s x \leq \rho\}$. If $\Omega = \mathcal{S}[Y_s,\rho]$, then $\beta\Omega = \mathcal{S}[Y_s,\beta\rho] = \{x : Y_s x \leq \beta\rho\}$, for any $\beta > 0$
FLC	Fuzzy Logic Control
PDC	Parallel Distributed Compensator
LMI	Linear Matrix Inequalities
LPV	Linear Parameter-Varying
T-S	Takagi-Sugeno
PI	Positively Invariant
RPI	Robust Positively Invariant
I-P	Integral-Proportional
PPgEEC	Programa de Pós-Graduação em Engenharia Elétrica e de Computação
UFRN	Universidade Federal do Rio Grande do Norte
w.r.t.	with respected to

Chapter 1

Introduction

The human decision-making process is often influenced by the subconscious utilization of past experiences. Inspired by the human ability to solve problems, researchers have strived to develop a mathematical approach that replicates the human inference process, especially in situations involving uncertainty. This effort has resulted in the emergence of fuzzy logic, which provides a formal framework for handling and reasoning with imprecise information.

The first notions about fuzzy logic were developed in 1920 by the polish philosopher and logician Jan Lukasiewicz. He introduced sets with degrees of membership being 0, $1/2$, and 1 and later proposed using the range of values $[0,1]$. In 1965, Lofti Zadeh published the article entitled *Fuzzy Sets* [Zadeh, L.A. 1965], in which he was able to combine the concepts of classical logic and Lukasiewicz sets. This article became known as the origin of fuzzy logic.

The fuzzy set is a generalization of the classical set, well known to mathematics and engineering students. In set theory, the belonging relation between an element and a given set is a fundamental concept, typically expressed using a membership function. Classical or boolean logic employs a bivalent membership function where an element is either a member or not of the set. However, in fuzzy logic, one takes values within the interval $[0,1]$. This allows for partial membership, where an element can have a fractional value within this numerical range. Fuzzy systems consist of rules, which can be segmented into two parts: the antecedent and the consequent. The most used fuzzy models are Mamdani (classic model) and Takagi-Sugeno (interpolation model). Unlike the Mamdani model, the Takagi-Sugeno model does not involve fuzzy sets for its consequents; instead, mathematical expressions are employed.

The first successful applications in fuzzy logic are found in the field of control systems. The *Fuzzy Logic Control* (FLC) have been applied to a wide variety of practical systems with complex nonlinear dynamics, e.g., in automobile engineering [Zimmermann, H.J. 1991], in power systems and nuclear reactors [Bernard, J.A. 1988], in robotics [Isik, C. 1987], in adjusting the step angle of a wind turbine [Civelek, Z. et al. 2016] and controlling a distillation column used to separate methanol from water [Drgoňa, J. et al. 2017]. Despite the visible success of these applications, many fundamental questions still need to be addressed. These include stability analysis, systematic design, and controller performance analysis, which are crucial to the validity and applicability of any design methodology.

Moreover, controlling practical systems with nonlinear dynamics is often challenging due to the complexity of their mathematical descriptions. However, the Takagi-Sugeno (T-S) model [Takagi, T., & Sugeno, M. 1985] offers a viable approach for representing and controlling such systems. The T-S model utilizes fuzzy IF-THEN rules to describe the local dynamics of each rule (implication) using a linear system model. The T-S model provides an exact local representation of nonlinear plants by employing a convex combination of these linear models, as presented in [Wang, H.O., & Tanaka, K. 2004]. This feature allows the application of less involved techniques used, e.g., in linear parameter-varying systems, to analyze and design nonlinear control systems.

In the literature on fuzzy T-S systems, numerous studies deal with the control design based on the solution of optimization problems that include *Linear Matrix Inequalities* (LMIs) [Wang, H.O., & Tanaka, K. 2004, Feng, G. 2018] and a control structure named *Parallel Distributed Compensator* (PDC). The PDC provides a straightforward procedure to design controllers for nonlinear systems compared to other nonlinear control techniques. In the PDC, each control rule is designed from the corresponding rule of the T-S model and the controller shares the same fuzzy sets in the premise parts as the model and employs a linear control law in the consequent parts. However, these techniques do not consider that the controller is designed for a locally-defined model. Thus, control objectives can only be guaranteed if the trajectory of states is included within the validity region of the T-S model.

On the other hand, forcing state trajectories to remain within a given region is an objective that can be achieved through so-called *invariant sets*. The theory of positively invariant sets is one of the main tools used in the context of systems subject to constraints on their state, output, and control variables, enabling the analysis and design of this type of system. A non-empty subset in state-space is said to be a *Positively Invariant* (PI) set of a given dynamical system if any trajectory originating from this set does not leave it. Thus, the property of positive invariance plays a fundamental role in this context, given that the constraints can be guaranteed for all time if and only if the initial state is contained in the invariant set.

In general, linear constraints associated with state and control variables are mathematically expressed by polyhedral sets. When a given polyhedral set defined by such constraints is not positively invariant concerning a given dynamic, a possible solution to the constrained regulator problem is to obtain a *controlled invariant* polyhedron contained in the set defined by the state constraints [Blanchini, F. 1994, Dórea, C.E.T., & Hennet, J.C. 1999]. In this case, a corresponding state feedback control action is capable of guiding any state trajectory to the origin without violating the constraints, whose initial condition belongs to the controlled invariant set. Invariant set theory is well established and widely applied to linear systems subject to constraints. An overview of works in this area is presented in [Blanchini, F., & Miani, S. 2015].

More recently, the invariance theory has been successfully applied to constrained nonlinear systems, in particular, from the perspective of the T-S model. In [Ariño, C. et al. 2013], a technique is proposed to compute a polytopic approximation of invariant sets for discrete-time fuzzy T-S systems with amplitude-bounded disturbances. In [Ariño, C. et al. 2014], another method is proposed to calculate the maximal PI polyhedron contained in the set defined by the state and control constraints. In both techniques, the local control gains are obtained based on the solution of LMIs, such that a PDC controller can be designed to guarantee local asymptotic stability of the T-S model. These techniques were developed from a pre-calculated controller, and the general objective is to determine the positively invariant set.

Currently, a new approach has been developed to deal with the problem of simultaneously computing control gains and the invariant polyhedron. In [Dórea, C.E. et al. 2020], for discrete-time fuzzy T-S systems with amplitude-bounded disturbances subject to state and control constraints, a bilinear programming approach was proposed to compute both stabilizing feedback gains and two associated positive invariant polyhedral sets, a larger one which stands for the region of attraction and a smaller one standing for the set where the state trajectories converge despite the disturbances. The state feedback case and a dynamic output feedback scheme were considered, but only when the membership functions depend on the measured output. Another example of using the bilinear programming approach is found in [Ernesto, J.G. et al. 2021], which investigates an incremental stabilizing output feedback control law for discrete-time *Linear Parameter-Varying* (LPV) systems subject to state constraints, control, and control-rate variation bounds. In this work, the constraints in an extended state-space based on the state and control variables are defined, aiming at simultaneously calculating the controller and the associated PI polyhedron.

The problem of regulating a fuzzy T-S system under state and control constraints via output feedback becomes more challenging when the membership functions depend on unmeasured variables. A natural way to deal with this situation is to design a state observer. Similarly to PDC, the fuzzy T-S observer can be designed to guarantee local asymptotic convergence of the observation/estimation error. The fuzzy T-S observer is obtained from local observers associated with the same sets as the fuzzy T-S model. In output feedback control, the membership functions are defined based, in general, on estimates of state variables. Such estimates are necessary because the output may not contain information about all the states used by the membership functions. In [Wang, H.O., & Tanaka, K. 2004, Tanaka, K., & Sano, M. 1994], an approach is presented that deals with the observer design and the controller. This approach uses an augmented system that considers the representation of the estimation error and state variables in a single augmented vector. The control and observation gains are then computed based on LMIs. This design methodology does not consider the fact that the fuzzy model is valid only locally and lacks a mechanism that makes it possible to determine an estimate of the region of attraction, which generally contains a subset of the validity region. Furthermore, when the T-S model is obtained from the nonlinear model, the estimated premise variables may assume values outside the region for which the fuzzy model was initially conceived. In this case, guaranteeing the control objectives is no longer possible.

Model Predictive Control (MPC) approaches have been proposed to tackle the output feedback control problem under constraints, as in [Ding, B., & Pan, H. 2016, Ping, X. et al. 2021]. Such approaches directly incorporate state and control constraints in the computation of the control action, which is performed online. However, due to the high complexity of the related optimization problem, the control implementation can become infeasible for systems that operate with large sampling rates.

For the most part, the approaches developed for fuzzy T-S systems subject to constraints are concerned with regulator control design, leaving a gap to be filled concerning the reference tracking problem. The study carried out by [Figueiredo, L.S. et al. 2020] goes in the opposite direction, presenting the solution to the tracking problem of constant references for LPV systems subject to control constraints based on an Integral-Proportional (I-P)-Like controller. The fuzzy T-S model consists of a LPV system whose time-varying parameters depend on the state variables or their estimates. Adapting this technique to the fuzzy case is almost immediate, as seen in [Lopes, A.N. et al. 2020]. On the other hand, the perspective adopted to solve the tracking problem in [Lopes, A.N. et al. 2020] only take into account the case where the membership functions depend on accessible states.

Thus, aiming at developing a general and uncomplicated application approach, a control design methodology for discrete-time fuzzy T-S systems subject to constraints is proposed in this thesis, which seeks to solve the constrained regulator and tracking problems based on the theory of invariant sets. Situations in which the membership functions depend only on the output or variables that cannot be directly measured are considered. A dynamic PDC control law and the fuzzy T-S observer characterize the proposed technique. The estimation process is essential for the general case, when the membership functions depend on non-accessible variables. In the simplest case, although the membership functions depend only on the output, the estimated state feedback results, in general, in controllers with better performance and with larger sets of admissible states associated with them than the static output feedback control. The bilinear programming approach [Brião, S.L. et al. 2021, Dórea, C.E. et al. 2020] is used to determine the stabilizing gains and the positively invariant polyhedron simultaneously.

This work builds on previous study on positive invariant polyhedral sets for discrete-time fuzzy T-S systems. The outcomes presented herein encompass and offer substantial contributions to the literature concerning fuzzy T-S systems in many ways:

- The proposed approach involves the computation of both the PI set and the corresponding controller, a distinction that sets it apart from the methodologies outlined in the works of [Ariño, C. et al. 2013, Ariño, C. et al. 2014].
- With regard to output feedback control, we approach both cases found in the literature of fuzzy T-S systems, characterized by:
  - Membership functions that depend only on the system output. In this case, the proposed approach allows obtaining larger sets of admissible states when compared to static output feedback control techniques;
  - Membership functions that depend on variables non-accessible to direct measurement. Within the framework presented, the resolution for this case serves to address a deficiency within the literature, thereby distinguishing itself from the methodologies proposed in the studies of [Dórea, C.E. et al. 2020] and [Lopes, A.N. et al. 2020].

The key results presented here are grounded in novel invariance conditions for polyhedral sets. These conditions form the foundation for the stability issues discussed throughout this work.

- In terms of computational cost and ease of implementation, the proposed method hinges on the resolution of an offline optimization problem, yielding the gains of both the controller and the observer, alongside the associated invariant set. Consequently, the predominant computational burden is not placed upon the online implementation of the controller, in contrast to Model Predictive Control approaches, as in [Ding, B., & Pan, H. 2016, Ping, X. et al. 2021], which require the online resolution of optimization problems.

1.1 Thesis Outline

The thesis is organized as follows:

- Chapter 2 provides the main concepts and mathematical fundamentals about invariant polyhedral sets and their relationship with the control of fuzzy T-S systems subject to constraints.

- Chapter 3 address the constrained regulator problem from the theory of positively invariant polyhedral sets under a dynamic output feedback control law. In this regard, sufficient conditions are presented for a polyhedron defined in the augmented state-space (state + estimation error) to be PI, thus ensuring that the state and input control constraints are satisfied.

- Chapter 4 deals with the problem of tracking a piecewise constant reference for which the conception of the tracking controller is motivated from the stabilizing I-P controller. Sufficient conditions are established for a polyhedron defined in the augmented state-space (state + estimation error + tracking error integral state) to be PI in the presence of a piecewise constant reference, which can be interpreted as a bounded disturbance.

- Chapter 5 presents the conclusions of this thesis and considerations are made for possible future works in this area.

Chapter 2

Positive Invariance of Polyhedral Sets

The problem of keeping the state trajectory of a discrete-time fuzzy T-S system inside a given polyhedral set is addressed using the set invariance theory. Initially, the property of positive invariance of convex polyhedra is introduced in linear systems' context and then generalized to T-S systems. This property will be geometrically and analytically characterized from the State Feedback (StF) and Static Output Feedback (SOF) approaches. The essential element of this characterization is based on a description of the one-step admissible domain, which results in a geometric property that can be analytically characterized by the application of Farkas' Lemma. Then, sufficient conditions under which a convex polyhedron (general and symmetric cases) is Positively Invariant (PI) are established from matrix relations.

Furthermore, the invariance relations can be used to compute the PI set included in a given polyhedron defined by the state constraints so that the trajectory of the state vector can be maintained in this PI polyhedron and the control constraints are satisfied. If the state constraints are symmetric, it is natural that the polyhedron is as well. Otherwise, a generically shaped convex polyhedron is a better candidate for a PI set, as it can capture the asymmetric geometry of the state constraints.

2.1 Preliminaries

Consider the following terminology and definitions about sets, which will be used throughout this chapter:

Definition 2.1.1 (Bounded Set) *A set $S \subset \mathbb{R}^n$ is bounded if there is a positive number k such that for every $s \in S$, $|s| \leq k$.*

Definition 2.1.2 (Closed set) *It is any subset of $\mathbb{R}^n$ that contains its boundary points;*

Definition 2.1.3 (Compact set) *A set $S \subset \mathbb{R}^n$ is compact if it is a closed and bounded set.*

Definition 2.1.4 (Convex Set) *A set $S \in \mathbb{R}^n$ is said to be convex if for any $s^1 \in S,\ s^2 \in S$, we have*

$$\alpha s^1 + (1-\alpha)s^2 \in S,\ \forall 0 \leq \alpha \leq 1.$$

The point

$$s = \alpha s^1 + (1-\alpha)s^2,$$

with $0 \leq \alpha \leq 1$ *is said to be a convex combination of the pair* $[s^1, s^2]$*. The set of all these points is the line segment that connects* s^1 *to* s^2*. Given that* s^1 *and* s^2 *are any two points of S, a set is convex if it includes all line segments connecting all pairs of points* $[s^1, s^2]$*.*

Definition 2.1.5 (Convex Combination) *A convex combination of a finite number of points* $x^1, x^2, \cdots, x^t \in \mathbb{R}^n$ *is defined as a point:*

$$x = \sum_{i=1}^{t} \alpha_i x^i, \ \alpha_i \geq 0, \ i = 1, \cdots, t, \ \sum_{i=1}^{t} \alpha_i = 1.$$

Definition 2.1.6 (Hyperplane) *A hyperplane in* $\mathbb{R}^n$ *is defined to be the set of points*

$$X = \{x \in \mathbb{R}^n : c^{\mathrm{T}} x = z\},$$

where $c^{\mathrm{T}} \neq 0$ *is a given n-component arrow vector and z is a given scalar.*

If the equation for a hyperplane is written out, one obtains:

$$c^{\mathrm{T}} x = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n = z,$$

and any x satisfying this equation lies on the hyperplane. A hyperplane passes through the origin if and only if $z = 0$*. A hyperplane is a convex set.*

Definition 2.1.7 (Half-space) *A (closed) half-space in* $\mathbb{R}^n$ *is defined to be the set of points* $c^{\mathrm{T}} x \leq z$*, with* $c^{\mathrm{T}} \neq 0$ *being a given n-component arrow vector and z is given a scalar. A closed half-space is also a convex set.*

Definition 2.1.8 (Polyhedral Set) *A convex polyhedral set is a set of the form:*

$$\Omega = R[G, \rho] = \{x : Gx \leq \rho\} = \{x : G_i x \leq \rho_i, \ i = 1, \cdots, g\},$$

where G_i *denotes the i-th row of the matrix* $G \in \mathbb{R}^{g \times n}$*,* $x \in \mathbb{R}^n$ *and* $\rho \in \mathbb{R}^g$*.*

Each inequality $G_i \leq \rho_i$ defines a closed half-space in $\mathbb{R}^n$. Thus, a convex polyhedral set is given by the nonempty intersection of a finite number of closed half-spaces, that is, given a set defined by $F_i = \{x : G_i x \leq \rho_i\}$, such that $G_i \neq 0$, $x \in \mathbb{R}^n$, and ρ_i, $i = 1, \ldots, g$, where

$$\Omega = \bigcap_{i=1}^{g} F_i$$

is a polyhedral set if $\Omega \neq \emptyset$.

"A polyhedral set includes the origin if and only if $\rho \geq 0$ and includes the origin as an interior point if and only if $\rho > 0$" [Blanchini, F., & Miani, S. 2015]. The polyhedral sets considered in this context are defined as sets that contain the origin as an interior point, defined by

$$\Omega = R[G, \rho] = \{x : Gx \leq \rho\}, \ \rho > 0. \tag{2.1}$$

A convex polyhedral set that contains the origin as an interior point can be equivalently represented by

$$\Omega = R[G,\mathbf{1}] = \{x : Gx \leq \mathbf{1}\} = \{x : G_i x \leq 1,\ i = 1,\cdots,g\},$$

which can be achieved from (2.1) by dividing both sides of each inequality by $\rho_i > 0$.

Definition 2.1.9 (Polyhedral cone) *A convex polyhedral cone in $\mathbb{R}^n$ is a set defined by*

$$C = R[G,0] = \{x : Gx \leq 0\},$$

such that $G \in \mathbb{R}^{g\times n}$ and $x \in \mathbb{R}^n$.

From this definition, it is clear that polyhedral cones are a particular class of polyhedral sets.

Definition 2.1.10 (Symmetric Polyhedral Set) *A symmetric polyhedral set is a set of the form:*

$$\Omega = \mathcal{S}[G_s,\rho] = \{x : |G_s x| \leq \rho\} = \{x : |G_{s_i} x| \leq \rho_i,\ i = 1,\cdots,g_s\},$$

where G_{s_i} denotes the i-th row of the matrix $G_s \in \mathbb{R}^{g_s\times n}$, $x \in \mathbb{R}^n$ and $\rho \in \mathbb{R}^{g_s}$.

Definition 2.1.11 (Extreme point (or Vertex)) *Let $S \subset \mathbb{R}^n$ be a polyhedral set. A point v is a vertex or an extreme point of S if and only if, there are no points s^1, s^2, with $s^1 \neq s^2$ in the set, such that*

$$v = \alpha s^1 + (1-\alpha)s^2,\ 0 < \alpha < 1.$$

According to the Definition 2.1.11, an extreme point cannot be located between any two different points of the set; that is, it is not located on the line segment that joins these points $(0 < \alpha < 1)$.

Definition 2.1.12 (Polytope) *A compact polyhedral set is called a polytope.*

When a convex polyhedral set is compact, it is considered a polytope. In this context, it is possible to verify that in the case of a polytope S, if all its vertices v^j are known, one can express any $s \in S$ as a convex combination of the vertices of S, of according to [Maculan, N., & Fampa, M.H.C. 2006],

$$s = \sum_{j=1}^{q} \alpha_j v^j,\ \sum_{j=1}^{q} \alpha_j = 1 \text{ and } \alpha_j \geq 0,\ j = 1,\cdots,q.$$

Lemma 2.1.1 (Farkas' Lemma (variant)) *Given a matrix M and a vector v. Then, the system $Mx \leq v$ of linear inequalities has a solution x if and only if, $yv \geq 0$ for every row vector $y \geq 0$ with $yM = 0$.*

The set of row vectors y such that $y \geq 0$ and $yM = 0$ forms a polyhedral cone. This cone is called the non-negative left kernel of matrix M. These vectors can all be expressed as non-negative linear combinations of the generators of this cone.

The matrix formulation of the Lemma 2.1.2 provides a set of sufficient conditions on Q, P, ϕ, Ψ under which $R[Q,\phi] \subset R[P,\Psi]$, such that $Q \in \mathbb{R}^{q\times n}$, $\phi \in \mathbb{R}^q$, $P \in \mathbb{R}^{p\times n}$ and $\Psi \in \mathbb{R}^p$.

Lemma 2.1.2 (Farkas' Lemma (extended) [Hennet, J.C. 1995]) *For $x \in \mathbb{R}^n$, the system $Px \leq \Psi$ is satisfied by any point of the nonempty convex polyhedral set defined by the system $Qx \leq \phi$ if and only if, there exists a (dual) matrix $U \in \mathbb{R}^{p\times q}$, with all its components nonnegative and satisfying the conditions:*

$$\begin{aligned} UQ &= P, \\ U\phi &\leq \Psi. \end{aligned}$$

2.2 Positively Invariant Domains

Although less popular than ellipsoidal sets, polyhedral sets have been widely accepted as candidates for invariant sets. The main advantage of using polyhedral invariant sets in solving control problems under constraints lies in their capability to represent the linear physical constraints associated with the state variables of dynamic systems. These sets figure as an important family of so-called convex sets.

The main definitions and properties related to the invariance of polyhedral sets for linear systems will be presented in this section.

Consider the linear, discrete, time-invariant system model described by:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k), \end{aligned} \tag{2.2}$$

where $k \in \mathbb{N}$ is the time index, $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^m$ is the control input, $y(k) \in \mathbb{R}^p$ is the measured output, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ are, respectively, the state, control and output matrices. If $B = 0$, then (2.2) is equivalent to autonomous system $x(k+1) = Ax(k)$.

In the following, some basic definitions and properties related to invariant sets w.r.t. system (2.2) will be introduced [Blanchini, F. 1994, Dórea, C.E.T., & Hennet, J.C. 1999].

Definition 2.2.1 (Positive Invariance) *A non-empty closed set $\Omega \subset \mathbb{R}^n$ is said to be positively invariant w.r.t. system $x(k+1) = Ax(k)$, if and only if:*

$$\forall x(0) \in \Omega,\ x(k) \in \Omega,\ \forall k \in \mathbb{N}. \tag{2.3}$$

This way, for any initial condition imposed on the system (2.2), as long as $x(0) \in \Omega$, the trajectory of the state vector will remain within the set Ω.

For the system (2.2), where $u(k)$ is the control vector subject to linear constraints:

$$u(k) \in \mathcal{U} = R[S_u, \mathbf{1}] = \{u(k) \in \mathbb{R}^m : S_u u(k) \leq \mathbf{1}\},\ S_u \in \mathbb{R}^{g_u \times m}, \forall k \in \mathbb{N}, \tag{2.4}$$

the following definition is introduced.

Definition 2.2.2 (Controlled Invariance) *A non-empty closed set $\Omega \subset \mathbb{R}^n$ is said to be controlled-invariant w.r.t. system* (2.2)*, if:*

$$\forall x(0) \in \Omega,\ \exists u(k) \in \mathcal{U}:\ Ax(k) + Bu(k) \in \Omega. \tag{2.5}$$

According to [Dórea, C.E.T., & Hennet, J.C. 1999], Ω is a Positively Invariant (PI) set if, $\forall x(0) \in \Omega$, the trajectory of the state vector of the controlled system remains wholly contained in Ω.

In this case, it is possible to characterize the one-step admissible set by considering state and control constraints as follows:

Definition 2.2.3 (The one-step admissible set) *The one-step admissible set to Ω is defined as follows:*

$$\mathcal{L}(\Omega) = \{x(k) \in \mathbb{R}^n : \exists u(k) \in \mathcal{U} : Ax(k) + Bu(k) \in \Omega\}. \tag{2.6}$$

Theorem 2.2.1 *The set $\Omega \in \mathbb{R}^n$ is positively invariant w.r.t. system* (2.2) *if and only if, $\Omega \subseteq \mathcal{L}(\Omega)$.*

The set $\mathcal{L}(\Omega)$ can be defined as the set of all states that can be transferred to Ω in one step. Definition 2.2.1 can be extended to the system $x(k+1) = Ax(k) + Bu(k)$, provided that $u(k)$ is determined on the basis of a controller that has been previously computed. Within this framework, the positive invariance of Ω is tantamount to the geometric condition set forth in Theorem 2.2.1

2.3 Positive Invariance of Convex Polyhedra for Linear Systems

The problem of confining the trajectory of a discrete-time linear system to a given polyhedral set is addressed through the property of positive invariance (see, e.g., [Dórea, C.E.T., & Hennet, J.C. 1999]). Characterizing positive invariance for linear systems provides a valuable basis for understanding this property in fuzzy T-S systems.

In this section, sufficient and necessary conditions under which a general polyhedral set defined on the state-space (and on an augmented space) is Positively Invariant (PI) to linear systems under control laws based on State Feedback (StF) and Static Output Feedback (SOF) are presented. The conditions that guarantee the existence of a symmetric

PI polyhedron can be found in [Dórea, C.E.T., & Hennet, J.C. 1999, Brião, S.L. et al. 2018].

In practical problems, it is common for state and control variables to be physically limited. Therefore, we also present the algebraic relations that characterize the inclusion of the PI polyhedron into the set of state constraints and those that characterize the control constraints from applying Farkas' lemma.

2.3.1 The One-Step Admissible Set

Consider the system described by (2.2), such that

$$\Omega = R[G, \mathbf{1}] = \{x(k) : Gx(k) \leq \mathbf{1}\}, \tag{2.7}$$

characterizes the set of state constraints, where $x(k) \in \mathbb{R}^n$, $G \in \mathbb{R}^{g \times n}$ and $\mathbf{1}$ is a vector of ones of appropriate size.

For a given time k, admissibility of the state vector at time $k+1$ is characterized by the set of constraints:

$$Gx(k) \leq \mathbf{1} \rightarrow \text{at instant } k+1 \rightarrow Gx(k+1) \leq \mathbf{1}. \tag{2.8}$$

Substituting (2.2) into (2.8), one has:

$$G(Ax(k) + Bu(k)) \leq \mathbf{1}. \tag{2.9}$$

Consider now a State Feedback (StF) control law, described by:

$$u(k) = Fx(k), \; K \in \mathbb{R}^{m \times n}, \tag{2.10}$$

so that the corresponding closed-loop system is defined as

$$x(k+1) = (A + BF)x(k). \tag{2.11}$$

Substituting (2.10) in (2.9), one has:

$$G(Ax(k) + BFx(k)) \leq \mathbf{1} \Longrightarrow G(A + BF)x(k) \leq \mathbf{1}, \tag{2.12}$$

which characterizes the one-step admissible set.

Remark 2.3.1 *The Static Output Feedback (SOF) approach corresponds to consider a control law:*

$$u(k) = Ky(k) = KCx(k), \; K \in \mathbb{R}^{m \times p}, \; C \in \mathbb{R}^{p \times n}. \tag{2.13}$$

Then, the one-step admissible set is characterized by

$$G(A + BKC)x(k) \leq \mathbf{1}. \tag{2.14}$$

These constraints define a convex polyhedron in the linear space of the vector $x(k)$ so that the one-step admissible set is $\mathcal{L}(\Omega) = R[G(A+BF),\mathbf{1}]$. Furthermore, according to Theorem 2.2.1, the positive invariance of $R[G,\mathbf{1}]$ can be geometrically characterized by $\Omega \subseteq \mathcal{L}(\Omega)$.

2.3.2 Positively Invariant λ-Contractive Sets

Given a compact PI set containing the origin, it is possible to adjust the convergence of the trajectory of the state vector to the equilibrium point w.r.t. the system (2.2). In this case, the set is referred to as PI λ-contractive. The positive invariance and the convergence rate can be related according to the following definition [Dórea, C.E.T., & Hennet, J.C. 1999].

Definition 2.3.1 *A compact set $\Omega \subset \mathbb{R}^n$ is said to be positively invariant with contraction rate λ, $0 < \lambda < 1$, w.r.t. system* (2.2) *if it there exists $u(k) \in \mathcal{U}$, such that*

$$Ax(k) + Bu(k) \in \lambda\Omega, \ \forall x(k) \in \Omega. \tag{2.15}$$

As a consequence, let $\Omega \subset \mathbb{R}^n$ be a non-empty, positively invariant compact set, then, if Ω is λ-contractive, $\lambda\Omega$ is also so. If Ω is PI with contraction rate λ, then, $\forall x(k) \in \Omega$, $x(k+1) \in \lambda\Omega$ under a control law $u(x(k))$.

The one-step admissible set is now defined by:

$$\mathcal{L}(\Omega,\lambda) = \{x(k) \in \mathbb{R}^n : \exists u(k) \in \mathcal{U} : G(A+BF)x(k) \in \lambda\Omega\}. \tag{2.16}$$

Next, we present the sufficient and necessary conditions that characterize the existence of a PI λ-contractive polyhedron according to the Definition 2.3.1.

Lemma 2.3.1 (Positive Invariance) *The convex polyhedron $R[G,\mathbf{1}] \subset \mathbb{R}^n$ is positively invariant with contraction rate λ, $0 < \lambda < 1$, w.r.t. system* (2.2) *if and only if there is a matrix $H \in \mathbb{R}^{g\times g}$, with $H \geq 0$, such that:*

$$\begin{aligned} HG &= G(A+BF), \\ H\mathbf{1} &\leq \lambda\mathbf{1}. \end{aligned} \tag{2.17}$$

Lemma 2.3.1 is related to the existence of a PI λ-contractive polyhedron, assuring that if $x(0) \in R[G,\mathbf{1}]$, then $x(k) \in R[G,\mathbf{1}]$, for $k = 1,2,\cdots$. Furthermore, with $0 < \lambda < 1$, it guarantees the contration of $R[G,\mathbf{1}]$, i.e., if $x(k) \in R[G,\mathbf{1}]$, then $x(k+1) \in \lambda R[G,\mathbf{1}]$. The results presented in this section can be found in [Hennet, J.C. 1995, Dórea, C.E.T., & Hennet, J.C. 1999].

The positive invariance of the polyhedral set $R[G,\mathbf{1}]$ w.r.t. system (2.2) implies the existence of a non-negative matrix $H \in \mathbb{R}^{g\times g}$ such that: $HG = GA_0$, with $A_0 = A+BF \in \mathbb{R}^{n\times n}$. The matrix H being non-negative, its spectral radius, $\rho(H)$, is an eigenvalue of H

and, from the Perron-Frobenius Lemma [Berman, A., & Plemmons, R.J. 1979], an associated eigenvector of H is non-negative and non-null. Matrices H and A_0 being similar, where $g > n$, any eigenvalue of A_0 is also an eigenvalue of H. Asymptotic stability of system (2.2) is characterized by $\rho(H) \leq ||H||_\infty \leq \lambda$, such that $||H||_\infty \leq \lambda$ is equivalent to $H\mathbf{1} \leq \lambda\mathbf{1}$ and from a classical result on matrices, $\rho(H) \leq ||H||_\infty$. Therefore, the contractivity coefficient λ is associated with the speed of convergence of the system response (see, e.g., [Dórea, C.E.T., & Hennet, J.C. 1999, Brião, S.L. et al. 2021]).

Furthermore, if $R[G,\mathbf{1}]$ is a PI λ-contractive polyhedron, the discrete-time system (2.2) is locally asymptotically stable and it admits the polyhedral norm $||Gx(k)||_\infty$ as a Lyapunov function [Blanchini, F., & Miani, S. 2015]. Precisely, $\forall x(k) \in R[G,\mathbf{1}]$,

$$||Gx(k+1)||_\infty = ||GA_0x(k)||_\infty \leq \lambda||Gx(k)||_\infty.$$

Moreover, $R[G,\mathbf{1}]$ is an estimate of the region of attraction of the system with respect to the origin.

For the SOF approach, replace F by KC in the expression derived in Lemma 2.3.1, according to Remark 2.3.1.

2.3.3 Constrained Control Problem

In the following, we present the polyhedral inclusion conditions that, when satisfied, guarantee the fulfillment of state and control constraints. To this end, let us consider the linear constraints associated with the state and control variables, which can be mathematically expressed by polyhedral sets, as follows:

$$\mathcal{X} = R[S_x,\mathbf{1}] = \{x(k) \in \mathbb{R}^n : S_x x(k) \leq \mathbf{1}\}, \quad (2.18)$$

$$\mathcal{U} = R[S_u,\mathbf{1}] = \{u(k) \in \mathbb{R}^m : S_u u(k) \leq \mathbf{1}\}, \quad (2.19)$$

where $S_x \in \mathbb{R}^{g_x \times n}$, $S_u \in \mathbb{R}^{g_u \times m}$ and $\mathbf{1}$ is a vector of ones of appropriate size.

The inclusion of the polyhedral domain given by $R[G,\mathbf{1}] \subseteq R[S_x,\mathbf{1}]$ is guaranteed, and the state constraints are ensured, if and only if there exists $M \geq 0$, with $M \in \mathbb{R}^{g_x \times g}$, such that

$$\begin{aligned} MG &= S_x, \\ M\mathbf{1} &\leq \mathbf{1}, \end{aligned} \quad (2.20)$$

are satisfied.

Now, consider the control constraints expressed in state-space by:

$$R[S_u,\mathbf{1}] = \{u(k) \in \mathbb{R}^m : S_u u(k) \leq \mathbf{1}\} \Longrightarrow R[S_uF,\mathbf{1}] = \{x(k) \in \mathbb{R}^n : S_uFx(k) \leq \mathbf{1}\}. \quad (2.21)$$

As can be seen in (2.21), it is possible to directly characterize the control constraints in the state-space because $u(k) = Fx(k)$. Thus, $x(k) \in R[S_uF,\mathbf{1}]$ implies $u(k) \in \mathcal{U}$. Moreover, the control constraints can be guaranteed through the polyhedral domain inclusion given by $R[G,\mathbf{1}] \subseteq R[S_uF,\mathbf{1}]$, since the polyhedral set $R[G,\mathbf{1}]$ is PI λ-contractive.

The inclusion of polyhedral sets given by $R[G,\mathbf{1}] \subseteq R[S_uF,\mathbf{1}]$ is guaranteed, with $S_u \in \mathbb{R}^{g_u \times m}$, and the control constraints are ensured, if there exists $Q \geq 0$, with $Q \in \mathbb{R}^{g_u \times g}$, such that

$$\begin{aligned} QG &= S_u F, \\ Q\mathbf{1} &\leq \mathbf{1}, \end{aligned} \tag{2.22}$$

are satisfied.

If the conditions presented in (2.20) and (2.22) are satisfied, a PI polyhedron $R[G,\mathbf{1}]$ is simultaneously contained in $R[S_x,\mathbf{1}]$ and $R[S_uF,\mathbf{1}]$. In other words, it can be stated that $R[G,\mathbf{1}]$ is a subset of the intersection of $R[S_x,\mathbf{1}]$ and $R[S_uF,\mathbf{1}]$, which can be expressed as $R[G,\mathbf{1}] \subseteq (R[S_x,\mathbf{1}] \cap R[S_uF,\mathbf{1}])$. Consequently, the satisfaction of state and control constraints is guaranteed as long as the initial state $x(0)$ belongs to the polyhedral set $R[G,\mathbf{1}]$. The results presented in this section can be found in [Dórea, C.E.T., & Hennet, J.C. 1999].

For the SOF approach, replace F by KC in (2.21) and (2.22), according to Remark 2.3.1.

2.4 Positively Invariant Sets for Admissible References

In this section, we define the positively invariant sets for admissible references, drawing an analogy to the concept of robust invariance of polyhedral sets. Specifically, we consider a piecewise constant reference signal denoted as $r(k)$ and interpret it as a bounded disturbance.

The positive invariance property is established within this context for a specific closed-loop system in the augmented state space. The augmented state vector is denoted as $x_a(k)^{\mathrm{T}} = [x(k)\ v(k)]$, where $x(k)$ represents the state vector, and $v(k)$ means the tracking error integral state. The inclusion of the dynamic behavior of $v(k)$ is a consequence of employing an Integral-Proportional (I-P)-Like controller for the system (2.2). This controller is designed to ensure that the tracking error is null for piecewise constant references. The error is given by

$$e_t(k) = r(k) - y(k), \tag{2.23}$$

where $r(k) \in \mathbb{R}^p$ is the desired output of the system.

Based on the topology of an integral action state feedback LPV controller presented in [Figueiredo, L.S. et al. 2020], the tracking error in $k+1$ can be expressed in form of the finite differences:

$$v(k+1) - v(k) = e_t(k+1), \forall k \geq 0. \tag{2.24}$$

Therefore, the tracking error integral state can be defined by

$$v(k+1) = v(k) - Cx(k+1) + r(k), \tag{2.25}$$

where $y(k+1) = Cx(k+1) \in \mathbb{R}^p$ and the reference signal is consider constant during transient response, with $r(k) \equiv r(k+1)$.

The closed-loop system incorporates a representation of both the state and tracking error integral variables within a single augmented vector. This augmentation is derived from equations (2.2) and (2.25), and it results in a system order higher than that of the original system described by equation (2.2) due to the inclusion of integral action. The closed-loop system can be expressed in the following form:

$$\begin{bmatrix} x(k+1) \\ v(k+1) \end{bmatrix} = \begin{bmatrix} A & 0 \\ -CA & \mathbf{I} \end{bmatrix} \underbrace{\begin{bmatrix} x(k) \\ v(k) \end{bmatrix}}_{x_a(k)} + \begin{bmatrix} B \\ -CB \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ \mathbf{I} \end{bmatrix} r(k), \tag{2.26}$$

where $x(k) \in \mathbb{R}^n$, $v(k) \in \mathbb{R}^p$, $u(k) \in \mathbb{R}^m$, $r(k) \in \mathbb{R}^p$ and $\mathbf{I} \in \mathbb{R}^{p\times p}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$.

Now, consider the following Integral-Proportional (I-P)-Like control law with a feed-forward term, given by:

$$u(k) = Fx(k) + K_I v(k) + K_R r(k), \tag{2.27}$$

with $F \in \mathbb{R}^{m\times n}$, $K_I \in \mathbb{R}^{m\times p}$ and $K_R \in \mathbb{R}^{m\times p}$. The terms F and K_I are associated with the proportional and integral control actions, while K_R is a feedforward term used to improve the response according to the set-point [Åström, K. J., & Hägglund, T. 2006].

Substituting (2.27) in (2.26), we obtain the following description of the closed-loop system:

$$\begin{bmatrix} x(k+1) \\ v(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} A+BF & BK_I \\ -C(A+BF) & \mathbf{I}-CBK_I \end{bmatrix}}_{A_{cl}} \underbrace{\begin{bmatrix} x(k) \\ v(k) \end{bmatrix}}_{x_a(k)} + \underbrace{\begin{bmatrix} BK_R \\ \mathbf{I} \end{bmatrix}}_{B_{cl}} r(k) \tag{2.28}$$

where $x_a(k) \in \mathbb{R}^{n_a}$ is the augmented state, $A_{cl} \in \mathbb{R}^{n_a\times n_a}$, $B_{cl} \in \mathbb{R}^{n_a\times p}$, $n_a = n+p$ and $r(k) \in \mathfrak{R} \subset \mathbb{R}^p$ is the set of admissible references.

Remark 2.4.1 *The output feedback approach corresponds to consider a control law:*

$$u(k) = K_P C x(k) + K_I v(k) + K_R r(k), \tag{2.29}$$

with $K_P \in \mathbb{R}^{m\times p}$, $C \in \mathbb{R}^{p\times n}$, $K_I \in \mathbb{R}^{m\times p}$ *and* $K_R \in \mathbb{R}^{m\times p}$.

For the system described by (2.28), consider that the polyhedral set

$$\Omega = R[G, \mathbf{1}] = \{x_a(k) : Gx_a(k) \leq \mathbf{1}\}, \tag{2.30}$$

characterizes the set of augmented state constraints, where $x_a(k) \in \mathbb{R}^{n_a}$, $G \in \mathbb{R}^{g\times n_a}$ and $\mathbf{1}$ is a vector of ones of appropriate size.

Definition 2.4.1 (Positive Invariance) *Given* $0 < \lambda < 1$*, the set* $\Omega \in \mathbb{R}^{n_a}$ *is said to be positively invariant with contraction rate* λ*, w.r.t. system* (2.28)*, if it there exists* $u(k) \in \mathcal{U}$ *such that* $\forall x_a(k) \in \Omega$, $A_{cl}(k)x_a + B_{cl}r(k) \in \lambda\Omega$, $\forall r(k) \in \mathfrak{R}$.

If Ω is positively invariant with contraction rate λ, then, $\forall x_a(k) \in \Omega$, $x_a(k+1) \in \lambda\Omega$ under a control law $u(x(k), v(k))$, for any admissible reference $r(k) \in \mathfrak{R}$.

The one-step admissible set is now defined by:

$$\mathcal{L}(\Omega,\mathfrak{R},\lambda) = \{x_a(k) \in \mathbb{R}^{n_a} : \exists u(k) \in \mathcal{U} : GA_{cl}x_a(k) + GB_{cl}r(k) \in \lambda\Omega, \forall r(k) \in \mathfrak{R}\}. \quad (2.31)$$

For a given time k, admissibility of the state vector at time $k+1$ is characterized by:

$$G(A_{cl}x_a(k) + B_{cl}r(k)) \leq \lambda\mathbf{1}. \quad (2.32)$$

According to the *Internal Model Principle* (IMP), any stabilizing controller having the structure of (2.27), guarantees asymptotic offset-free set-point tracking [Chen, B.M. et al. 2004, 1, Section 9.2.2]. However, since the considered system is subject to state and input constraints, the validity of such a result may be restricted to a bounded state-space region. The results presented in this section can be found in [dos Santos, G.F. et al. 2023].

Characterizing the region of attraction becomes more complex in the context of tracking because the equilibrium point is not exclusively the origin. For $r(k)$ constant and admissible, the equilibrium point of (2.28) is characterized by $x(k+1) = x(k) = \bar{x}$ and $v(k+1) = v(k) = \bar{v}$. From (2.24), this last expression implies the tracking error in the equilibrium: $\bar{e}_t = 0$. Therefore, the determination of the equilibrium point of (2.28) depends directly on the admissible reference $r(k)$. As $r(k)$ can take on a range of values within the set $\mathfrak{R}$, this adds complexity to the computation of the region of attraction.

2.4.1 Constrained Control Problem

Now consider the following sets of constraints for the reference signal and the tracking error integral:

$$\mathcal{V} = R[S_v,\mathbf{1}] = \{v(k) \in \mathbb{R}^p : S_v v(k) \leq \mathbf{1}\},\ S_v \in \mathbb{R}^{g_v \times p}, \quad (2.33)$$

$$\mathfrak{R} = R[S_r,\mathbf{1}] = \{r(k) \in \mathbb{R}^p : S_r r(k) \leq \mathbf{1}\},\ S_r \in \mathbb{R}^{g_r \times p}. \quad (2.34)$$

and the sets $\mathcal{X}$ and $\mathcal{U}$, defined in Section 2.3.3.

The set $\mathcal{V}$ limits the magnitude of the variable $v(k)$. In general, the integral control action tends to make the response slower. The constraints on $v(k)$ can be tightened to reduce this effect. The set $\mathfrak{R}$ is used to establish the admissible values for the signal $r(k)$.

To deal with the state and tracking error integral constraints in the closed-loop state, from (2.18) and (2.33), we define the following set

$$R[S_a,\mathbf{1}] = \{x_a(k) \in \mathbb{R}^{n_a} : S_a x_a(k) \leq \mathbf{1}\}, \quad (2.35)$$

where $S_a = \begin{bmatrix} S_x & 0 \\ 0 & S_v \end{bmatrix}$, $S_a \in \mathbb{R}^{g_a \times n_a}$ and $g_a = g_x + g_v$.

A polyhedron $R[G,\mathbf{1}] = \{x_a(k) \in \mathbb{R}^{n_a} : Gx_a(k) \leq \mathbf{1}\}$, with $G \in \mathbb{R}^{g \times n_a}$. The inclusion of the polyhedral domain given by $R[G,\mathbf{1}] \subseteq R[S_a,\mathbf{1}]$ is guaranteed, if and only if there exists $M \geq 0$, with $M \in \mathbb{R}^{g_a \times g}$, such that

$$\begin{aligned} MG &= S_a, \\ M\mathbf{1} &\leq \mathbf{1}, \end{aligned} \quad (2.36)$$

are satisfied.

The linear control constraints related to the system (2.28), can be expressed from (2.19) and (2.27), by:

$$S_u u(k) = S_u K_a x_a(k) + S_u K_R r(k) \tag{2.37}$$

such that

$$R[S_u K_a, \mathbf{1}] = \{x_a(k) \in \mathbb{R}^{n_a} : S_u K_a x_a(k) \leq \mathbf{1}\}, \tag{2.38}$$

$$R[S_u K_R, \mathbf{1}] = \{r(k) \in \mathbb{R}^{p} : S_u K_R r(k) \leq \mathbf{1}\}. \tag{2.39}$$

with $K_a = [F \; K_I]$, $K_a \in \mathbb{R}^{m \times n_a}$ and $K_R \in \mathbb{R}^{m \times p}$.

On the other hand, a polyhedron defined by $R[S_u K_a, \mathbf{1}]$ characterizes the control constraints defined in the state-space. The inclusion of polyhedral sets given by $R[G, \mathbf{1}] \subseteq R[S_u K_a, \mathbf{1}]$ is guaranteed, with $S_u \in \mathbb{R}^{g_u \times m}$, $K_a = [F \; K_I]$, $K_a \in \mathbb{R}^{m \times n_a}$ and $K_R \in \mathbb{R}^{m \times p}$, if there exists $Q \geq 0$, $Q \in \mathbb{R}^{g_u \times g}$ and $Q_r \geq 0$, $Q_r \in \mathbb{R}^{g_u \times g_r}$, such that

$$\begin{array}{l} QG = S_u K_a, \\ Q_r S_r = S_u K_R, \\ Q\mathbf{1} + Q_r \mathbf{1} \leq \mathbf{1}. \end{array} \tag{2.40}$$

Then, the polyhedral set $R[G, \mathbf{1}]$ is PI λ-contractive and such that $R[G, \mathbf{1}] \subseteq R[S_a, \mathbf{1}]$ and $K_a R[G, \mathbf{1}] \oplus K_R \mathfrak{R} \subseteq \mathcal{U}$, where denotes $\oplus$ the Minkowski set sum operator. The results presented in this section can be found in [dos Santos, G.F. et al. 2023].

For the output feedback approach, consider $K_a = [K_p C \; K_I]$ in (2.38) and (2.40), according to Remark 2.4.1.

2.5 The Fuzzy Takagi-Sugeno (T-S) Model

The fuzzy Takagi-Sugeno (T-S) model proposed by [Takagi, T., & Sugeno, M. 1985] is described by fuzzy IF-THEN rules, which represent the local dynamics of each fuzzy implication (rule) by a linear system model.

Consider the discrete-time nonlinear system given by:

$$\begin{array}{l} x(k+1) = f(x(k), u(k)), \\ y(k) = Cx(k), \end{array} \tag{2.41}$$

where $f(\cdot) : \mathbb{R}^{n+m} \to \mathbb{R}^n$, $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$ and $y(k) \in \mathbb{R}^p$ are, respectively, the state, control input and output vectors, with $C \in \mathbb{R}^{p \times n}$.

In a limited region of state-space, the system (2.41) can be locally expressed from the fuzzy T-S model with r rules (see, e.g., [Wang, H.O. et al. 1996, Wang, H.O., & Tanaka, K. 2004]).

The i-th rule of the fuzzy T-S model for the discrete case is defined as follows:

Model i rule:

IF $z_1(k)$ *is* $\mathcal{M}_{i1}$ and $\cdots$ and $z_p(k)$ *is* $\mathcal{M}_{ip}$,

$$\textbf{\textit{THEN}} \begin{cases} x(k+1) = A_i x(k) + B_i u(k) \\ y(k) = Cx(k), \end{cases} \tag{2.42}$$

where $\mathcal{M}_{ij}(z_j(k))$ is s the grade of membership of $z_j(k)$ associated with the fuzzy sets $\mathcal{M}_{ij}$, $z(k) \in \mathbb{R}^p$ is the vector of premise variables and $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$, the matrices that define the consequent part of the i-th rule.

Therefore, given a pair $(x(k), u(k))$, the output of the fuzzy system is inferred as follows:

$$\begin{aligned} x(k+1) &= \sum_{i=1}^{r} \alpha_i(z(k))(A_i x(k) + B_i u(k)), \\ y(k) &= Cx(k), \end{aligned} \tag{2.43}$$

with

$$z(k) = [z_1(k)\ z_2(k)\ \cdots\ z_p(k)],$$

$$\alpha_i(z(k)) = \frac{\prod_{j=1}^{p} \mathcal{M}_{ij}(z_j(k))}{\sum_{i=1}^{r} \prod_{j=1}^{p} \mathcal{M}_{ij}(z_j(k))}, \tag{2.44}$$

where $\alpha_i(z(k))$ are the normalized membership functions (abbreviated as membership functions).

Since

$$\begin{cases} \sum_{i=1}^{r} \prod_{j=1}^{p} \mathcal{M}_{ij}(z_j(k)) > 0, \\ \\ \prod_{j=1}^{p} \mathcal{M}_{ij}(z_j(k)) \geq 0,\ i = 1, 2, \cdots, r, \end{cases} \tag{2.45}$$

we have

$$\begin{cases} \sum_{i=1}^{r} \alpha_i(z(k)) = 1, \\ \\ \alpha_i(z(k)) \geq 0,\ i = 1, 2, \cdots, r. \end{cases} \tag{2.46}$$

In general, the premise variables $z_1(k), z_2(k), \cdots, z_p(k)$ are functions of the state variables. Therefore, the fuzzy T-S model in (2.43) can be rewritten as follows:

$$\begin{aligned} x(k+1) &= \sum_{i=1}^{r} \alpha_i(x(k))(A_i x(k) + B_i u(k)), \\ y(k) &= Cx(k), \end{aligned} \tag{2.47}$$

with $\alpha(x(k)) \equiv \alpha(z(k))$.

2.5.1 Construction of the Fuzzy T-S Model

In the approach presented in this thesis, to design a fuzzy controller, we need a fuzzy T-S model for a nonlinear system. Therefore the construction of a fuzzy model represents an important and basic procedure. In general there are two approaches for constructing fuzzy models:

- Identification (fuzzy modeling) using input-output data and
- Derivation from given nonlinear system equations.

The identification (fuzzy modeling) is suitable for plants that are unable or too difficult to be represented by physical models. On the other hand, the second approach is more appropriated for the case where the nonlinear model is available.

In this thesis, we will not focus on the construction of the fuzzy T-S model. However, for the purpose of illustration, we shall herein present the idea of the *sector nonlinearity*, which defines a fundamental aspect of the second approach.

Sector Nonlinearity

The idea of using sector nonlinearity in fuzzy model construction first appeared in [Kawamoto, S. et al. 1992]. In this approach, the local representation of the component nonlinearities of the system is considered, in view of the difficulty to find global sectors for general nonlinear systems. The following example illustrate the concrete steps to construct fuzzy models.

Example 2.5.1 *Consider the following discrete-time nonlinear system [Tanaka, K. et al. 1997, Wang, H.O., & Tanaka, K. 2004, Song, W., & Liang, J. 2013]:*

$$\begin{aligned} x_1(k+1) &= (1-T\sigma)x_1(k) + T\sigma x_2(k) \\ x_2(k+1) &= T\rho x_1(k) + (1-T)x_2(k) - Tx_1(k)x_3(k) \\ x_3(k+1) &= Tx_1(k)x_2(k) + (1-T\beta)x_3(k) \end{aligned} \tag{2.48}$$

where $T = 0.01s$, $\sigma = 10$, $\rho = 28$ and $\beta = 8/3$.

We assume that $x_1(k) \in [-2,\ 2]$. Of course, we can assume any range for $x_1(k)$ to construct a fuzzy model.

Equation (2.48) *can be written as*

$$x(k+1) = \begin{bmatrix} 1-T\sigma & T\sigma & 0 \\ T\rho & 1-T & -z(k)T \\ 0 & z(k)T & 1-T\beta \end{bmatrix} x(k). \tag{2.49}$$

where $x(k)^{\mathrm{T}} = [x_1(k)\ x_2(k)\ x_3(k)]$ and $x_1(k)$ is associated to the nonlinear terms $-Tx_1(k)x_3(k)$ and $Tx_1(k)x_2(k)$, and $z(k) \equiv x_1(k)$ was chosen as the premise variable.

The minimum and maximum values of $z(k)$ under $x_1(k) \in [-2,\ 2]$ are denoted by $\min(z(k)) = -2$ *and* $\max(z(k)) = 2$. *From the maximum and minimum values, $z(k)$ can* *be represented by*

$$z(k) = x_1(k) = \mathcal{M}_{11}(z(k)) \cdot (2) + \mathcal{M}_{21}(z(k)) \cdot (-2) \tag{2.50}$$

where

$$\mathcal{M}_{11}(z(k)) + \mathcal{M}_{21}(z(k)) = 1. \tag{2.51}$$

The nonlinear system (2.48) *is the discrete-time representation of the Lorenz system based on Euler's method and it can be represented by the following fuzzy T-S model:*

Model* 1 *rule:

IF *$z(k)$ is $\mathcal{M}_{11}$,*

$$\textbf{\textit{THEN}} x(k+1) = A_1 x(k), \tag{2.52}$$

Model* 2 *rule:

IF *$z(k)$ is $\mathcal{M}_{21}$,*

$$\textbf{\textit{THEN}} x(k+1) = A_2 x(k), \tag{2.53}$$

where

$$A_1 = \begin{bmatrix} 1-T\sigma & T\sigma & 0 \\ T\rho & 1-T & -2T \\ 0 & 2T & 1-T\beta \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1-T\sigma & T\sigma & 0 \\ T\rho & 1-T & 2T \\ 0 & -2T & 1-T\beta \end{bmatrix}. \tag{2.54}$$

The membership functions can be calculated as

$$\begin{array}{l} \alpha_1(z(k)) = \mathcal{M}_{11}(z(k)) = \frac{z(k)+2}{4}, \\ \alpha_2(z(k)) = \mathcal{M}_{21}(z(k)) = \frac{2-z(k)}{4}. \end{array} \tag{2.55}$$

The defuzzification is carried out as

$$x(k) = \sum_{i=1}^{2} \alpha_i(z(k)) A_i x(k). \tag{2.56}$$

Figure (2.1) *shows the membership functions.*

This fuzzy model exactly represents the nonlinear system in the region defined by $x_1(k) \in [-2,\ 2]$.

If $x_1(k) \in [-2,\ 2]$ then the membership functions of the T-S model are guaranteed to be non-negative and $\sum_{i=1}^{r} \alpha_i(z(k)) = 1$. It is important to note that the dynamics of the nonlinear system represented by (2.48) can still be exactly represented even beyond the region defined by $x_1(k) \in [-2,\ 2]$, provided that the membership functions are allowed to take negative values.

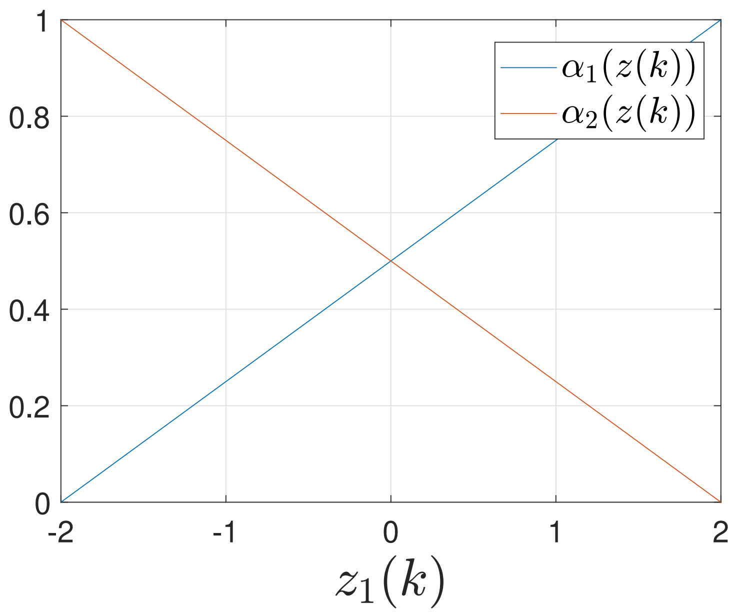


Figure 2.1: Membership functions $\alpha_1(z(k))$ and $\alpha_2(z(k))$.

For system control strategies based on the T-S model, it is crucial to ensure that the trajectory of states remains within the region for which the the model T-S was defined, denoted as validity region. Therefore, the control objectives can only be guaranteed if the trajectory of states is included within the validity region of the T-S model, which implies $\alpha_i(z(k)) \geq 0$ and $\sum_{i=1}^{r} \alpha_i(z(k)) = 1$. For further insights and detailed explanations, Chapter 3 will provide comprehensive discussions on this topic.

2.6 Parallel Distributed Compensation

Parallel Distributed Compensation (PDC) offers a procedure for schematizing the fuzzy controller based on the model proposed by Takagi and Sugeno. In PDC, each control rule is designed from the corresponding rule of the fuzzy T-S model. The designed fuzzy controller shares the same fuzzy sets with the model in the antecedent part [Wang, H.O., & Tanaka, K. 2004].

Initially, consider a simple state feedback. The i-th PDC rule is defined by:

Control rule i:

$$\textbf{\textit{IF}}\ z_1(k)\ is\ \mathcal{M}_{i1}\ \text{and} \cdots \text{and}\ z_p(k)\ is\ \mathcal{M}_{ip},$$

$$\textbf{\textit{THEN}}\ u(k) = F_i x(k),\ i = 1, 2, \cdots, r. \tag{2.57}$$

Then, a PDC control law is performed from the convex combination of the local controllers, so that

$$u(k) = \sum_{i=1}^{r} \alpha_i(x(k))F_i x(k),\ F_i \in \mathbb{R}^{m\times n}. \tag{2.58}$$

A PDC controller is designed for a locally-defined model (T-S model) and the control objectives can only be guaranteed if the trajectory of states is included within the validity region of the T-S model.

In practical problems, only the output variables are expected to be available for measurement. On the other hand, the membership functions of the T-S model (and, by extension, of the PDC controller) may only depend on the output or non-accessible states. For the simplest case, an static output feedback controller can be used to achieve the control objectives and the T-S model and the PDC controller are characterized by membership functions dependent on the measured output, such that $\alpha(y(k)) \equiv \alpha(z(k))$.

Remark 2.6.1 *For the static output feedback approach, the control law* (2.58) *can be replaced by*

$$u(k) = \sum_{i=1}^{r} \alpha_i(y(k))K_i C x(k), \tag{2.59}$$

with $K_i \in \mathbb{R}^{m\times p}$.

2.7 Positive Invariance for Fuzzy T-S Systems (General Case)

This section presents sufficient conditions for a polyhedral set defined on the state-space (and on augmented space) to be positively invariant for the fuzzy T-S system under control laws based on StF (and, by extension, under SOF control laws).

PDC controllers are designed for a locally-defined model (T-S model), and the control objectives can only be guaranteed if the trajectory of states is included within the validity region Ω of the T-S model. Furthermore, in practical problems, it is common that the state and the control input are limited. Then, we also present the algebraic relations that characterize the inclusion of the PI polyhedron into the set of state constraints and those that characterize the control constraints.

For the sake of simplicity, when necessary, we drop the explicit dependency of x on variable k inside the memberships functions.

2.7.1 Positively Invariant λ-Contractive Sets

Consider the discrete-time T-S fuzzy system, given by:

$$\begin{aligned} x(k+1) &= \underbrace{\sum_{i=1}^{r} \alpha_i(x(k))(A_i x(k) + B_i u(k)),}_{A(\alpha)x(k)+B(\alpha)u(k)} \\ y(k) &= Cx(k), \end{aligned} \tag{2.60}$$

with $A(\alpha) = \sum_{i=1}^{r} \alpha_i(x(k))A_i$ and $B(\alpha) = \sum_{i=1}^{r} \alpha_i(x(k))B_i$, such that $\alpha(x(k))$ belongs to the standard $(r-1)$-dimensional simplex $\Delta \in \mathbb{R}^r$, defined by:

$$\Delta = \{\alpha(x(k)) \in \mathbb{R}^r : \sum_{i=1}^{r} \alpha_i(x(k)) = 1,\ \alpha_i(x(k)) \geq 0\}.$$

For the system described by (2.60), the polyhedral set

$$\Omega = R[G, \mathbf{1}] = \{x(k) : Gx(k) \leq \mathbf{1}\}, \tag{2.61}$$

characterizes the set of state constraints, where $x(k) \in \mathbb{R}^n$, $G \in \mathbb{R}^{g\times n}$ and $\mathbf{1}$ is a vector of ones of appropriate size.

The possibility of imposing trajectories of $x_a(k)$ within a given polyhedral set $R[G,\mathbf{1}] \subset \Omega$ w.r.t. closed-loop system (2.60) can be achieved from the Definition (2.3.1). If $R[G,\mathbf{1}]$ is positively invariant with contraction rate λ, then, $\forall x_a(k) \in R[G,\mathbf{1}]$, $x_a(k+1) \in \lambda R[G,\mathbf{1}]$ under a control law $u(x(k))$.

The one-step admissible set is now defined by:

$$\mathcal{L}(\Omega,\lambda) = \{x(k) \in \mathbb{R}^n : \exists u(k) \in \mathcal{U} : GA(\alpha)x(k) + GB(\alpha)u(k) \in \lambda\Omega, \forall \alpha(x(k)) \in \Delta\}, \tag{2.62}$$

where $u(k)$ is the control vector subject to linear constraints in (2.4).

For a given time k, admissibility of the state vector at time $k+1$ is characterized by:

$$G(A(\alpha)x(k) + B(\alpha)u(k)) \leq \lambda\mathbf{1}. \tag{2.63}$$

Now, consider the PDC controller, described by:

$$u(k) = \sum_{i=1}^{r} \alpha_i(x(k))F_i x(k), \tag{2.64}$$

where $F_i \in \mathbb{R}^{m\times n}$, such that

$$x(k+1) = \sum_{i=1}^{r}\sum_{j}^{r} \alpha_i\alpha_j(A_i + B_iF_j)x(k) \tag{2.65}$$

is the StF closed-loop model, obtained by substituting the Equation (2.64) into (2.60).

The T-S model (2.65) can be equivalently represented by:

$$x(k+1) = \sum_{i=1}^{r} \alpha_i^2(A_i + B_iF_i)x(k) + \sum_{i=1}^{r}\sum_{i<j}^{r} \alpha_i\alpha_j(A_i + B_iF_j + A_j + B_jF_i)x(k), \tag{2.66}$$

The Equation (2.66) is useful because makes it possible to reduce the number of bilinear constraints that constitute the invariance conditions presented below.

From the Definition (2.3.1), consider the following proposition (see, e.g., [Dórea, C.E. et al. 2020]):

Proposition 2.7.1 *The polyhedral set $R[G,\mathbf{1}]$ is positively invariant with contraction rate λ, $0<\lambda<1$, w.r.t. the closed-loop system* (2.66)*, if there exist matrices $H_{ii} \in \mathbb{R}^{g\times g}$, $i=1,\cdots,r$ and $H_{ij} \in \mathbb{R}^{g\times g}$, with $i=1,\cdots,r$, $j=i+1,\cdots,r$, such that:*

$$\begin{array}{c} H_{ii}G = G(A_i+B_iF_i),\ H_{ii} \geq 0, \\ H_{ii}\mathbf{1} \leq \lambda\mathbf{1}, \\ H_{ij}G = G\frac{(A_i+B_iF_j+A_j+B_jF_i)}{2},\ H_{ij} \geq \mathbf{0}, \\ H_{ij}\mathbf{1} \leq \lambda\mathbf{1}. \end{array} \tag{2.67}$$

Consider $Gx(k) \leq \mathbf{1}$. Then:

$$Gx(k+1) = \sum_{i=1}^{r} \alpha_i^2 G(A_i+B_iF_i)x(k) + 2\sum_{i=1}^{r}\sum_{i<j}^{r} \alpha_i\alpha_j G\frac{(A_i+B_iF_j+A_j+B_jF_i)}{2}x(k)$$

$$= \sum_{i=1}^{r} \alpha_i^2 H_{ii}Gx(k) + 2\sum_{i=1}^{r}\sum_{i<j}^{r} \alpha_i\alpha_j H_{ij}Gx(k) \leq \sum_{i=1}^{r} \alpha_i^2 H_{ii}\mathbf{1} + 2\sum_{i=1}^{r}\sum_{i<j}^{r} \alpha_i\alpha_j H_{ij}\mathbf{1} \leq$$

$$\sum_{i=1}^{r} \alpha_i^2 \lambda\mathbf{1} + 2\sum_{i=1}^{r}\sum_{i<j}^{r} \alpha_i\alpha_j\lambda\mathbf{1} = \lambda(\sum_{i=1}^{r}\alpha_i)^2\mathbf{1} = \lambda\mathbf{1}.$$

This proves that the polyhedron $R[G,\mathbf{1}]$ is PI λ-contractive.

Proposition 2.7.1 is related to the existence of a PI λ-contractive polyhedron, assuring that if $x(0) \in R[G,\mathbf{1}]$, then $x(k) \in R[G,\mathbf{1}]$, for $k=1,2,\cdots$. Furthermore, with $0<\lambda<1$, it guarantees the contration of $R[G,\mathbf{1}]$, i.e., if $x(k) \in R[G,\mathbf{1}]$, then $x(k+1) \in \lambda R[G,\mathbf{1}]$. The results presented in this section can be found in [Dórea, C.E. et al. 2020].

If $R[G,\mathbf{1}]$ is a PI λ-contractive polyhedron, the discrete-time system (2.65) is locally asymptotically stable and it admits the polyhedral norm $||Gx(k)||_\infty$ as a Lyapunov function. Precisely, $\forall x(k) \in R[G,\mathbf{1}]$,

$$||Gx(k+1)||_\infty = ||GA_0(\alpha)x(k)||_\infty \leq \lambda||Gx(k)||_\infty,$$

with $A_0(\alpha) = \sum_{i=1}^{r}\sum_{j}^{r}\alpha_i\alpha_j(A_i+B_iF_j)$.

In this case, all the vertex systems $x(k+1) = (A_i+B_iF_j)x(k)$ share a common polyhedral Lyapunov function $||Gx(k)||_\infty$. Convexity of the fuzzy T-S model ensures that it is a Lyapunov function $\forall\alpha$ in the unity simplex [Blanchini, F., & Miani, S. 2015, p. 261].

Moreover, $R[G,\mathbf{1}]$ is an estimate of the region of attraction of the system with respect to the origin.

For the SOF approach, replace F_i by K_iC in (2.64), (2.65), (2.66) and in the expressions derived in Proposition 2.7.1, according to Remark 2.6.1.

2.7.2 Constrained Control Problem

The set of state constraints $R[S_x,\mathbf{1}]$ is defined based on the region of validity of the T-S model, ensuring that $R[S_x,\mathbf{1}] \subseteq \Omega$. It is important to note that, generally, the polyhedral set $R[S_x, \mathbf{1}]$ is not positively invariant under a given dynamic. Nonetheless, there is a polyhedron $R[G, \mathbf{1}] \subseteq R[S_x, \mathbf{1}]$ that exhibits the property of positive invariance.

The inclusion of the polyhedral domain given by $R[G, \mathbf{1}] \subseteq R[S_x, \mathbf{1}]$ is guaranteed as well as for the linear case. Therefore, the state constraints are respected as long as the conditions presented in (2.20) of the Section 2.3.3 are satisfied and, consequently, $R[G, \mathbf{1}]$ is contained in the validity region for T-S model.

The control constraints related to the system (2.66) can be expressed from (2.19) and (2.64) in the state-space, by:

$$S_u u(k) = \sum_{i=1}^{r} \alpha_i(x(k)) S_u F_i x(k), \tag{2.68}$$

such that

$$R[S_u F_i, \mathbf{1}] = \{x(k) \in \mathbb{R}^n : S_u F_i x(k) \leq \mathbf{1}\}. \tag{2.69}$$

As can be seen in (2.69), it is possible to directly characterize the control constraints in the state-space because $u(k) = \sum_{i=1}^{r} \alpha_i(x(k)) F_i x(k)$. Thus, if $x(k) \in R[S_u F_i, \mathbf{1}]$, $i = 1, 2, \cdots, r$, then $u(k) \in \mathcal{U}$. Moreover, the control constraints can be guaranteed through the polyhedral domain inclusions given by $R[G, \mathbf{1}] \subseteq R[S_u F_i, \mathbf{1}]$, since the polyhedral set $R[G, \mathbf{1}]$ is PI λ-contractive.

In this regard, consider the following proposition (see, e.g., [Dórea, C.E. et al. 2020]):

Proposition 2.7.2 *A polyhedron defined by $R[S_u F_i, \mathbf{1}]$ characterizes the control constraints defined in the state-space. The inclusion of polyhedral sets given by $R[G, \mathbf{1}] \subseteq R[S_u F_i, \mathbf{1}]$ is guaranteed, with $F_i \in \mathbb{R}^{m \times n}$, if there exist matrices $Q_i \geq 0$, with $Q_i \in \mathbb{R}^{g_u \times g}$, such that*

$$\begin{aligned} Q_i G &= S_u F_i, \\ Q_i \mathbf{1} &\leq \mathbf{1}. \end{aligned} \tag{2.70}$$

Consider $Gx(k) \leq \mathbf{1}$. Then, from (2.68) and (2.70):

$$\begin{aligned} S_u u(k) = \sum_{i=1}^{r} \alpha_i(x(k)) S_u F_i x(k) = \sum_{i=1}^{r} \alpha_i(x(k)) Q_i G x(k) \leq \sum_{i=1}^{r} \alpha_i(x(k)) Q_i \mathbf{1} \leq \\ \sum_{i=1}^{r} \alpha_i(x(k)) \mathbf{1} = \mathbf{1}. \end{aligned} \tag{2.71}$$

This proves the inclusion of the polyhedral domain given by $R[G, \mathbf{1}] \subseteq R[S_u F_i, \mathbf{1}]$.

If the conditions presented in (2.20) and (2.70) are satisfied, a PI polyhedron $R[G, \mathbf{1}]$ is simultaneously contained in $R[S_x, \mathbf{1}]$ and $R[S_u F_i, \mathbf{1}]$. The intersections of the set $R[S_x, \mathbf{1}]$ with $R[S_u F_i, \mathbf{1}]$ form a compact polytope that includes the origin. It indicates that the polyhedral domain $R[G, \mathbf{1}]$ is contained into the intersections of the polyhedral sets $R[S_x, \mathbf{1}]$ and $R[S_u F_i, \mathbf{1}]$. Consequently, the satisfaction of state and control constraints is guaranteed as long as the initial state $x(0)$ belongs to the polyhedral set $R[G, \mathbf{1}]$. The results presented in this section can be found in [Dórea, C.E. et al. 2020].

For the SOF approach, replace F_i by K_iC in (2.68), (2.69) and in the expressions derived in Proposition 2.7.2, according to Remark 2.6.1.

2.8 Positively Invariant Sets for Admissible References

The inclusion of the dynamic behavior of $v(k)$ is a consequence of employing an Integral-Proportional (I-P)-Like controller for the system (2.47), with the aim to ensure that the tracking error is null for piecewise constant references. The error is given by

$$e_t(k) = r(k) - y(k), \tag{2.72}$$

where $r(k) \in \mathbb{R}^p$ is the desired output of the system.

Based on the topology of an integral action state feedback LPV controller presented in [Figueiredo, L.S. et al. 2020], the tracking error in $k+1$ can be expressed in form of the finite differences:

$$v(k+1) - v(k) = e_t(k+1), \forall k \geq 0. \tag{2.73}$$

Therefore, the tracking error integral state can be defined by

$$v(k+1) = v(k) - Cx(k+1) + r(k), \tag{2.74}$$

where $y(k+1) = Cx(k+1) \in \mathbb{R}^p$ and the reference signal is consider constant during transient response, with $r(k) \equiv r(k+1)$.

The closed-loop system incorporates a representation of both the state and tracking error integral variables within a single augmented vector. This augmentation is derived from (2.47) and (2.74), and it results in a system order higher than that of the original system described by (2.47) due to the inclusion of integral action. The closed-loop system can be expressed in the following form:

$$\begin{bmatrix} x(k+1) \\ v(k+1) \end{bmatrix} = \sum_{i=1}^{r} \alpha_i(x(k)) \left(\begin{bmatrix} A_i & 0 \\ -CA_i & \mathbf{I} \end{bmatrix} \underbrace{\begin{bmatrix} x(k) \\ v(k) \end{bmatrix}}_{x_a(k)} + \begin{bmatrix} B_i \\ -CB_i \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ \mathbf{I} \end{bmatrix} r(k) \right) \tag{2.75}$$

where $x(k) \in \mathbb{R}^n$, $v(k) \in \mathbb{R}^p$, $u(k) \in \mathbb{R}^m$, $r(k) \in \mathbb{R}^p$ and $\mathbf{I} \in \mathbb{R}^{p \times p}$, $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{p \times n}$, with $\alpha(x(k)) \in \Delta$.

Now, consider a PDC (I-P)-Like control law, such that

$$u(k) = F_i x(k) + K_{I_i} v(k) + K_{R_i} r(k),\ i = 1, 2, \cdots, r, \tag{2.76}$$

correspond to the consequents of local rules of the PDC controller, given by

$$u(k) = \sum_{i=1}^{r} \alpha_i(x(k))(F_i x(k) + K_{I_i} v(k) + K_{R_i} r(k)), \tag{2.77}$$

where $F_i \in \mathbb{R}^{m \times n}$, $K_{I_i} \in \mathbb{R}^{m \times p}$ and $K_{R_i} \in \mathbb{R}^{m \times p}$ are, respectively, the proportional, integral and feedforward terms.

Remark 2.8.1 *For the output feedback approach, the control law 2.77 can be replaced by*

$$u(k) = \sum_{i=1}^{r} \alpha_i(y(k))(K_{P_i}Cx(k) + K_{I_i}v(k) + K_{R_i}r(k)), \tag{2.78}$$

with $K_i \in \mathbb{R}^{m\times p}$, $K_{P_i} \in \mathbb{R}^{m\times p}$, $K_{I_i} \in \mathbb{R}^{m\times p}$ *and* $K_{R_i} \in \mathbb{R}^{m\times p}$.

Substituting (2.77) into (2.75), we obtain the following description of the closed-loop system:

$$\begin{bmatrix} x(k+1) \\ v(k+1) \end{bmatrix} = \sum_{i=1}^{r}\sum_{i=1}^{r} \alpha_i(x)\alpha_j(x)\left(\underbrace{\begin{bmatrix} A_i + B_iF_j & B_iK_{I_j} \\ -C(A_i+B_iF_j) & \mathbf{I} - CB_iK_{I_j} \end{bmatrix}}_{A^a_{ij}} \underbrace{\begin{bmatrix} x(k) \\ v(k) \end{bmatrix}}_{x_a(k)} + \underbrace{\begin{bmatrix} B_iK_{R_j} \\ \mathbf{I} \end{bmatrix}}_{B^a_{ij}} r(k)\right), \tag{2.79}$$

where $x_a(k) \in \mathbb{R}^{n_a}$ is the augmented state, $A(\alpha) = \sum_{i=1}^{r}\sum_{i=1}^{r} \alpha_i(x)\alpha_j(x)A^a_{ij}$, $A^a_{ij} \in \mathbb{R}^{n_a\times n_a}$, $\mathcal{B}(\alpha) = \sum_{i=1}^{r}\sum_{i=1}^{r} \alpha_i(x)\alpha_j(x)B^a_{ij}$, $B^a_{ij} \in \mathbb{R}^{n_a\times p}$, $n_a = n + p$ and $r(k) \in \mathfrak{R} \subset \mathbb{R}^p$ is the set of admissible references.

For the system described by (2.79), consider that the polyhedral set

$$\Omega = R[G, \mathbf{1}] = \{x_a(k) : Gx_a(k) \leq \mathbf{1}\}, \tag{2.80}$$

characterizes the set of augmented state constraints, where $x_a(k) \in \mathbb{R}^{n_a}$, $G \in \mathbb{R}^{g\times n_a}$ and $\mathbf{1}$ is a vector of ones of appropriate size.

The possibility of imposing trajectories of $x_a(k)$ within a given polyhedral set $R[G, \mathbf{1}] \subset \Omega$ w.r.t. closed-loop system (2.79) can be achieved from the Definition (2.4.1). If $R[G, \mathbf{1}]$ is positively invariant with contraction rate λ, then, $\forall x_a(k) \in R[G, \mathbf{1}]$, $x_a(k+1) \in \lambda R[G, \mathbf{1}]$ under a control law $u(x(k), v(k))$, for any admissible reference $r(k) \in \mathfrak{R}$, with

$$\mathfrak{R} = R[S_r, \mathbf{1}] = \{r(k) \in \mathrm{R}^p : S_r r(k) \leq \mathbf{1}\},\ S_r \in \mathbb{R}^{g_r\times p}. \tag{2.81}$$

The one-step admissible set is now defined by:

$$\mathcal{L}(\Omega, \mathfrak{R}, \lambda) = \{x_a(k) \in \mathbb{R}^{n_a} : \exists u(k) \in \mathcal{U} : GA(\alpha)x_a(k) + G\mathcal{B}(\alpha)r(k) \in \lambda\Omega, \forall r(k) \in \mathfrak{R}, \forall \alpha(x(k)) \in \Delta\}, \tag{2.82}$$

where $u(k)$ is the control vector subject to linear constraints in (2.4).

For a given time k, admissibility of the state vector at time $k+1$ is characterized by:

$$G(A(\alpha)x_a(k) + \mathcal{B}(\alpha)r(k)) \leq \lambda\mathbf{1}; \tag{2.83}$$

The model T-S (2.79) can be rewritten as follows:

$$x_a(k+1) = \sum_{i=1}^{r} \alpha_i^2(x)(A_{ii}^a x_a(k) + B_{ii}^a r(k)) + \\ 2\sum_{i=1}^{r}\sum_{i<j}^{r} \alpha_i(x)\alpha_j(x)\left(\left(\frac{A_{ij}^a + A_{ji}^a}{2}\right)x_a(k) + \left(\frac{B_{ij}^a + B_{ji}^a}{2}\right)r(k)\right). \tag{2.84}$$

The Equation (2.84) is equivalent to (2.79) and makes it possible to reduce the number of bilinear constraints that constitute the invariance conditions presented below.

In the following, we propose sufficient conditions for the invariance of polyhedral sets w.r.t. closed-loop system (2.84). These conditions draw inspiration from the ones previously presented for the linear case [dos Santos, G.F. et al. 2023].

Theorem 2.8.1 *A polyhedral set $R[G,\mathbf{1}]$ is positively invariant with contraction rate λ, $0 < \lambda < 1$, w.r.t. closed-loop system (2.84), if there are matrices $H_{ii} \in \mathbb{R}^{g\times g}$, $Z_{ii} \in \mathbb{R}^{g\times g_r}$, $i = 1,\cdots,r$ and $H_{ij} \in \mathbb{R}^{g\times g}$, $Z_{ij} \in \mathbb{R}^{g\times g_r}$, with $i = 1,\cdots,r$, $j = i+1,\cdots,r$, such that:*

$$\begin{gathered} H_{ii}G = GA_{ii}^a,\ H_{ii} \geq 0, \\ Z_{ii}S_r = GB_{ii}^a,\ Z_{ii} \geq 0, \\ H_{ii}\mathbf{1} + Z_{ii}\mathbf{1} \leq \lambda\mathbf{1}, \\ H_{ij}G = G\left(\frac{A_{ij}^a + A_{ji}^a}{2}\right),\ H_{ij} \geq 0, \\ Z_{ij}S_r = G\left(\frac{B_{ij}^a + B_{ji}^a}{2}\right),\ Z_{ij} \geq 0, \\ H_{ij}\mathbf{1} + Z_{ij}\mathbf{1} \leq \lambda\mathbf{1}. \end{gathered} \tag{2.85}$$

Proof 2.8.1 *Consider $Gx_a(k) \leq \mathbf{1}$ and $S_r r(k) \leq \mathbf{1}$. Then:*

$$Gx_a(k+1) = \sum_{i=1}^{r}\alpha_i^2 G(A_{ii}^a x_a(k) + B_{ii}^a r(k)) + 2\sum_{i=1}^{r}\sum_{i<j}^{r}\alpha_i\alpha_j G\left(\left(\frac{A_{ij}^a + A_{ji}^a}{2}\right)x_a(k) + \left(\frac{B_{ij}^a + B_{ji}^a}{2}\right)r(k)\right) = \sum_{i=1}^{r}\alpha_i^2(H_{ii}Gx_a(k) + Z_{ii}S_r r(k)) + 2\sum_{i=1}^{r}\sum_{i<j}^{r}\alpha_i\alpha_j(H_{ij}Gx_a(k) + Z_{ij}S_r r(k)) \leq \sum_{i=1}^{r}\alpha_i^2 H_{ii}\mathbf{1} + 2\sum_{i=1}^{r}\sum_{i<j}^{r}\alpha_i\alpha_j H_{ij}\mathbf{1} \leq \sum_{i=1}^{r}\alpha_i^2\lambda\mathbf{1} + 2\sum_{i=1}^{r}\sum_{i<j}^{r}\alpha_i\alpha_j\lambda\mathbf{1} = \lambda(\sum_{i=1}^{r}\alpha_i)^2\mathbf{1} = \lambda\mathbf{1}.$$

This proves that the polyhedron $R[G,\mathbf{1}]$ is PI λ-contractive. □

Positive invariance of $R[G,\mathbf{1}]$ ensures $x(k) \in R[G,\mathbf{1}]$ $\forall k \geq 0$ and for all admissible reference, as long as $x(0) \in R[G,\mathbf{1}]$. If $R[G,\mathbf{1}]$ is compact, that implies, at least, closed-loop local stability. Positive invariance with a contraction rate $\lambda < 1$, implies asymptotic stability for $r(k) = 0$. For $r(k)$ constant and admissible, the equilibrium point of (2.75) is characterized by $x(k+1) = x(k) = \bar{x}$ and $v(k+1) = v(k) = \bar{v}$. From (2.73), this last expression implies for the tracking error in the equilibrium: $\bar{e}_t = 0$.

For the output feedback approach, replace F_i by $K_{P_i}C$ in (2.77) and (2.79), according to Remark 2.8.1.

2.8.1 Constrained Control Problem

Consider the following sets of constraints for the tracking error integral and the reference signal:

$$\mathcal{V} = R[S_v, \mathbf{1}] = \{v(k) \in \mathrm{R}^p : S_v v(k) \leq \mathbf{1}\},\ S_v \in \mathbb{R}^{g_v \times p}, \tag{2.86}$$

$$\mathfrak{R} = R[S_r, \mathbf{1}] = \{r(k) \in \mathrm{R}^p : S_r r(k) \leq \mathbf{1}\},\ S_r \in \mathbb{R}^{g_r \times p}. \tag{2.87}$$

and the sets $\mathcal{X}$ and $\mathcal{U}$, defined in Section 2.3.3.

To deal with the state and tracking error integral constraints in the closed-loop, from (2.18) and (2.86), consider the following set

$$R[S_a, \mathbf{1}] = \{x_a(k) \in \mathbb{R}^{n_a} : S_a x_a(k) \leq \mathbf{1}\}, \tag{2.88}$$

where $S_a = \begin{bmatrix} S_x & 0 \\ 0 & S_v \end{bmatrix}$, $S_a \in \mathbb{R}^{g_a \times n_a}$ and $g_a = g_x + g_v$.

Since $R[G, \mathbf{1}] = \{x_a(k) \in \mathbb{R}^{n_a} : G x_a(k) \leq \mathbf{1}\}$ with $G \in \mathbb{R}^{g \times n_a}$, the inclusion of the polyhedral domain given by $R[G, \mathbf{1}] \subseteq R[S_a, \mathbf{1}]$ is guaranteed as well as for the linear case. Therefore, the constraints on the augmented state are respected as long as the conditions presented in (2.36) of the Section 2.4.1 are satisfied and, consequently, $R[G, \mathbf{1}]$ is contained in the validity region for T-S model.

The control constraints related to the system (2.79), can be expressed from (2.19) and (2.77), by:

$$S_u u(k) = \sum_{i=1}^{r} \alpha_i(x(k))(S_u K_{a,i} x_a(k) + S_u K_{R_i} r(k)) \tag{2.89}$$

such that

$$R[S_u K_{a,i}, \mathbf{1}] = \{x_a(k) \in \mathbb{R}^{n_a} : S_u K_{a,i} x_a(k) \leq \mathbf{1}\}, \tag{2.90}$$

$$R[S_u K_{R_i}, \mathbf{1}] = \{r(k) \in \mathbb{R}^p : S_u K_{R_i} r(k) \leq \mathbf{1}\}. \tag{2.91}$$

with $K_{a,i} = [F_i\ K_{I_i}]$, $K_{a,i} \in \mathbb{R}^{m \times n_a}$ and $K_{R_i} \in \mathbb{R}^{m \times p}$.

In the following, we propose sufficient conditions that, when satisfied, guarantee that the control constraints are respected. These conditions draw inspiration from the ones previously presented for the linear case [dos Santos, G.F. et al. 2023].

Theorem 2.8.2 *A polyhedron defined by $R[S_u K_{a,i}, \mathbf{1}]$ characterize the control constraints defined in the state-space. The inclusion of polyhedral sets given by $R[G, \mathbf{1}] \subseteq R[S_u K_{a,i}, \mathbf{1}]$ is guaranteed, with $K_{a,i} = [F_i\ K_{I_i}]$, $K_{a,i} \in \mathbb{R}^{m \times n_a}$ and $K_{R_i} \in \mathbb{R}^{m \times p}$, if there exist matrices*

$Q_i \geq 0,\ Q_i \in \mathbb{R}^{g_u \times g}$ *and* $Q_{r,i} \geq 0, Q_{r,i} \in \mathbb{R}^{g_u \times g_r}$*, such that*

$$\begin{array}{c} Q_i G = S_u K_{a,i}, \\ Q_{r,i} S_r = S_u K_{R_i}, \\ Q_i \mathbf{1} + Q_{r,i} \mathbf{1} \leq \mathbf{1}. \end{array} \tag{2.92}$$

Proof 2.8.2 *Consider* $G x_a(k) \leq \mathbf{1}$ *and* $S_r r(k) \leq \mathbf{1}$*. Then, from* (2.89) *and* (2.92)*:*

$$\begin{array}{l} S_u u(k) = \sum\limits_{i=1}^{r} \alpha_i(x(k))(S_u K_{a,i} x_a(k) + S_u K_{R_i} r(k)) = \sum\limits_{i=1}^{r} \alpha_i(x(k))(Q_i G x_a(k) + Q_{r,i} S_r r(k)) \\ \leq \sum\limits_{i=1}^{r} \alpha_i(x(k))(Q_i \mathbf{1} + Q_{r,i} \mathbf{1}) \leq \sum\limits_{i=1}^{r} \alpha_i(x(k)) \mathbf{1} \leq \mathbf{1}. \end{array} \tag{2.93}$$

This proves the inclusion of the polyhedral domain given by $K_{a,i} R[G, \mathbf{1}] \oplus K_{R_i} \mathfrak{R} \subseteq \mathcal{U}$*.*

Then, the polyhedral set $R[G, \mathbf{1}]$ is PI λ-contractive and such that $R[G, \mathbf{1}] \subseteq R[S_a, \mathbf{1}]$ and $K_{a,i} R[G, \mathbf{1}] \oplus K_{R_i} \mathfrak{R} \subseteq \mathcal{U}$, where denotes $\oplus$ the Minkowski set sum operator.

For the SOF approach, consider $K_{a,i} = [K_{P_i} C\ K_{I_i}]$ in (2.89) and in the expressions derived in Theorem 2.8.2, according to Remark 2.6.1.

2.9 Positive Invariance for Fuzzy T-S Systems (Symmetric Case)

When the set of state constraints is symmetric, it is natural that the PI polyhedron is as well. Therefore, in this section, sufficient conditions for a symmetric polyhedral set defined on the state-space (and on augmented space) to be PI for the fuzzy T-S system under control laws based on StF (and, by extension, under SOF control laws) are presented. The algebraic relations that characterize the inclusion of the PI polyhedron to the set of symmetric state constraints and those that characterize the symmetric control constraints are established. The present formulation extends the linear case, as described in [Dórea, C.E.T., & Hennet, J.C. 1999, Brião, S.L. et al. 2018].

For the sake of simplicity, when necessary, we drop the explicit dependency of x on variable k inside the memberships functions.

2.9.1 Positively Invariant λ-Contractive Sets

For the system described by (2.60), a symmetric polyhedral set

$$\Omega = \mathcal{S}[G_s, \mathbf{1}] = \{x(k) : |G_s x(k)| \leq \mathbf{1}\}, \tag{2.94}$$

characterizes a set of state constraints, where $x(k) \in \mathbb{R}^n$, $G_s \in \mathbb{R}^{g_s \times n}$ and $\mathbf{1}$ is a vector of ones of appropriate size.

The one-step admissible set is now defined by:

$$\mathcal{L}(\Omega,\lambda) = \{x(k) \in \mathbb{R}^n : \exists u(k) \in \mathcal{U}_s : G_s A(\alpha)x(k) + G_s B(\alpha)u(k) \in \lambda\Omega, \forall \alpha(x(k)) \in \Delta\}, \tag{2.95}$$

where $u(k)$ is the control vector subject to linear constraints:

$$u(k) \in \mathcal{U}_s = \mathcal{S}[S_u^s, \mathbf{1}] = \{u(k) \in \mathbb{R}^m : |S_u^s u(k)| \leq \mathbf{1}\},\ S_u^s \in \mathbb{R}^{g_u^s \times m}, \forall k \in \mathbb{N}. \tag{2.96}$$

For a given time k, admissibility of the state vector at time $k+1$ is characterized by:

$$G_s(A(\alpha)x(k) + B(\alpha)u(k)) \leq \lambda \mathbf{1}. \tag{2.97}$$

Now, consider a StF closed-loop T-S model is given by

$$x(k+1) = \sum_{i=1}^{r} \alpha_i^2 (A_i + B_i F_i)x(k) + \sum_{i=1}^{r}\sum_{i<j}^{r} \alpha_i \alpha_j (A_i + B_i F_j + A_j + B_j F_i)x(k), \tag{2.98}$$

as presented in Section 2.7.1.

The possibility of imposing trajectories of $x_a(k)$ within a given polyhedral set $\mathcal{S}[G_s, \mathbf{1}] \subset \Omega$ w.r.t. closed-loop system (2.98) can be achieved from the Definition (2.3.1), where $u(k) \in \mathcal{U}_s$, $\forall k \in \mathbb{N}$.

In the following, we propose sufficient conditions for the invariance of symmetric polyhedral sets w.r.t. closed-loop system (2.98). These conditions draw inspiration from the ones presented for the linear case (see, e.g., [Brião, S.L. et al. 2021]).

Proposition 2.9.1 *A polyhedral set $R[G,\mathbf{1}]$ is positively invariant with contraction rate λ, $0 < \lambda < 1$, w.r.t. closed-loop system (2.98), if there are matrices $H_{ii} \in \mathbb{R}^{g_s \times g_s}$, $i = 1, \cdots, r$ and $H_{ij} \in \mathbb{R}^{g_s \times g_s}$, with $i = 1, \cdots, r$, $j = i+1, \cdots, r$, such that:*

$$\begin{array}{c} H_{ii} G_s = G_s(A_i + B_i F_i), \\ ||H_{ii}||_\infty \leq \lambda, \\ H_{ij} G_s = G_s \frac{(A_i + B_i F_j + A_j + B_j F_i)}{2}, \\ ||H_{ij}||_\infty \leq \lambda. \end{array} \tag{2.99}$$

Consider $|G_s x(k)| \leq \mathbf{1}$. Then:

$$|G_s x(k+1)| = \left| \sum_{i=1}^{r} \alpha_i^2 H_{ii} G_s x(k) + 2\sum_{i=1}^{r}\sum_{i<j}^{r} \alpha_i \alpha_j H_{ij} G_s x(k) \right| \leq \sum_{i=1}^{r} \alpha_i^2 |H_{ii} G_s x(k)| +$$
$$2\sum_{i=1}^{r}\sum_{i<j}^{r} \alpha_i \alpha_j |H_{ij} G_s x(k)| \leq \sum_{i=1}^{r} \alpha_i^2 |H_{ii}|\mathbf{1} + 2\sum_{i=1}^{r}\sum_{i<j}^{r} \alpha_i \alpha_j |H_{ij}|\mathbf{1} \leq \sum_{i=1}^{r} \alpha_i^2 \lambda \mathbf{1} +$$
$$2\sum_{i=1}^{r}\sum_{i<j}^{r} \alpha_i \alpha_j \lambda \mathbf{1} = \lambda (\sum_{i=1}^{r} \alpha_i)^2 \mathbf{1} = \lambda \mathbf{1}.$$

given that $||H_{ii}||_\infty \leq \lambda$ and $||H_{ij}||_\infty \leq \lambda$ are equivalent to $|H_{ii}|\mathbf{1} \leq \lambda \mathbf{1}$ and $|H_{ij}|\mathbf{1} \leq \lambda \mathbf{1}$, respectively.

This proves that the polyhedron $\mathcal{S}[G_s,\mathbf{1}]$ is PI λ-contractive.

The Proposition 2.9.1 is related to the existence of a symmetric PI λ-contractive polyhedron, assuring that if $x(0)\in\mathcal{S}[G_s,\mathbf{1}]$, then $x(k)\in\mathcal{S}[G_s,\mathbf{1}]$, for $k=1,2,\cdots$. Furthermore, with $0<\lambda<1$, it guarantees the contration of $\mathcal{S}[G_s,\mathbf{1}]$, i.e., if $x(k)\in\mathcal{S}[G_s,\mathbf{1}]$, then $x(k+1)\in\lambda\mathcal{S}[G_s,\mathbf{1}]$.

Similarly to the case of general compact polyhedral sets, if $\mathcal{S}[G_s,\mathbf{1}]$ is a PI λ-contractive polyhedron, the discrete-time system (2.65) is locally asymptotically stable and it admits the polyhedral norm $||G_sx(k)||_\infty$ as a Lyapunov function [Blanchini, F., & Miani, S. 2015, p. 261].

For the SOF approach, replace F_i by K_iC in (2.98) and in the expressions derived in Proposition 2.9.1, according to Remark 2.6.1.

2.9.2 Constrained Control Problem

In the following, we present the polyhedral inclusion conditions that, when satisfied, guarantee the fulfillment of state and control constraints. To this end, let us consider the symmetric constraints associated with the state and control variables, which can be mathematically expressed by polyhedral sets, as follows:

$$\mathcal{X}_s=\mathcal{S}[S_x^s,\mathbf{1}]=\{x(k)\in\mathbb{R}^n:|S_x^sx(k)|\leq\mathbf{1}\}, \tag{2.100}$$
$$\mathcal{U}_s=\mathcal{S}[S_u^s,\mathbf{1}]=\{u(k)\in\mathbb{R}^m:|S_u^su(k)|\leq\mathbf{1}\}, \tag{2.101}$$

such that $S_x^s\in\mathbb{R}^{g_x^s\times n}$, $S_u^s\in\mathbb{R}^{g_u^s\times m}$ and $\mathbf{1}$ is a vector of ones of appropriate size.

The inclusion of the polyhedral domain given by $R[G_s,\mathbf{1}]\subseteq R[S_x^s,\mathbf{1}]$ is guaranteed, if and only if there exists $M\in\mathbb{R}^{g_x^s\times g_s}$, such that

$$\begin{aligned} MG_s&=S_x^s,\\ ||M||_\infty&\leq\mathbf{1}, \end{aligned} \tag{2.102}$$

are satisfied.

The conditions presented in (2.102) guarantee that the symmetric constraints $\mathcal{S}[S_x^s,\mathbf{1}]$ are satisfied, since $\mathcal{S}[G_s,\mathbf{1}]$ is PI λ-contractive (see, e.g., [Brião, S.L. et al. 2021]).

The symmetric control constraints related to the system (2.98), can be expressed from (2.64) and (2.101), by:

$$|S_u^su(k)|=\sum_{i=1}^{r}\alpha_i(x(k))|S_u^sF_ix(k)|, \tag{2.103}$$

such that

$$\mathcal{S}[S_u^sF_i,\mathbf{1}]=\{x(k)\in\mathbb{R}^n:|S_u^sF_ix(k)|\leq\mathbf{1}\}. \tag{2.104}$$

As can be seen in (2.104), it is possible to directly characterize the control constraints in the state-space because $u(k)=\sum_{i=1}^{r}\alpha_i(x(k))F_ix(k)$. Thus, if $x(k)\in\mathcal{S}[S_u^sF_i,\mathbf{1}]$, $i=1,2,\cdots,r$, then $u(k)\in\mathcal{U}_s$. Moreover, the control constraints can be guaranteed through the polyhedral domain inclusions given by $\mathcal{S}[G_s,\mathbf{1}]\subseteq\mathcal{S}[S_u^sF_i,\mathbf{1}]$, since the polyhedral set $\mathcal{S}[G_s,\mathbf{1}]$ is PI λ-contractive.

In the following, we propose sufficient conditions that, when satisfied, guarantee that the control constraints are respected. These conditions draw inspiration from the ones previously presented for the linear case (see, e.g., [Brião, S.L. et al. 2021]).

Proposition 2.9.2 *A polyhedron defined by $\mathcal{S}[S_u^s F_i, \boldsymbol{1}]$ characterizes the control constraints defined in the state-space. The inclusion of polyhedral sets given by $\mathcal{S}[G_s, \boldsymbol{1}] \subseteq \mathcal{S}[S_u^s F_i, \boldsymbol{1}]$ is guaranteed, with $F_i \in \mathbb{R}^{m\times n}$, if there exist matrices $Q_i \in \mathbb{R}^{g_u^s \times g_s}$, such that*

$$\begin{aligned} Q_i G_s &= S_u^s F_i, \\ ||Q_i||_\infty &\leq 1. \end{aligned} \tag{2.105}$$

Consider $|G_s x(k)| \leq \mathbf{1}$. Then, from (2.68) and (2.70):

$$|S_u^s u(k)| = \left|\sum_{i=1}^{r} \alpha_i(x(k)) S_u^s F_i x(k)\right| = \sum_{i=1}^{r} \alpha_i(x(k))|Q_i G_s x(k)| \leq \sum_{i=1}^{r} \alpha_i(x(k))|Q_i|\mathbf{1} \leq \sum_{i=1}^{r} \alpha_i(x(k))\mathbf{1} = \mathbf{1}. \tag{2.106}$$

given that $||Q_i||_\infty \leq \lambda$ is equivalent to $|Q_i|\mathbf{1} \leq \lambda\mathbf{1}$.

This proves the inclusion of the polyhedral domain given by $\mathcal{S}[G_s, \mathbf{1}] \subseteq \mathcal{S}[S_u^s F_i, \mathbf{1}]$.

If the conditions presented in (2.102) and (2.105) are satisfied, a PI polyhedron $\mathcal{S}[G_s, \mathbf{1}]$ is simultaneously contained in $\mathcal{S}[S_x^s, \mathbf{1}]$ and $\mathcal{S}[S_u^s F_i, \mathbf{1}]$. The intersections of the set $\mathcal{S}[S_x^s, \mathbf{1}]$ with $\mathcal{S}[S_u^s F_i, \mathbf{1}]$ form a compact polytope that includes the origin. It indicates that the polyhedral domain $\mathcal{S}[G_s, \mathbf{1}]$ is contained into the intersections of the polyhedral sets $\mathcal{S}[S_x^s, \mathbf{1}]$ and $\mathcal{S}[S_u^s F_i, \mathbf{1}]$. Consequently, the satisfaction of state and control constraints is guaranteed as long as the initial state $x(0)$ belongs to the polyhedral set $\mathcal{S}[G_s, \mathbf{1}]$.

For the SOF approach, replace F_i by $K_i C$ in (2.103) and in the expressions derived in Proposition 2.9.2, according to Remark 2.6.1.

2.9.3 Positively Invariant Sets for Admissible References

For the system described by (2.75), a symmetric polyhedral set

$$\Omega = \mathcal{S}[G_s, \mathbf{1}] = \{x_a(k) : |G_s x_a(k)| \leq \mathbf{1}\}, \tag{2.107}$$

characterizes a set of augmented state constraints, where $x_a(k) \in \mathbb{R}^{n_a}$, $G_s \in \mathbb{R}^{g_s \times n_a}$ and $\mathbf{1}$ is a vector of ones of appropriate size.

A StF closed-loop T-S model is given by

$$\begin{aligned} x_a(k+1) = &\sum_{i=1}^{r} \alpha_i^2(x)(A_{ii}^a x_a(k) + B_{ii}^a r(k)) + \\ &2\sum_{i=1}^{r}\sum_{i<j}^{r} \alpha_i(x)\alpha_j(x)\left(\left(\frac{A_{ij}^a + A_{ji}^a}{2}\right)x_a(k) + \left(\frac{B_{ij}^a + B_{ji}^a}{2}\right)r(k)\right). \end{aligned} \tag{2.108}$$

as presented in Section 2.108.

The one-step admissible set is now defined by:

$$\mathcal{L}(\Omega,\mathfrak{R}_s,\lambda) = \{x_a(k) \in \mathbb{R}^{n_a} : \exists u(k) \in \mathcal{U}_s : G_sA(\alpha)x_a(k) + G_s\mathcal{B}(\alpha)r(k) \in \lambda\Omega, \forall r(k) \in \mathfrak{R}_s, \forall\alpha(x(k)) \in \Delta\}, \tag{2.109}$$

such that

$$\mathfrak{R}_s = R[S_r^s,\mathbf{1}] = \{r(k) \in \mathrm{R}^p : S_r^s r(k) \leq \mathbf{1}\},\ S_r^s \in \mathbb{R}^{g_r^s \times p}, \tag{2.110}$$

is the symmetric polyhedral set of admissible references and $u(k)$ is the control vector subject to linear constraints in (2.96).

For a given time k, admissibility of the state vector at time $k+1$ is characterized by:

$$G_s(A(\alpha)x_a(k) + \mathcal{B}(\alpha)r(k)) \leq \lambda\mathbf{1}. \tag{2.111}$$

The possibility of imposing trajectories of $x_a(k)$ within a given polyhedral set $\mathcal{S}[G_s,\mathbf{1}] \subset \Omega$ w.r.t. closed-loop system (2.108) can be achieved from the Definition (2.4.1), where $u(k) \in \mathcal{U}_s$, $\forall k \in \mathbb{N}$. In this regard, consider the following theorem:

Theorem 2.9.1 *A polyhedral set $\mathcal{S}[G_s,\mathbf{1}]$ is positively invariant with contraction rate λ, $0 < \lambda < 1$, w.r.t. closed-loop system* (2.108), *if there are matrices $H_{ii} \in \mathbb{R}^{g_s \times g_s}$, $Z_{ii} \in \mathbb{R}^{g_s \times g_r^s}$, $i = 1,\cdots,r$ and $H_{ij} \in \mathbb{R}^{g_s \times g_r^s}$, $Z_{ij} \in \mathbb{R}^{g_s \times g_r^s}$, with $i = 1,\cdots,r,\ j = i+1,\cdots,r$, such that:*

$$\begin{gathered} H_{ii}G_s = G_sA_{ii}^a, \\ Z_{ii}S_r^s = G_sB_{ii}^a, \\ |H_{ii}|_\infty + |Z_{ii}|_\infty \leq \lambda, \\ H_{ij}G_s = G_s\left(\frac{A_{ij}^a + A_{ji}^a}{2}\right), \\ Z_{ij}S_r^s = G_s\left(\frac{B_{ij}^a + B_{ji}^a}{2}\right), \\ |H_{ij}|_\infty + |Z_{ij}|_\infty \leq \lambda. \end{gathered} \tag{2.112}$$

Proof 2.9.1 *Consider $|G_sx_a(k)| \leq \mathbf{1}$ and $|S_r^s r(k)| \leq \mathbf{1}$. Then:*

$$\begin{aligned} |G_sx_a(k+1)| &= \left|\sum_{i=1}^r \alpha_i^2(H_{ii}Gx_a(k) + Z_{ii}S_rr(k)) + 2\sum_{i=1}^r\sum_{i<j}^r \alpha_i\alpha_j(H_{ij}Gx_a(k) + Z_{ij}S_rr(k))\right| \\ &\leq \sum_{i=1}^r \alpha_i^2|(H_{ii}Gx_a(k) + Z_{ii}S_rr(k))| + 2\sum_{i=1}^r\sum_{i<j}^r \alpha_i\alpha_j|(H_{ij}Gx_a(k) + Z_{ij}S_rr(k))| \\ &\leq \sum_{i=1}^r \alpha_i^2(|H_{ii}|\mathbf{1} + |Z_{ii}|\mathbf{1}) + 2\sum_{i=1}^r\sum_{i<j}^r \alpha_i\alpha_j(|H_{ij}|\mathbf{1} + |Z_{ij}|\mathbf{1}) \\ &\leq \sum_{i=1}^r \alpha_i^2\lambda\mathbf{1} + 2\sum_{i=1}^r\sum_{i<j}^r \alpha_i\alpha_j\lambda\mathbf{1} = \lambda(\sum_{i=1}^r \alpha_i)^2\mathbf{1} = \lambda\mathbf{1}. \end{aligned}$$

given that $||H_{ii}||_\infty + ||Z_{ii}||_\infty \leq \lambda$ and $||H_{ij}||_\infty + ||Z_{ij}||_\infty \leq \lambda$ are equivalent to $|H_{ii}|\mathbf{1} + |Z_{ii}|\mathbf{1} \leq \lambda\mathbf{1}$ and $|H_{ij}|\mathbf{1} + |Z_{ij}|\mathbf{1} \leq \lambda\mathbf{1}$, respectively.

This proves that the symmetric polyhedron $\mathcal{S}[G_s, \mathbf{1}]$ is PI λ-contractive. $\square$

2.9.4 Constrained Control Problem

Now consider the following sets of constraints for the reference signal and the tracking error integral:

$$\mathcal{V}_s = R[S_v^s, \mathbf{1}] = \{v(k) \in \mathbb{R}^p : |S_v^s v(k)| \leq \mathbf{1}\},\ S_v \in \mathbb{R}^{g_v^s \times p}, \tag{2.113}$$

$$\mathfrak{R}_s = R[S_r^s, \mathbf{1}] = \{r(k) \in \mathbb{R}^p : |S_r^s r(k)| \leq \mathbf{1}\},\ S_r^s \in \mathbb{R}^{g_r^s \times p}. \tag{2.114}$$

and the sets $\mathcal{X}_s$ and $\mathcal{U}_s$, defined in Section 2.9.2. The set $\mathfrak{R}_s$ is used to establish the admissible values for the signal $r(k)$.

To deal with the state and tracking error integral constraints in the augmented state, from (2.100) and (2.113), we define the following set

$$\mathcal{S}[S_a^s, \mathbf{1}] = \{x_a(k) \in \mathbb{R}^{n_a} : |S_a^s x_a(k)| \leq \mathbf{1}\}, \tag{2.115}$$

where $S_a^s = \begin{bmatrix} S_x^s & 0 \\ 0 & S_v^s \end{bmatrix}$, $S_a^s \in \mathbb{R}^{g_a^s \times n_a}$ and $g_a^s = g_x^s + g_v^s$.

The inclusion of the polyhedral domain given by $\mathcal{S}[G_s, \mathbf{1}] \subseteq \mathcal{S}[S_a^s, \mathbf{1}]$ is guaranteed, if and only if there exists $M \in \mathbb{R}^{g_a^s \times g_s}$, such that

$$\begin{aligned} MG_s &= S_a^s, \\ ||M||_\infty &\leq \mathbf{1}, \end{aligned} \tag{2.116}$$

are satisfied.

The conditions presented in (2.116) guarantee that the symmetric constraints $\mathcal{S}[S_a^s, \mathbf{1}]$ are satisfied, since $\mathcal{S}[G_s, \mathbf{1}]$ is PI λ-contractive (see, e.g., [Brião, S.L. et al. 2021]).

The linear control constraints related to the system (2.79), can be expressed from (2.77) and (2.101), by:

$$|S_u^s u(k)| = \sum_{i=1}^{r} \alpha_i(x(k))(|S_u^s K_{a,i} x_a(k) + S_u^s K_{R_i} r(k)|) \tag{2.117}$$

such that

$$\mathcal{S}[S_u^s K_{a,i}, \mathbf{1}] = \{x_a(k) \in \mathbb{R}^{n_a} : |S_u^s K_{a,i} x_a(k)| \leq \mathbf{1}\}, \tag{2.118}$$

$$\mathcal{S}[S_u^s K_{R_i}, \mathbf{1}] = \{r(k) \in \mathbb{R}^p : |S_u^s K_{R_i} r(k)| \leq \mathbf{1}\}. \tag{2.119}$$

with $K_{a,i} = [F_i\ K_{I_i}]$, $K_{a,i} \in \mathbb{R}^{m \times n_a}$ and $K_{R_i} \in \mathbb{R}^{m \times p}$.

In the following, we propose sufficient conditions that, when satisfied, guarantee that the control constraints are respected. These conditions draw inspiration from the ones previously presented for the linear case (see, e.g., [Brião, S.L. et al. 2021] and [dos Santos, G.F. et al. 2023]).

Theorem 2.9.2 *A polyhedron defined by $\mathcal{S}[S_u^s K_{a,i}, \mathbf{1}]$ characterize the control constraints* *defined in the state-space. The inclusion of polyhedral sets given by* $\mathcal{S}[G_s, \mathbf{1}] \subseteq \mathcal{S}[S_u^s K_{a,i}, \mathbf{1}]$ *is guaranteed, with* $K_{a,i} = [F_i \; K_{I_i}]$, $K_{a,i} \in \mathbb{R}^{m \times n_a}$ *and* $K_{R_i} \in \mathbb{R}^{m \times p}$, *if there exist matrices* $Q_i \in \mathbb{R}^{g_u^s \times g_s}$ *and* $Q_{r,i} \in \mathbb{R}^{g_u^s \times g_r^s}$, *such that*

$$\begin{aligned} Q_i G_s &= S_u^s K_{a,i}, \\ Q_{r,i} S_r^s &= S_u^s K_{R_i}, \\ \|Q_i\|_\infty + \|Q_{r,i}\|_\infty &\leq 1. \end{aligned} \tag{2.120}$$

Proof 2.9.2 *Consider* $|G_s x_a(k)| \leq \mathbf{1}$ *and* $|S_r^s r(k)| \leq \mathbf{1}$. *Then, from* (2.117) *and* (2.120)*:*

$$|S_u^s u(k)| = \sum_{i=1}^{r} \alpha_i(x(k))(|S_u^s K_{a,i} x_a(k) + S_u^s K_{R_i} r(k)|) = \sum_{i=1}^{r} \alpha_i(x(k))(|Q_i G_s x_a(k) + Q_{r,i} S_r^s r(k)|) \leq \sum_{i=1}^{r} \alpha_i(x(k))(|Q_i|\mathbf{1} + |Q_{r,i}|\mathbf{1}) \leq \sum_{i=1}^{r} \alpha_i(x(k))\mathbf{1} = \mathbf{1}. \tag{2.121}$$

given that $\|Q_i\|_\infty + \|Q_{r,i}\|_\infty \leq 1$ *is equivalent to* $|Q_i|\mathbf{1} + |Q_{r,i}|\mathbf{1} \leq \mathbf{1}$.

This proves the inclusion of the polyhedral domain given by $K_{a,i}\mathcal{S}[G_s, \mathbf{1}] \oplus K_{R_i}\mathfrak{R}_s \subseteq \mathcal{U}_s$.

Then, the polyhedral set $\mathcal{S}[G_s, \mathbf{1}]$ is PI λ-contractive and such that $\mathcal{S}[G_s, \mathbf{1}] \subseteq \mathcal{S}[S_a^s, \mathbf{1}]$ and $K_{a,i}\mathcal{S}[G_s, \mathbf{1}] \oplus K_{R_i}\mathfrak{R}_s \subseteq \mathcal{U}_s$, where denotes $\oplus$ the Minkowski set sum operator.

For the output feedback approach, consider $K_{a,i} = [K_{P_i}C \; K_{I_i}]$ in (2.118) and in the expressions derived in Theorem 2.9.2, according to Remark 2.8.1.

2.10 The Fuzzy T-S Observer

In practice, the state of a system is often not readily available. Thus, the question arises whether it is possible to determine the state of a given system by measuring its output. For linear systems, a linear observer [Kalman, R.E. 1960] gives an affirmative answer if the system is observable. Similarly, a method of systematically designing fuzzy controllers and observers fills an essential gap concerning fuzzy T-S systems, as stated in [Tanaka, K., & Wang, H.O. 1997, Tanaka, K. et al. 1998, Wang, H.O., & Tanaka, K. 2004].

The fuzzy T-S observer is designed to satisfy the following condition:

$$e(k) \to 0 \text{ as } k \to \infty, \tag{2.122}$$

with

$$e(k) = x(k) - \hat{x}(k), \tag{2.123}$$

such that $x(k) \in \mathbb{R}^n$, $e(k) \in \mathbb{R}^n$, and $\hat{x}(k) \in \mathbb{R}^n$ denote, respectively, the state, the observation error, and the estimated state vectors. This condition ensures that the steady-state error between $x(k)$ and $\hat{x}(k)$ converges to 0.

The fuzzy observer shares the same fuzzy sets in the antecedent part with the T-S model, and its consequent is related to the respective local linear observers. Then, a procedure analogous to the one employed in the design of the PDC controller is used for the fuzzy T-S observer and the condition 2.122 can only be guaranteed if the trajectory of states is included within the validity region of the T-S model.

The i-th fuzzy T-S observer rule is defined as follows:

Observer rule i**:**

$$\textbf{\textit{IF}}\ z_1(k)\ is\ \mathcal{M}_{i1}\ \text{and} \cdots \text{and}\ z_p(k)\ is\ \mathcal{M}_{ip},$$

$$\textbf{\textit{THEN}} \begin{cases} \hat{x}(k+1) = A_i\hat{x}(k) + B_iu(k) + L_i(y(k) - \hat{y}(k)) \\ \hat{y}(k) = C\hat{x}(k),\ i = 1, 2, \ldots, r. \end{cases} \tag{2.124}$$

The dependence of the premise variables on the state variables makes it necessary to consider two cases for fuzzy observer design:

2.10.1 Case A

In this case, it is considered that $\alpha_i(x(k))$ is, by hypothesis, given by $\alpha_i(y(k))$. The fuzzy observer T-S for the case **A** is represented as follows:

$$\begin{aligned} \hat{x}(k+1) &= \sum_{i=1}^{r} \alpha_i(y(k))[A_i\hat{x}(k) + B_iu(k) + L_i(y(k) - \hat{y}(k))], \\ \hat{y}(k) &= C\hat{x}(k), \end{aligned} \tag{2.125}$$

where $\hat{x}(k) \in \mathbb{R}^n$, $\hat{y}(k) \in \mathbb{R}^p$ is the measured output vector, $L_i \in \mathbb{R}^{n\times p}$ and $\alpha_i(\cdot)$ are the same membership functions that define the T-S model.

2.10.2 Case B

For the general case, as the membership functions depend on non-accessible states, it is necessary to use an estimation mechanism so that these variables can be calculated. The fuzzy T-S observer for the case **B** is represented as follows:

$$\begin{aligned} \hat{x}(k+1) &= \sum_{i=1}^{r} \alpha_i(\hat{x}(k))[A_i\hat{x}(k) + B_iu(k) + L_i(y(k) - \hat{y}(k))], \\ \hat{y}(k) &= C\hat{x}(k), \end{aligned} \tag{2.126}$$

where $\alpha_i(\hat{x}(k))$ are membership functions applied to the vector of estimated premise variables $\hat{x}(k)$, which must belong to the simplex:

$$\Delta = \{\alpha(\hat{x}(k)) \in \mathbb{R}^r : \sum_{i=1}^{r} \alpha_i(\hat{x}(k)) = 1,\ \alpha_i(\hat{x}(k)) \geq 0\}. \tag{2.127}$$

In this Thesis, an approach based on set-invariance theory is proposed for the augmented system containing both the controller and the observer. Conditions of positive invariance of convex polyhedra in the augmented state-space are proposed, similar to those presented in this chapter, considering the cases **A** and **B**.

In the case **A**, although the membership functions depend only on the output, the estimated state feedback results, in general, in controllers with better performance and with larger sets of admissible states associated with them than output static feedback control. For the general case (case **B**), as membership functions depend on non-accessible states, an estimation mechanism is needed to calculate these variables. In both cases, this role is played by the fuzzy T-S observer.

2.11 Conclusions

In this chapter, the properties of positive invariance of polyhedral sets and their relationship with constrained control problems for discrete-time linear systems are presented and characterized geometrically (by an inclusion condition) and algebraically (by matrix relations). These concepts are generalized for discrete-time fuzzy T-S systems.

Conditions under which a polyhedron defined in the state-space (and in the augmented state-space (state + tracking error integral) is PI λ-contractive are presented. Once these conditions are satisfied, the state trajectory converges in one-step to a subset of the PI polyhedron. Furthermore, the invariance relations make it possible to characterize the inclusion of the PI set in a given polyhedron defined by the state constraints. If the state constraints are symmetric, it is natural that the PI polyhedron is as well. Otherwise, a generically shaped convex polyhedron is a better candidate for a PI set.

A brief introduction to the Parallel Distributed Compensation was presented. PDC controller is designed for a locally-defined model and the control objectives can only be guaranteed if the trajectory of states is included within the validity region of the T-S model. Conditions under which a PI polyhedron is contained in this region were presented.

Finally, two types of fuzzy T-S observers were described. In the first type, the premise variables depend only on the system output, while in the second, the premise variables depend on non-accessible states. The fuzzy observer is designed analogously to the PDC controller and is also only valid in a local context. The forthcoming chapter will provide a comprehensive examination of the fuzzy T-S observers discussed previously, explicitly emphasizing the design of controllers that utilize estimated state feedback.

Chapter 3

Output Feedback Positive Invariance

The problem of maintaining the state trajectory of a discrete-time fuzzy T-S system within a given polyhedral set is addressed using set invariance theory based on the positive invariance property of convex polyhedra. When considering a polytopic set that contains the origin as a interior point, it is also possible to incorporate the contraction effect to the set. This effect associated with the PI set enables the solution of the constrained regulator problem. This problem consists of designing a controller capable of guiding the state trajectory to the origin without violating the limits established by a given PI λ-contractive set. This property is geometrically and analytically characterized for the linear case in [Blanchini, F. 1994, Dórea, C.E.T., & Hennet, J.C. 1999].

In [Dórea, C.E. et al. 2020], an adaptation of these conditions to discrete-time fuzzy T-S systems is presented. In this approach, the membership functions that constitute the model only depend on the system output (case **A**). However, membership functions can show dependence on states that cannot be directly measured (case **B**). For the general case, as the membership functions depend on non-accessible states, an estimation mechanism is needed to calculate these variables. In case **A**, although the membership functions depend only on the output, the estimated state feedback results in better-performing controllers with larger sets of admissible states associated with them than static output feedback.

Motivated by these considerations, an observer-based output feedback approach is proposed for fuzzy T-S systems under constraints from the invariance theory. Next, the conditions that characterize the existence of general and symmetric PI polyhedra will be presented. These conditions are chosen based on the constraints that define the validity region in the augmented state-space. When these constraints are symmetric, it is natural that the polyhedron PI is also symmetric. Otherwise, the invariance is characterized considering the conditions associated with general polyhedra.

Here, the control design methodology involves formulating the constrained regulator problem as an optimization problem subject to constraints. The conditions that guarantee local asymptotic stability and the augmented state (state + estimation error) and control constraints are incorporated as constraints of the optimization problem.

The solution to the optimization problem is obtained offline to the control implementation and provides the gains for the controller and the observer, as well as the associated PI λ-contractive polyhedron. Furthermore, state and error dynamics are represented from a model described in the augmented state-space. Therefore, the constraints of the optimization problem are formulated according to this representation.

3.1 Preliminaries

Consider a discrete-time nonlinear system, given by:

$$\begin{aligned} x(k+1) &= f(x(k),u(k)), \\ y(k) &= Cx(k), \end{aligned} \tag{3.1}$$

where $f(\cdot): \mathbb{R}^{n+m} \to \mathbb{R}^n$, $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^m$ is the control input and $y(k) \in \mathbb{R}^p$ the measured output and $C \in \mathbb{R}^{p \times n}$ is the output matrix.

In a compact region of state-space, the system (3.1) can be expressed locally as a fuzzy (T-S) Takagi-Sugeno system with r rules in the form (see, e.g., [Wang, H.O., & Tanaka, K. 2004, Wang, H.O. et al. 1996]):

$$\begin{aligned} x(k+1) &= \sum_{i=1}^{r} \alpha_i(x(k))(A_i x(k) + B_i u(k)), \\ y(k) &= Cx(k), \end{aligned} \tag{3.2}$$

where $\alpha_i(x(k))$ represent the membership functions such that the vector $\alpha(x(k))$ belongs to the standard simplex $\Delta \in \mathbb{R}^r$, defined as:

$$\Delta = \{\alpha(x(k)) \in \mathbb{R}^r : \sum_{i=1}^{r} \alpha_i(x(k)) = 1,\ \alpha_i(x(k)) \geq 0\}. \tag{3.3}$$

The system (3.2) is subject to state and control constraints, represented, respectively, by the closed polyhedral sets containing the origin:

$$\mathcal{X} = R[S_x, \mathbf{1}] = \{x(k) \in \mathbb{R}^n : S_x x(k) \leq \mathbf{1}\}, \tag{3.4}$$

$$\mathfrak{U} = R[S_u, \mathbf{1}] = \{u(k) \in \mathbb{R}^m : S_u u(k) \leq \mathbf{1}\}, \tag{3.5}$$

where $S_x \in \mathbb{R}^{g_x \times n}$, $S_u \in \mathbb{R}^{g_u \times m}$, and $\mathbf{1}$ is a vector of dimensions suitable whose elements are all equal to 1.

The approach outlined in this chapter incorporates the utilization of the state observer in the estimation process, as specified by Equations (2.125) and (2.126). Consequently, it becomes required to establish a set of constraints for the estimation error, denoted as $e(k) \in \mathbb{R}^n$:

$$\mathcal{E} = R[S_e, \mathbf{1}] = \{e(k) \in \mathbb{R}^n : S_e e(k) \leq \mathbf{1}\},\ S_e \in \mathbb{R}^{g_e \times n}. \tag{3.6}$$

In principle, the constraints defined by the set $\mathcal{E}$ can be specified as desired, i.e., the observation error can be contained in a much larger region than that specified for the states, as long as the state constraints and control are satisfied. Although the specification of the set $\mathcal{E}$ is flexible, it is indispensable since the approach to be discussed in this chapter is based on the existence of compact polyhedral invariant sets, for which the origin is an interior point.

The augmented state, in turn, can be defined by $x_a(k)^{\mathrm{T}} = [x(k)\ e(k)]$, $x_a(k) \in \mathbb{R}^{2n}$, where $x(k)$ and $e(k)$, respectively, are the state and observation error vectors. Thus, it is possible to write a set of constraints in the augmented state-space:

$$R[S,\mathbf{1}] = \{x_a(k) \in \mathbb{R}^{2n} : Sx_a(k) \leq \mathbf{1}\},\ S = \begin{bmatrix} S_x & 0 \\ 0 & S_e \end{bmatrix}, \tag{3.7}$$

where $S \in \mathbb{R}^{g_a \times 2n}$, with $g_a = g_x + g_e$.

Equations (3.5) and (3.7) characterize generic shaped polyhedrons associated with control constraints and augmented state-space, respectively.

When we consider the symmetric form, the set of constraints in the augmented state-space can be defined by:

$$\mathcal{S}[S_s,\mathbf{1}] = \{x_a(k) \in \mathbb{R}^{2n} : |S_s x_a(k)| \leq \mathbf{1}\},\ S_s = \begin{bmatrix} S_x^s & 0 \\ 0 & S_e^s \end{bmatrix}, \tag{3.8}$$

with $S_s \in \mathrm{R}^{g_a^s \times 2n}$, $g_a^s = g_x^s + g_e^s$, such that

$$\mathcal{X}_s = \mathcal{S}[S_x^s,\mathbf{1}] = \{x(k) \in \mathbb{R}^n : |S_x^s x(k)| \leq \mathbf{1}\},\ S_x^s \in \mathbb{R}^{g_x^s \times n}, \tag{3.9}$$

$$\mathfrak{U}_s = \mathcal{S}[S_u^s,\mathbf{1}] = \{u(k) \in \mathbb{R}^m : |S_u^s u(k)| \leq \mathbf{1}\},\ S_u^s \in \mathbb{R}^{g_u^s \times p}, \tag{3.10}$$

$$\mathcal{E}_s = \mathcal{S}[S_e^s,\mathbf{1}] = \{e(k) \in \mathbb{R}^n : |S_e^s e(k)| \leq \mathbf{1}\},\ S_e^s \in \mathbb{R}^{g_e^s \times n}, \tag{3.11}$$

describe polyhedra with symmetric shapes in the spaces associated with the variables $x(k)$, $u(k)$, and $e(k)$, respectively.

As previously discussed, in the fuzzy T-S model, the membership functions can depend solely on the output variables or on states that are not directly measurable. In this regard, we consider the two situations: in the first, the membership functions depend only on the output (case **A**), and in the second, they rely on states that cannot be directly measured (case **B**) [Tanaka, K. et al. 1997, Wang, H.O., & Tanaka, K. 2004]. Initially, the invariance conditions for generic polyhedra will be presented, and then the conditions associated with symmetric polyhedral sets. The concepts discussed in this chapter have been used as a basis for the publications of the articles [Isidório, I.D. et al. 2022, Isidório, I.D. et al. 2023].

3.2 Positive Invariance of General Polyhedra

3.2.1 Case A

In this case, we consider that $\alpha_i(x(k))$ is, by hypothesis, given by $\alpha_i(y(k))$. A fuzzy T-S observer can be expressed with r rules in the form:

$$\begin{aligned} \hat{x}(k+1) &= \sum_{i=1}^{r} \alpha_i(y(k))[A_i\hat{x}(k) + B_i u(k) + L_i(y(k) - \hat{y}(k))], \\ \hat{y}(k) &= C\hat{x}(k), \end{aligned} \tag{3.12}$$

where $\hat{x}(k) \in \mathbb{R}^n$ is the estimated state vector, $\hat{y}(k) \in \mathbb{R}^p$ is the measured output, $L_i \in \mathbb{R}^{n \times p}$ and $\alpha_i(\cdot)$ are the same membership functions that define the model (3.2).

Note that, in this case, we consider that the membership functions depend only on the measured output.

The estimation error is defined by $e(k) = x(k) - \hat{x}(k)$ and its dynamics is given by:

$$e(k+1) = \sum_{i=1}^{r} \alpha_i(y(k))(A_i - L_i C)e(k).$$

As in [Wang, H.O., & Tanaka, K. 2004], we define the augmented state $x_a(k)^{\mathrm{T}} = [x(k)\ e(k)]$, $x_a(k) \in \mathbb{R}^{2n}$.

Let us now consider the following estimated state feedback control law, according to the scheme PDC, given by:

$$u(k) = -\sum_{i=1}^{r} \alpha_i(y(k))K_i\hat{x}(k), \tag{3.13}$$

with $K_i \in \mathbb{R}^{m\times n}$, such that the corresponding augmented closed-loop system is given by:

$$\begin{aligned} x_a(k+1) = &\sum_{i=1}^{r} \alpha_i^2(y(k))G_{ii}^a x_a(k) + \\ &2\sum_{i=1}^{r}\sum_{i<j}^{r} \alpha_i(y(k))\alpha_j(y(k))\frac{G_{ij}^a + G_{ji}^a}{2} x_a(k), \end{aligned} \tag{3.14}$$

such that

$$G_{ij}^a = \begin{bmatrix} A_i - B_iK_j & B_iK_j \\ 0 & A_i - L_iC \end{bmatrix}, \tag{3.15}$$

$x_a(k)^{\mathrm{T}} = [x(k)\ e(k)]$, with $x_a(k) \in \mathbb{R}^{2n}$ and $e(k) = x(k) - \hat{x}(k)$.

Linear control constraints related to (3.14)-(3.15) can be expressed from (3.5) and (3.7) by:

$$\begin{aligned} S_u u(k) &= -S_u \left[\sum_{i=1}^{r} \alpha_i(y(k))K_i x(k) - \sum_{i=1}^{r} \alpha_i(y(k))K_i e(k) \right] \\ &= \sum_{i=1}^{r} \alpha_i(y(k))S_uK_i[-\mathbf{I}\ \mathbf{I}]x_a(k) \le \mathbf{1},\ \text{with } S_u \in \mathbb{R}^{g_u\times m}. \end{aligned} \tag{3.16}$$

We then define the extension to the augmented state-space of the constraints on the control input:

$$R[S_uK_i^a, \mathbf{1}] = \{x_a(k) \in \mathbb{R}^{2n} : S_uK_i^a x_a(k) \le \mathbf{1}\} \tag{3.17}$$

with $K_i^a = K_i[-\mathbf{I}\ \mathbf{I}]$, $K_i^a \in \mathbb{R}^{m\times 2n}$.

Note that, due to the fact that $\alpha_i(y(k)) \in \Delta$ (3.3), we have that if $x_a(k) \in R[S_uK_i^a, \mathbf{1}]$, $i = 1, \cdots, r$, then $Su(k) \le \mathbf{1}$, thus ensuring respect for control constraints.

The intersection of the sets Ω, $R[S, \mathbf{1}]$ and $R[S_uK_i^a, \mathbf{1}]$ forms a compact polyhedron containing the origin, with Ω denoting the region of validity of the fuzzy T-S model in the augmented space.

Our objective then becomes to obtain matrices K_i, L_i, $i = 1, \cdots, r$ and a polyhedral set

$$R[Y, \mathbf{1}] = \{x_a(k) \in \mathbb{R}^{2n} : Yx_a(k) \le \mathbf{1}\}, \tag{3.18}$$

contained in the set of constraints $\Omega \cap R[S,\mathbf{1}] \cap R[S_u K_i^a,\mathbf{1}]$, which is PI λ-contractive under the control law (3.7), such that any trajectory starting at $R[Y,\mathbf{1}]$ asymptotically converges to the origin without violating the constraints.

The possibility of imposing trajectories of $x_a(k)$ within a given polyhedral set $R[Y,\mathbf{1}] \subset \Omega$ w.r.t. closed-loop system (3.14)-(3.15) can be achieved from the Definition (2.3.1).

Next, we present sufficient conditions to guarantee that a polyhedral set $R[Y,\mathbf{1}]$ is PI λ-contractive w.r.t. fuzzy T-S system under a PDC control law. To simplify the notation, we drop the explicit dependence of x_a on the variable k in the membership functions.

Theorem 3.2.1 *The polyhedron $R[Y,\mathbf{1}]$ is PI λ-contractive w.r.t. closed-loop system* (3.14)-(3.15)*, if there are matrices $H_{ii} \in \mathbb{R}^{g\times g}$, $i = 1,2,\cdots,r$ and $H_{ij} \in \mathbb{R}^{g\times g}$, $i = 1,2,\cdots,r$ and $j = i+1,\cdots,r$, such that:*

$$\begin{array}{c} H_{ii}Y = YG_{ii}^a,\; H_{ii} \geq 0, \\ H_{ii}\mathbf{1} \leq \lambda \mathbf{1}, \\ H_{ij}Y = Y\left(\frac{G_{ij}^a+G_{ji}^a}{2}\right),\; H_{ij} \geq 0, \\ H_{ij}\mathbf{1} \leq \lambda\mathbf{1}. \end{array} \tag{3.19}$$

Proof 3.2.1 *Consider $Yx_a(k) \leq \mathbf{1}$. Then, from* (3.14)*:*

$$\begin{aligned} Yx_a(k+1) &= \left(\sum_{i=1}^{r}\alpha_i^2(y)YG_{ii}^a + 2\sum_{i=1}^{r}\sum_{i<j}^{r}\alpha_i(y)\alpha_j(y)Y\left(\frac{G_{ij}^a+G_{ji}^a}{2}\right)\right)x_a(k) \\ &= \left(\sum_{i=1}^{r}\alpha_i^2(y)H_{ii}Y + 2\sum_{i=1}^{r}\sum_{i<j}^{r}\alpha_i(y)\alpha_j(y)H_{ij}Y\right)x_a(k) \\ &\leq \left(\sum_{i=1}^{r}\alpha_i^2(y)H_{ii}\mathbf{1} + 2\sum_{i=1}^{r}\sum_{i<j}^{r}\alpha_i(y)\alpha_j(y)H_{ij}\mathbf{1}\right) \\ &\leq \left(\sum_{j=1}^{r}\alpha_i^2(y)\lambda\mathbf{1} + 2\sum_{i<j}^{r}\alpha_i(y)\alpha_j(y)\lambda\mathbf{1}\right) = \lambda\left(\sum_{j=1}^{r}\alpha_i(y)\right)^2\mathbf{1} = \lambda\mathbf{1} < \mathbf{1}. \end{aligned}$$

This proves that the polyhedron $R[Y,\mathbf{1}]$ is PI and λ-contractive. □

The matrix $Y \in \mathbb{R}^{g\times 2n}$ must have full column rank to prevent the solution from having a degenerate polyhedron. It is equivalent to the existence of the pseudo-inverse matrix $P \in \mathbb{R}^{2n\times g}$. Then, the equality:

$$PY = \mathbf{I} \tag{3.20}$$

must be verified, with $\mathbf{I} \in \mathbb{R}^{2n\times 2n}$.

The Theorem 3.2.1 is related to the existence of a PI polyhedron, guaranteeing that if $x_a(0) \in R[Y,\mathbf{1}]$, then $x_a(k) \in R[Y,\mathbf{1}]$, for $k = 1,2,\cdots$. Furthermore, $0 < \lambda < 1$ guarantees the contraction of trajectories in the augmented state-space within $R[Y,\mathbf{1}]$, that is, if $x_a(k) \in R[Y,\mathbf{1}]$, then $x_a(k+1) \in \lambda R[Y,\mathbf{1}]$. Considering the effect of contraction and the conditions in (3.19), it is possible to verify that for $x_a(k) \in R_\lambda[Y,\mathbf{1}]$, with $R_\lambda[Y,\mathbf{1}] \equiv \lambda R[Y,\mathbf{1}]$, $x_a(k+1) \in \lambda R_\lambda[Y,\mathbf{1}], \forall k \geq 0$. If $x_a(0) \in R[Y,\mathbf{1}]$, then $x_a(k) \in \lambda^k R[Y,\mathbf{1}]$. Thus, if the conditions presented in Theorem 3.2.1 are satisfied, then $x_a(k) \to 0$ when $k \to \infty$.

Given that $R[Y,\mathbf{1}]$ is a PI λ-contractive polyhedron, the discrete-time system (3.14)-(3.15) is locally asymptotically stable and it admits the polyhedral norm $||Yx_a(k)||_\infty$ as a Lyapunov function [Blanchini, F., & Miani, S. 2015].

Constrained Regulator Problem

The constrained regulator problem can be established from the following definition:

Definition 3.2.1 *Consider the output feedback closed-loop system in* (3.14)-(3.15). *Assume the system response under non-zero initial conditions. The constrained regulator problem consists in determining the gain matrices for the controller and the observer, respectively, K_i and L_i, such that:*

$$\lim_{k \to \infty} x_a(k) \to 0, \tag{3.21}$$

and the control and augmented state constraints given, respectively, by (3.5) *and* (3.7), *are satisfied.*

Under the conditions presented in section 3.2.1, it is possible to guarantee the contraction of $R[Y,\mathbf{1}]$, with $0 < \lambda < 1$, such that the trajectory of the augmented state stays within $R[Y,\mathbf{1}]$ and $x_a(k) \to 0$ as $k \to \infty$.

Furthermore, the characterization of the positive invariance of $R[Y,\mathbf{1}]$ under the PDC law (3.13) allows proposing a solution that takes into account both the control constraints and the constraints in the augmented state-space, as defined in (3.5) and (3.7), respectively. The conditions under which such restrictions can be applied are based on the Farkas Lemma in its matrix version [Hennet, J.C. 1989, Schrijver, A. 1998].

Constraints in the augmented state can be satisfied if the PI polyhedron is contained in the state constraints polyhedron, that is if $R[Y,\mathbf{1}] \subseteq R[S,\mathbf{1}]$. It is guaranteed, if and only if there exists a matrix $M \geq 0$, with $M \in \mathbb{R}^{g_a \times g}$, such that

$$\begin{aligned} MY &= S, \\ M\mathbf{1} &\leq \mathbf{1}. \end{aligned} \tag{3.22}$$

Thus, the characterization of the polyhedral inclusion $R[Y,\mathbf{1}] \subseteq R[S,\mathbf{1}]$ for fuzzy T-S systems is analogous to the linear case [Hennet, J.C. 1989, Schrijver, A. 1998].

In turn, the control constraints can be established from the following theorem:

Theorem 3.2.2 *The polyhedron defined by $R[S_u K_i^a, \mathbf{1}]$ characterizes the control constraints defined in the augmented state-space. The inclusion of the polyhedral domain given by $R[Y,\mathbf{1}] \subseteq R[S_u K_i^a, \mathbf{1}]$ is guaranteed, with $K_i^a = K_i[-\mathbf{I}\ \mathbf{I}]$, $K_i^a \in \mathbb{R}^{m \times 2n}$ and $S_u \in \mathbb{R}^{g_u \times m}$, if there exist matrices $Q_i \geq 0$, $i = 1, \cdots, r$, with $Q_i \in \mathbb{R}^{g_u \times g}$, such that*

$$\begin{aligned} Q_i Y &= S_u K_i^a, \\ Q_i \mathbf{1} &\leq \mathbf{1}. \end{aligned} \tag{3.23}$$

Proof 3.2.2 *Consider $Yx_a(k) \leq \mathbf{1}$. So, from* (3.13) *and* (3.17)*:*

$$S_u u(k) = \sum_{i=1}^{r} \alpha(y) S_u K_i^a x_a(k) \leq \sum_{i=1}^{r} \alpha(y) Q_i Y x_a(k) \leq \sum_{i=1}^{r} \alpha(y) Q_i \mathbf{1} \leq \mathbf{1}.$$

This proves the inclusion of the polyhedral domain given by $R[Y,\mathbf{1}] \subseteq R[S_u K_i^a, \mathbf{1}]$. □

Note that, due to the fact that $\alpha_i(y(k)) \in \Delta$ (3.3), we have that if $x_a(k) \in R[S_u K_i^a, \mathbf{1}]$, $i = 1, \cdots, r$, then $S_u u(k) \leq \mathbf{1}$, thus ensuring respect for control constraints.

3.2.2 Case B

For this case, as the membership functions depend on non-accessible states, it is necessary to use an estimation mechanism so that these variables can be calculated.

The fuzzy observer T-S can be expressed with r rules in the form:

$$\begin{aligned} \hat{x}(k+1) &= \sum_{i=1}^{r} \alpha_i(\hat{x}(k))[A_i \hat{x}(k) + B_i u(k) + L_i(y(k) - \hat{y}(k))] \\ \hat{y}(k) &= C\hat{x}(k). \end{aligned} \tag{3.24}$$

where $\alpha_i(\hat{x}(k))$ are membership functions applied to the vector of estimated premise variables $\hat{x}(k)$, which must belong to the simplex:

$$\Delta = \{\alpha(\hat{x}(k)) \in \mathbb{R}^r : \sum_{i=1}^{r} \alpha_i(\hat{x}(k)) = 1,\ \alpha_i(\hat{x}(k)) \geq 0\}. \tag{3.25}$$

The PDC control law is now given by:

$$u(k) = -\sum_{i=1}^{r} \alpha_i(\hat{x}(k)) K_i \hat{x}(k), \tag{3.26}$$

with $K_i \in \mathbb{R}^{m \times n}$, such that the corresponding augmented closed-loop system is:

$$\begin{aligned} x_a(k+1) = \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i(x(k)) \alpha_j^2(\hat{x}(k)) G_{ijj}^a x_a(k) + \\ 2 \sum_{i=1}^{r} \sum_{j<c}^{r} \alpha_i(x(k)) \alpha_j(\hat{x}(k)) \alpha_c(\hat{x}(k)) \frac{G_{ijc}^a + G_{icj}^a}{2} x_a(k), \end{aligned} \tag{3.27}$$

such that

$$G_{ijc}^a = \begin{bmatrix} A_i - B_i K_c & B_i K_c \\ S_{ijc}^1 & S_{ijc}^2 \end{bmatrix}, \tag{3.28}$$

$$S_{ijc}^1 = (A_i - A_j) - (B_i - B_j) K_c, \tag{3.29}$$

$$S_{ijc}^2 = A_j - L_j C + (B_i - B_j) K_c, \tag{3.30}$$

where $x_a(k)^{\mathrm{T}} = [x(k)\ e(k)]$, with $x_a(k) \in \mathbb{R}^{2n}$ and $e(k) = x(k) - \hat{x}(k)$.

Now, the linear control constraints related to the system (3.27)-(3.30) can be expressed from (3.5) and (3.26) by:

$$\begin{aligned} S_u u(k) &= -S_u \left[\sum_{i=1}^{r} \alpha_i(\hat{x}(k)) K_i x(k) - \sum_{i=1}^{r} \alpha_i(\hat{x}(k)) K_i e(k) \right] \\ &= \sum_{i=1}^{r} \alpha_i(\hat{x}(k)) S_u K_i [-\mathbf{I}\ \mathbf{I}] x_a(k) \leq \mathbf{1}, \end{aligned} \tag{3.31}$$

such that $R[S_u K_i^a, \mathbf{1}]$ can be defined by the polyhedral set given in (3.17).

The control objectives under constraints are the same as those defined for the case **A** but considering the control law (3.26). Moreover, the possibility of imposing trajectories of $x_a(k)$ within a given polyhedral set $R[Y, \mathbf{1}] \subset \Omega$ w.r.t. closed-loop system (3.27)-(3.30) can be achieved from the Definition (2.3.1).

Next, we present sufficient conditions to guarantee that a polyhedral set $R[Y, \mathbf{1}]$ is PI λ-contractive w.r.t. fuzzy T-S system under a PDC control law. To simplify the notation, we drop the explicit dependence of x_a on the variable k in the membership functions.

Theorem 3.2.3 *The polyhedron $R[Y, \mathbf{1}]$ is PI λ-contractive w.r.t. closed-loop system* (3.27)-(3.30)*, if there are matrices $H_{ijj} \in \mathbb{R}^{g \times g}$, $i = 1, 2, \cdots, r$, $j = 1, 2, \cdots, r$ and $H_{ijc} \in \mathbb{R}^{g \times g}$, $i = 1, 2, \cdots, r$, $j = 1, \cdots, (r-1)$ and $c = j+1, \cdots, r$, such that:*

$$\begin{gathered} H_{ijj} Y = Y G_{ijj}^a,\ H_{ijj} \geq 0, \\ H_{ijj} \mathbf{1} \leq \lambda \mathbf{1}, \\ H_{ijc} Y = Y \left(\frac{G_{ijc}^a + G_{icj}^a}{2} \right),\ H_{ijc} \geq 0, \\ H_{ijc} \mathbf{1} \leq \lambda \mathbf{1}. \end{gathered} \tag{3.32}$$

Proof 3.2.3 *Consider $Y x_a(k) \leq \mathbf{1}$. Then, from* (3.27)*:*

$$\begin{aligned} Y x_a(k+1) &= \sum_{i=1}^{r} \alpha_i(x) \left(\sum_{j=1}^{r} \alpha_j^2(\hat{x}) Y G_{ijj}^a + 2 \sum_{j<c}^{r} \alpha_j(\hat{x}) \alpha_c(\hat{x}) Y \left(\frac{G_{ijc}^a + G_{icj}^a}{2} \right) \right) x_a(k) \\ &= \sum_{i=1}^{r} \alpha_i(x) \left(\sum_{j=1}^{r} \alpha_j^2(\hat{x}) H_{ijj} Y + 2 \sum_{j<c}^{r} \alpha_j(\hat{x}) \alpha_c(\hat{x}) H_{ijc} Y \right) x_a(k) \\ &\leq \sum_{i=1}^{r} \alpha_i(x) \left(\sum_{j=1}^{r} \alpha_j^2(\hat{x}) H_{ijj} \mathbf{1} + 2 \sum_{j<c}^{r} \alpha_j(\hat{x}) \alpha_c(\hat{x}) H_{ijc} \mathbf{1} \right) \\ &\leq \sum_{i=1}^{r} \alpha_i(x) \left(\sum_{j=1}^{r} \alpha_j^2(\hat{x}) \lambda \mathbf{1} + 2 \sum_{j<c}^{r} \alpha_j(\hat{x}) \alpha_c(\hat{x}) \lambda \mathbf{1} \right) \\ &= \sum_{i=1}^{r} \alpha_i(x) \left(\lambda \left(\sum_{j=1}^{r} \alpha_j(\hat{x}) \right)^2 \mathbf{1} \right) = \lambda \mathbf{1} < \mathbf{1}. \end{aligned}$$

This proves that the polyhedron $R[Y, \mathbf{1}]$ is PI and λ-contractive. □

For the case **B**, the conditions above are only valid if $\alpha(\hat{x}(k))$ belongs to the domain of validity Ω of the fuzzy T-S model. A possible way of guaranteeing that is to reduce the universes of discourses of $x(k)$ and $e(k)$. Consequently, the set $R[S, \mathbf{1}]$ needs to be reduced or the local models redefined. In the classical approach for fuzzy T-S systems, this problem is circumvented because the membership functions are defined for the interval $[0, 1]$, such that for values outside the expected universe of discourse, these functions present the values 0 or 1.

However, the translation of nonlinear dynamics through fuzzy T-S model presents membership functions with a form specified by such dynamics, as in the case of the sector nonlinearity approach [Wang, H.O., & Tanaka, K. 2004]. Local models will be defined based on the universe of discourse of the estimated state variables to work around this problem. As $\hat{x}(k) = x(k) - e(k)$, it is possible to infer the universe of discourse associated with the estimates from the extreme values for the state vector and observation error. This inference, together with the inclusion of $x_a(k)$ in Ω, through the computation of suitable invariant sets, will be enough to guarantee the validity of the model, such that $\alpha(x(k)) \in \Delta$ and $\alpha(\hat{x}(k)) \in \Delta$.

The comments made for the case **A**, referring to the invariance and contraction of the polyhedral set $R[Y, \mathbf{1}]$ can be extended to the case **B**. To do so, we must replace the conditions in (3.19) from Theorem 3.2.1, by the conditions in (3.32) from Theorem 3.2.3. Furthermore, the Equation (3.20) must be satisfied to avoid the computation of a degenerate polyhedron.

Given that $R[Y, \mathbf{1}]$ is a PI λ-contractive polyhedron, the discrete-time system (3.27)-(3.30) is locally asymptotically stable and it admits the polyhedral norm $||Yx_a(k)||_\infty$ as a Lyapunov function [Blanchini, F., & Miani, S. 2015, p. 261].

The estimated membership functions are calculated analogously to $\alpha_i(x(k))$, except for the fact that the estimates of the state variables are considered instead of $x(k)$. Given that $x_a(0) \in R[Y, \mathbf{1}]$, then $x_a(k) \to 0$ and $\alpha_i(\hat{x}(k)) \to \alpha_i(x(k))$, when $k \to \infty$. Therefore, the control objectives are satisfied simultaneously to the estimation process.

Constrained Regulator Problem

The constrained regulator problem can be stated from the following definition:

Definition 3.2.2 *Consider the output feedback closed-loop system in* (3.27)-(3.30). *Assume the system response under non-zero initial conditions. The constrained regulator problem consists in determining the gain matrices for the controller and the observer, respectively, K_i and L_i, such that:*

$$\lim_{k \to \infty} x_a(k) \to 0, \tag{3.33}$$

and the control and augmented state constraints given, respectively, by (3.5) *and* (3.7), *are satisfied.*

The comments about the positive invariance and contraction of the constraint set $R[Y,\mathbf{1}]$ made for the case **A** can be extended to the case **B**, considering, however, the PDC control law presented in (3.26).

Analogous to the case **A**, the constraints in the augmented state can be satisfied if the PI polyhedron is contained in the state constraints polyhedron, which, in turn, can be guaranteed if the conditions in (3.22) are satisfied. In turn, the control constraints are satisfied as long as the conditions in (3.23) are guaranteed, according to Theorem 3.2.2 and the control law (3.26).

Proof 3.2.4 *Consider* $Yx_a(k) \leq \mathbf{1}$. *Then, from* (3.17) *and* (3.26):

$$S_u u(k) = \sum_{i=1}^{r} \alpha(\hat{x}) S_u K_i^a x_a(k) \leq \sum_{i=1}^{r} \alpha(\hat{x}) Q_i Y x_a(k) \leq \sum_{i=1}^{r} \alpha(\hat{x}) Q_i \mathbf{1} \leq \mathbf{1}.$$

This proves the inclusion of the polyhedral domain given by $R[Y,\mathbf{1}] \subseteq R[S_u K_i^a, \mathbf{1}]$. □

One should note that for the case **B**, the above conditions are only valid if $\alpha(\hat{x}(k)) \in \Delta$, such that the fuzzy T-S model is defined within the validity domain Ω.

3.3 Positive Invariance of Symmetric Polyhedra

3.3.1 Case A

Consider the case in which the membership functions depend only on the output of the system, such that the discrete-time, closed-loop T-S fuzzy system can be expressed by:

$$\begin{aligned} x_a(k+1) = &\sum_{i=1}^{r} \alpha_i^2(y(k)) G_{ii}^a x_a(k) + \\ 2 &\sum_{i=1}^{r} \sum_{i<j}^{r} \alpha_i(y(k)) \alpha_j(y(k)) \frac{G_{ij}^a + G_{ji}^a}{2} x_a(k), \end{aligned} \tag{3.34}$$

where

$$G_{ij}^a = \begin{bmatrix} A_i - B_i K_j & B_i K_j \\ 0 & A_i - L_i C \end{bmatrix}, \tag{3.35}$$

with $x_a(k)^{\mathrm{T}} = [x(k)\ e(k)]$, $x_a(k) \in \mathbb{R}^{2n}$ e $e(k) = x(k) - \hat{x}(k)$.

Linear control constraints related to the system (3.34)-(3.35) can be expressed from (3.10) and (3.13) by:

$$|S_u^s u(k)| = \left| \sum_{i=1}^{r} \alpha_i(y(k)) S_u^s K_i^a x_a(k) \right| \leq \mathbf{1}, \tag{3.36}$$

where

$$\mathcal{S}[S_u^s K_i^a, \mathbf{1}] = \{x_a(k) \in \mathbb{R}^{2n} : |S_u^s K_i^a x_a(k)| \leq \mathbf{1}\} \tag{3.37}$$

defines the control constraints in the augmented state space $S_u^s \in \mathbb{R}^{g_u^s \times m}$, $K_i^a = K_i[-\mathbf{I}\ \mathbf{I}\,]$, $K_i^a \in \mathbb{R}^{m \times 2n}$.

The intersection of the sets Ω, $\mathcal{S}[S_s,\mathbf{1}]$ and $\mathcal{S}[S_u^s K_i^a,\mathbf{1}]$ forms a polyhedron symmetrical and compact containing the origin.

Our objective then becomes to obtain the matrices K_i, L_i, $i=1,\cdots,r$ and a polyhedral set

$$\mathcal{S}[Y_s,\mathbf{1}] = \{x_a(k) \in \mathbb{R}^{2n+p} : |Y_s x_a(k)| \leq \mathbf{1}\}, \tag{3.38}$$

contained in the constraint set $\Omega \cap \mathcal{S}[S_s,\mathbf{1}] \cap \mathcal{S}[S_u^s K_i^a,\mathbf{1}]$, that it is PI λ-contractive under the control law (3.13), such that for any trajectory starting at $\mathcal{S}[Y_s,\mathbf{1}]$, with $Y_s \in \mathbb{R}^{g_s \times 2n}$, the output $y(k)$ converges asymptotically to the origin without violating the constraints.

The possibility of imposing trajectories of $x_a(k)$ within a given polyhedral set $\mathcal{S}[Y_s,\mathbf{1}] \subset \Omega$ w.r.t. closed-loop system (3.34)-(3.35) can be achieved from the Definition (2.3.1), where $u(k) \in \mathcal{U}_s$, $\forall k \in \mathbb{N}$.

Next, we present sufficient conditions that guarantee that a symmetric polyhedral set $\mathcal{S}[Y_s,\mathbf{1}]$ is PI λ-contractive with respect to the fuzzy system T-S under a PDC control law. Once again, we will simplify the notation, discarding the explicit dependence of x_a on the variable k in the membership functions.

Theorem 3.3.1 *The polyhedron $\mathcal{S}[Y_s,\mathbf{1}]$ is PI λ-contractive with respect to the closed-loop system* (3.34)-(3.35)*, if there are matrices $H_{ii} \in \mathrm{R}^{g_s \times g_s}$, $i=1,2,\cdots,r$ and $H_{ij} \in \mathrm{R}^{g_s \times g_s}$, $i=1,2,\cdots,r$ and $j=i+1,\cdots,r$, such that:*

$$\begin{array}{c} H_{ii}Y_s = Y_s G_{ii}^a, \\ ||H_{ii}||_\infty \leq \lambda, \\ H_{ij}Y_s = Y_s \left(\frac{G_{ij}^a + G_{ji}^a}{2}\right), \\ ||H_{ij}||_\infty \leq \lambda. \end{array} \tag{3.39}$$

Proof 3.3.1 *Consider $|Y_s x_a(k)| \leq \mathbf{1}$. Then, from* (3.34)*:*

$$\begin{aligned} |Y_s x_a(k+1)| &= \left| \left(\sum_{i=1}^r \alpha_i^2(y) Y_s G_{ii}^a + 2\sum_{i=1}^r \sum_{i<j}^r \alpha_i(y)\alpha_j(y) Y_s \left(\frac{G_{ij}^a + G_{ji}^a}{2}\right)\right) x_a(k) \right| \\ &\leq \left(\sum_{i=1}^r \alpha_i^2(y)|H_{ii}Y_s| + 2\sum_{i=1}^r \sum_{i<j}^r \alpha_i(y)\alpha_j(y)|H_{ij}Y_s| \right)|x_a(k)| \\ &\leq \left(\sum_{i=1}^r \alpha_i^2(y)|H_{ii}|\mathbf{1} + 2\sum_{i=1}^r \sum_{i<j}^r \alpha_i(y)\alpha_j(y)|H_{ij}|\mathbf{1} \right) \\ &\leq \left(\sum_{j=1}^r \alpha_i^2(y)\lambda\mathbf{1} + 2\sum_{i<j}^r \alpha_i(y)\alpha_j(y)\lambda\mathbf{1} \right) = \lambda\left(\sum_{j=1}^r \alpha_i(y)\right)^2 \mathbf{1} = \lambda\mathbf{1} < \mathbf{1}, \end{aligned}$$

given that $||H_{ii}||_\infty \leq \lambda$ and $||H_{ij}||_\infty \leq \lambda$ are equivalent to $|H_{ii}|\mathbf{1} \leq \lambda\mathbf{1}$ and $|H_{ij}|\mathbf{1} \leq \lambda\mathbf{1}$, respectively.

This proves that the polyhedron $\mathcal{S}[Y_s,\mathbf{1}]$ is PI and λ-contractive.

The matrix $Y_s \in \mathbb{R}^{g_s \times 2n}$ must present full column rank to avoid the solution presenting a degenerate polyhedron. It is equivalent to the existence of the pseudo-inverse matrix $P_s \in \mathbb{R}^{2n \times g_s}$. Then, the equality:

$$P_s Y_s = \mathbf{I}, \tag{3.40}$$

must be verified, with $\mathbf{I} \in \mathbb{R}^{2n \times 2n}$.

Theorem 3.3.2 is related to the existence of a symmetric polyhedron PI, guaranteeing that if $x_a(0) \in \mathcal{S}[Y_s, \mathbf{1}]$, then $x_a(k) \in \mathcal{S}[Y_s, \mathbf{1}]$, for $k = 1, 2, \cdots$. Furthermore, $0 < \lambda < 1$ guarantees the contraction of the polyhedron $\mathcal{S}[Y_s, \mathbf{1}]$ in the augmented state space, so that if $x_a(k) \in \mathcal{S}[Y_s, \mathbf{1}]$, then $x_a(k+1) \in \lambda\mathcal{S}[Y_s, \mathbf{1}]$. Considering the effect of contraction and the conditions in (3.39), it is possible to verify that for $x_a(k) \in \mathcal{S}_\lambda[Y_s, \mathbf{1}]$, with $\mathcal{S}_\lambda[Y_s, \mathbf{1}] \equiv \lambda\mathcal{S}[Y_s, \mathbf{1}]$, $x_a(k+1) \in \lambda\mathcal{S}_\lambda[Y_s, \mathbf{1}], \forall k \geq 0$. So if $x_a(0) \in \mathcal{S}[Y_s, \mathbf{1}]$, then $x_a(k) \in \lambda^k \mathcal{S}[Y_s, \mathbf{1}]$. Thus, if the conditions presented in Theorem 3.3.2 are satisfied, then $x_a(k) \to 0$ when $k \to \infty$.

Given that $\mathcal{S}[Y_s, \mathbf{1}]$ is a PI λ-contractive polyhedron, the discrete-time system (3.34)-(3.35) is locally asymptotically stable and it admits the polyhedral norm $||Y_s x_a(k)||_\infty$ as a Lyapunov function [Blanchini, F., & Miani, S. 2015, p. 261].

Constrained Regulator Problem

The constrained regulator problem can be established based on the Definition 3.2.1 through the augmented state and control constraints for the symmetric case. This implies replacing the polyhedral sets presented in (3.5) and (3.7), respectively, by (3.10) and (3.8).

There are sufficient conditions under which the constrained regulator problem for symmetric polyhedra can be solved. These conditions are based on the matrix version of Farkas' lemma for the symmetric case [Hennet, J.C. 1989, Schrijver, A. 1998]. Constraints in the augmented state can be satisfied if the PI polyhedron is contained within the state constraints polyhedron, that is, if $\mathcal{S}[Y_s, \mathbf{1}] \subseteq \mathcal{S}[S_s, \mathbf{1}]$. It is guaranteed, if and only if there exists a matrix $M \in \mathbb{R}^{g_a^s \times g_s}$, such that

$$\begin{aligned} MY_s &= S_s, \\ ||M||_\infty &\leq 1. \end{aligned} \tag{3.41}$$

The characterization of the polyhedral inclusion $\mathcal{S}[Y_s, \mathbf{1}] \subseteq \mathcal{S}[S_s, \mathbf{1}]$ is analogous to the linear case [Hennet, J.C. 1989, Schrijver, A. 1998].

In turn, the control constraints can be established from the following theorem:

Theorem 3.3.2 *The polyhedron defined by $\mathcal{S}[S_u^s K_i^a, \mathbf{1}]$ characterizes the control constraints defined in the augmented state-space. The inclusion of the polyhedral domain given by $\mathcal{S}[Y_s, \mathbf{1}] \subseteq \mathcal{S}[S_u^s K_i^a, \mathbf{1}]$ is guaranteed, with $K_i^a = K_i[-\mathbf{I}, \mathbf{I}]$, $K_i^a \in \mathbb{R}^{m \times 2n}$ and $S_u^s \in \mathbb{R}^{g_u^s \times m}$, if there are $Q_i \in \mathbb{R}^{g_u^s \times g_s}$, such that*

$$\begin{aligned} Q_i Y_s &= S_u^s K_i^a, \\ ||Q_i||_\infty &\leq 1. \end{aligned} \tag{3.42}$$

Proof 3.3.2 *Consider $|Y_s x_a(k)| \leq \mathbf{1}$. Then, from* (3.13) *and* (3.37)*:*

$$|S_u^s u(k)| = \left| \sum_{i=1}^{r} \alpha(y) S_u^s K_i^a x_a(k) \right| \leq \sum_{i=1}^{r} \alpha(y) |Q_i Y_s x_a(k)| \leq \sum_{i=1}^{r} \alpha(y) |Q_i| \mathbf{1} \leq \mathbf{1},$$

since $||Q_i||_\infty \leq 1$ implies $|Q_i|\mathbf{1} \leq \mathbf{1}$.

This proves the inclusion of the polyhedral domain given by $\mathcal{S}[Y_s, \mathbf{1}] \subseteq R[S_u^s K_i^a, \mathbf{1}]$. □

Analogous to the case where polyhedrons of generic shape are considered, it is possible to observe that for symmetric control constraints, $\alpha_i(y(k)) \in \Delta$ (3.3), therefore, if $x_a(k) \in \mathcal{S}[S_u^s K_i^a,\ \mathbf{1}]$, $i = 1, \cdots, r$, so $S_u^s u(k) \leq \mathbf{1}$, thus ensuring respect for control constraints.

3.3.2 Case B

Consider the case where the membership functions depend on non-accessible states, such that the discrete-time, closed-loop fuzzy T-S system can be expressed by:

$$\begin{aligned} x_a(k+1) = & \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i(x(k))\alpha_j^2(\hat{x}(k)) G_{ijj}^a x_a(k) + \\ & 2\sum_{i=1}^{r}\sum_{j<c}^{r} \alpha_i(x(k))\alpha_j(\hat{x}(k))\alpha_c(\hat{x}(k)) \tfrac{G_{ijc}^a + G_{icj}^a}{2} x_a(k), \end{aligned} \tag{3.43}$$

where

$$G_{ijc}^a = \begin{bmatrix} A_i - B_i K_c & B_i K_c \\ S_{ijc}^1 & S_{ijc}^2 \end{bmatrix}, \tag{3.44}$$

$$S_{ijc}^1 = (A_i - A_j) - (B_i - B_j) K_c, \tag{3.45}$$

$$S_{ijc}^2 = A_j - L_j C + (B_i - B_j) K_c, \tag{3.46}$$

with $x_a(k)^{\mathrm{T}} = [x(k)\ e(k)]$, $x_a(k) \in \mathbb{R}^{2n}$ e $e(k) = x(k) - \hat{x}(k)$.

The linear control contraints related to the system (3.34)-(3.35) can be expressed from (3.10) and (3.13), by:

$$|S_u^s u(k)| = \left| \sum_{i=1}^{r} \alpha_i(\hat{x}(k)) S_u^s K_i^a x_a(k) \right| \leq \mathbf{1}, \tag{3.47}$$

such that

$$\mathcal{S}[S_u^s K_i^a, \mathbf{1}] = \{x_a(k) \in \mathbb{R}^{2n} : |S_u^s K_i^a x_a(k)| \leq \mathbf{1}\} \tag{3.48}$$

defines the control constraints in the augmented state space $S_u^s \in \mathbb{R}^{g_u^s \times m}$, $K_i^a = K_i[-\mathbf{I}\ \mathbf{I}]$, $K_i^a \in \mathbb{R}^{m \times 2n}$.

The control objectives under constraints are the same as those defined for the case **A** but considering the control law (3.26). Moreover, the possibility of imposing trajectories of $x_a(k)$ within a given polyhedral set $\mathcal{S}[Y_s, \mathbf{1}] \subset \Omega$ w.r.t. closed-loop system (3.43)-(3.46) can be achieved from the Definition (2.3.1), where $u(k) \in \mathcal{U}_s$, $\forall k \in \mathbb{N}$.

Next, we present sufficient conditions that guarantee that a polyhedral set $\mathcal{S}[Y_s, \mathbf{1}]$ is PI λ-contractive with respect to the fuzzy system T-S under a PDC control law. For simplicity, from this point on, we drop the explicit dependence of x_a on the variable k in the membership functions.

Theorem 3.3.3 *Given that* $0 < \lambda < 1$*, a polyhedron* $\mathcal{S}[Y_s, \mathbf{1}]$ *is PI* λ*-contractive to the closed-loop system* (3.43)-(3.46)*, if there are matrices* $H_{ijj} \in \mathbb{R}^{g_s \times g_s}$*,* $i = 1, 2, \cdots, r$*,* $j = 1, 2, \cdots, r$ *and* $H_{ijc} \in \mathbb{R}^{g_s \times g_s}$*,* $i = 1, 2, \cdots, r$*,* $j = 1, \cdots, (r-1)$ *and* $c = j+1, \cdots, r$*, such that:*

$$\begin{gathered} H_{ijj} Y_s = Y_s G^a_{ijj}, \\ \|H_{ijj}\|_\infty \leq \lambda, \\ H_{ijc} Y_s = Y_s \left(\frac{G^a_{ijc} + G^a_{icj}}{2} \right), \\ \|H_{ijc}\|_\infty \leq \lambda. \end{gathered} \tag{3.49}$$

Proof 3.3.3 *Consider* $|Y_s x_a(k)| \leq \mathbf{1}$*. Then, from* (3.43)*:*

$$\begin{aligned} |Y_s x_a(k+1)| &= \left| \sum_{i=1}^r \alpha_i(x) \left(\sum_{j=1}^r \alpha_j^2(\hat{x}) Y_s G^a_{ijj} + 2 \sum_{j<c}^r \alpha_j(\hat{x}) \alpha_c(\hat{x}) Y_s \left(\frac{G^a_{ijc} + G^a_{icj}}{2} \right) \right) x_a(k) \right| \\ &= \sum_{i=1}^r \alpha_i(x) \left(\sum_{j=1}^r \alpha_j^2(\hat{x}) |H_{ijj} Y_s| + 2 \sum_{j<c}^r \alpha_j(\hat{x}) \alpha_c(\hat{x}) |H_{ijc} Y_s| \right) |x_a(k)| \\ &\leq \sum_{i=1}^r \alpha_i(x) \left(\sum_{j=1}^r \alpha_j^2(\hat{x}) |H_{ijj}| \mathbf{1} + 2 \sum_{j<c}^r \alpha_j(\hat{x}) \alpha_c(\hat{x}) |H_{ijc}| \mathbf{1} \right) \\ &\leq \sum_{i=1}^r \alpha_i(x) \left(\sum_{j=1}^r \alpha_j^2(\hat{x}) \lambda \mathbf{1} + 2 \sum_{j<c}^r \alpha_j(\hat{x}) \alpha_c(\hat{x}) \lambda \mathbf{1} \right) \\ &= \sum_{i=1}^r \alpha_i(x) \left(\lambda \left(\sum_{j=1}^r \alpha_j(\hat{x}) \right)^2 \mathbf{1} \right) = \lambda \mathbf{1} < \mathbf{1}, \end{aligned}$$

given that $\|H_{ijj}\|_\infty \leq \lambda$ *and* $\|H_{ijc}\|_\infty \leq \lambda$ *are equivalent to* $|H_{ijj}| \mathbf{1} \leq \lambda \mathbf{1}$ *and* $|H_{ijc}| \mathbf{1} \leq \lambda \mathbf{1}$*, respectively.*

This proves that the polyhedron $\mathcal{S}[Y_s, \mathbf{1}]$ *is PI and* λ*-contractive.* □

The comments made for the case **A**, referring to the invariance and contraction of the polyhedral set $\mathcal{S}[Y_s, \mathbf{1}]$, can be extended to the case **B**. To do so, we must replace the conditions in (3.39) from Theorem 3.3.1, by the conditions in (3.49) from Theorem 3.3.3. To avoid that the solution results in a degenerate polyhedron, the condition imposed in (3.40) must be satisfied.

Given that $\mathcal{S}[Y_s, \mathbf{1}]$ is a PI λ-contractive polyhedron, the discrete-time system (3.43)-(3.46) is locally asymptotically stable and it admits the polyhedral norm $\|Y_s x_a(k)\|_\infty$ as a Lyapunov function [Blanchini, F., & Miani, S. 2015, p. 261].

Constrained Regulator Problem

The constrained regulator problem can be established based on Definition 3.2.2 through the augmented state and control constraints for the symmetric case. This implies replacing the polyhedral sets presented in (3.5) and (3.7), respectively, by (3.10) and (3.8).

Analogous to the case **A**, the constraints in the augmented state-space can be satisfied if the PI polyhedron is contained in the state constraints polyhedron, which, in turn, can be guaranteed if the conditions in (3.41) are satisfied. The control constraints are satisfied as long as the conditions in (3.42) are guaranteed, according to Theorem 3.3.2 and control law (3.26).

Proof 3.3.4 *Consider* $|Y_s x_a(k)| \leq \mathbf{1}$. *So, from* (3.26) *and* (3.48)*:*

$$|S_u^s u(k)| = \sum_{i=1}^{r} \alpha(\hat{x})|S_u^s K_i^a x_a(k)| \leq \sum_{i=1}^{r} \alpha(\hat{x})|Q_i Y_s x_a(k)| \leq \sum_{i=1}^{r} \alpha(\hat{x})|Q_i|\mathbf{1} \leq \mathbf{1},$$

given that $||Q_i||_\infty \leq 1$ *is equivalent to* $|Q_i|\mathbf{1} \leq \mathbf{1}$.

This proves the inclusion of the polyhedral domain given by $\mathcal{S}[Y_s, \mathbf{1}] \subseteq \mathcal{S}[S_u^s K_i^a, \mathbf{1}]$. □

One should note that also for the case **B**, the above conditions are only valid if $\alpha(\hat{x}(k)) \in \Delta$.

3.4 Design Strategy Using Bilinear Optimization

The conditions of positive invariance and respect for the constraints proposed above carry some products among pairs of matrix variables, with bilinear terms arising from the products between matrices. These bilinear products can be considered design constraints, and the problem discussed in this section can be treated as a nonlinear optimization problem.

The approach presented in this Thesis allows that G_{ij}^a, G_{ijc}^a, H_{ij} and H_{ijc} be not *a priori* given. The matrices K_i and L_i are computed explicitly by solving the nonlinear equations with additional constraints in (3.19), (3.39) and (3.32), (3.49) related, respectively, to cases **A** and **B**.

Here, we use two different optimization strategies. Both aim to enlarge the admissible initial states while minimizing the contraction rate. These strategies were called, respectively, *Homogeneous Expansion* and *Expansion via Vertices*. The first was proposed by [Brião, S.L. et al. 2018] and the second by [Ernesto, J.G. et al. 2021].

3.4.1 First Design Strategy: Homogeneous Expansion

Case A

Based on [Brião, S.L. et al. 2018], a convex objective function is chosen to obtain a compromise between the speed of convergence of the closed-loop augmented state trajectories and the size of the set of admissible initial states given by $R[Y, \mathbf{1}]$.

A shape set approach is used to optimize the size of $R[Y,\mathbf{1}]$. The basic idea is to make the invariant set candidate $R[Y,\mathbf{1}]$ contain a set of a given shape and seek to maximize the volume of this set. This way, the $R[Y,\mathbf{1}]$ volume is indirectly optimized.

Let the following hypercube represent the shape set:

$$R[\mathbf{I}^*,\gamma^{-1}\mathbf{1}] = \{x_a(k) \in \mathbb{R}^{2n} : \mathbf{I}^* x_a(k) \leq \gamma^{-1}\mathbf{1}\}, \tag{3.50}$$

with $\mathbf{I}^* = \begin{bmatrix} \mathbf{I} \\ -\mathbf{I} \end{bmatrix}$, where $\mathbf{I}$ is an identity matrix of appropriate dimension and $\gamma > 0$. Then, the application of the extended Farkas' lemma leads to the following conditions for inclusion $R[\mathbf{I}^*,\gamma^{-1}\mathbf{1}] \subseteq R[Y,\mathbf{1}]$: $\exists\ \Xi \in \mathbb{R}^{g\times 4n}$ such that:

$$\begin{aligned} &\Xi\ \mathbf{I}^* = Y,\ \Xi \geq 0, \\ &\Xi\ \mathbf{1} \leq \mathbf{1}\gamma, \end{aligned} \tag{3.51}$$

Then, a basic nonlinear optimization problem to find solutions to the proposed regulator problem for the general case is formulated as:

$$\begin{aligned} &\min_{\Gamma'}\ \mathcal{F}(\lambda,\gamma) = a\lambda + (1-a)\gamma \\ &\text{subject to}(3.19),(3.20),(3.22),(3.23),(3.51), \\ &\qquad \psi_l^{'} \leq f_l \leq \psi_l^{''},\ l = 1,\cdots,\bar{l}, \end{aligned} \tag{3.52}$$

with $0 \leq a \leq 1$ and $\Gamma^{'} = (G_{ij}^a,\ M,\ Y,\ H_{ij},\ Q_i,\ K_i,\ L_i,\ \Xi,\ P,\ \lambda,\gamma)$. For some non-negative integer $\bar{l}$, $\psi_l^{'} \leq f_l \leq \psi_l^{''}$ represent additional constraints that can be imposed on the variables of $\Gamma^{'}$ with lower and upper limits defined by, respectively, $\psi_l^{'}$ and $\psi_l^{''}$.

The variable λ is associated with the contraction of the PI set along the trajectories of the closed-loop system. To ensure local asymptotic convergence of trajectories in the augmented state space, $0 \leq \lambda < 1$. The smaller λ, the faster state trajectories converge to the origin. The variable γ is related to the description of a hypercube in the augmented state space, such that the inclusion $R[\mathbf{I}^*,\gamma^{-1}\mathbf{1}] \subseteq R[Y,\mathbf{1}]$ is characterized by (3.51), with $\gamma > 0$.

For symmetric polyhedra, a shape set approach can be reformulated based on the optimization of the size of $\mathcal{S}[Y_s,\mathbf{1}]$. Consider

$$||Y_s||_\infty \leq \gamma \tag{3.53}$$

for some positive scalar γ. The (3.53) condition is related to Farkas' lemma, and when satisfied, it guarantees the inclusion $\mathcal{S}[\mathbf{I},\gamma^{-1}\ \mathbf{1}] \subseteq \mathcal{S}[Y_s,\mathbf{1}]$.

The approach for symmetric polyhedra is based on the objective function $\mathcal{F}$, but subject to (3.39), (3.40), (3.41), (3.42) and (3.53). Additionally, the matrix P is replaced by P_s, and Ξ must also be removed from the optimization problem formulation so that the trivial solution $\Xi\ \mathbf{I} = Y_s,\ \Xi = Y_s$ can be suppressed.

Note that based on the value of γ, it is possible to expand (or contract) the polyhedron $\mathcal{S}[\mathbf{I},\gamma^{-1}\ \mathbf{1}]$. Thus, since $\mathcal{S}[Y_s,\mathbf{1}]$ must contain $\mathcal{S}[\mathbf{I},\gamma^{-1}\mathbf{1}]$, the PI polyhedron size $\mathcal{S}[Y_s,\mathbf{1}]$ can be adjusted indirectly according to the size of $\mathcal{S}[\mathbf{I},\gamma^{-1}\mathbf{1}]$. The same argument can be extended to the general case.

Different results can be obtained based on the choice of parameter a. As a increases, the optimal values of λ and γ tend to decrease and increase, respectively. The opposite is also true; the decrease in a implies an increase in the optimal values of λ and a reduction in γ. The terms λ and γ are inversely proportional to the speed of convergence and polyhedron size, respectively. For more details, see [Brião, S.L. et al. 2018].

Case B

For generically shaped polyhedra, the optimization problem can be described analogously, simply replacing (3.19) by (3.32) in the constraints presented in (3.52). To the symmetric polyhedra, the optimization problem is formulated based on the constraints in (3.40), (3.41), (3.42), (3.49) and (3.53). In both, the matrix variables G^a_{ij} and H_{ij} must be replaced, respectively, by G^a_{ijc} and H_{ijc}.

3.4.2 Second Design Strategy: Expansion via Vertices

Case A

Based on the approach proposed by [Ernesto, J.G. et al. 2021], the objective function is chosen to allow the maximization of the set $R[Y,\mathbf{1}]$ through scalar factors ϕ_j associated with the set of directions, which correspond to the vertices of the polyhedron $R[S,\mathbf{1}]$ defined by

$$\mathcal{V} = \{\phi_j v_j,\ j = 1,\dots,\bar{j}\}, \tag{3.54}$$

where $v_j \in \mathbb{R}^{2n}$ are known and $0 < \phi_j \in \mathbb{R}$ are scalar factors to be optimized, the inclusion being $\mathcal{V} \subseteq R[Y,1]$ characterized by:

$$Y\phi_j v_j \leq 1,\ j = 1,\dots,\bar{j}. \tag{3.55}$$

For generically shaped polyhedra, the basic nonlinear optimization problem to find solutions to the proposed regulator problem in the general case is formulated:

$$\begin{array}{l} \max\limits_{\Gamma''}\ \Phi(\phi_j) = \sum\limits_{j=1}^{\bar{j}} \phi_j \\ \text{subject to (3.19), (3.20), (3.22), (3.23), (3.55),} \\ \qquad \psi'_l \leq f_l \leq \psi''_l,\ l = 1,\cdots,\bar{l}, \end{array} \tag{3.56}$$

with $\Gamma'' = (G^a_{ij},\ M,\ Y,\ H_{ij},\ Q_i,\ K_i,\ L_i,\ P,\ \lambda,\ \phi_j)$. For some integer $\bar{l} \geq 1$, $\psi'_l \leq f_l \leq \psi''_l$ represent additional constraints that may be imposed on the variables in Γ'', with the lower and upper bounds defined by, respectively, ψ'_l and ψ''_l.

The approach for symmetric polyhedra is based on the objective function $\Phi(\phi_j)$, but subject to (3.39), (3.40), (3.41), (3.42) and (3.55). The matrix P is replaced by P_s, and Y_s is used instead of Y (except for Equation (3.55), in which a symmetric polyhedron is represented in generic form). As the PI polyhedron is symmetric, it is natural that the internal polyhedron defined from the set of vertices $\mathcal{V}$ is also symmetric. It is possible to guarantee this by associating a single element ϕ_j to each vertex v_j and its opposite.

The set $\mathcal{V}$ is commonly described from the vertices of the polyhedron $R[S,\mathbf{1}]$ (or $R[S_s,\mathbf{1}]$ for the symmetric case). This choice allows obtaining solutions with larger PI polyhedra compared to homogeneous expansion. More details will be presented in Subsection 3.4.3.

Case B

For generically shaped polyhedra, the optimization problem can be described analogously, simply replacing (3.19) by (3.32) in the constraints presented in (3.56). To the symmetric polyhedra, the optimization problem is formulated based on the constraints in (3.40), (3.41), (3.42), (3.49) and (3.55). In both, the matrix variables G^a_{ij} and H_{ij} must be replaced, respectively, by G^a_{ijc} and H_{ijc}.

3.4.3 Numerical Aspects

In this work, as in [Brião, S.L. et al. 2021], the KNITRO solver [Byrd, R.H. et al. 2000, Byrd, R.H. et al. 2003, Waltz, R.A. et al. 2006, Nocedal, J. 2006] was used to obtain the numerical results. It builds on the barrier and active set methods that can deal efficiently with problems involving bilinear products.

The KNITRO solver only guarantees convergence to a locally optimal solution. It is possible to determine the search region of the optimal local solution through the choice of lower and upper bounds for the variables associated with the matrices defining the problem.

In the numerical examples presented in the next section, all KNITRO algorithms were tested simultaneously. In general, if one is not sure about which algorithm works best for a particular application, a recommended strategy is to set KNITRO options: *alg* = 5, *ma_terminate* = 1 [Waltz, R.A., & Nocedal, J. 2004]. For this setting, a stop criterion was chosen to terminate the procedure as soon as the first local optimal solution was found.

For the case **B**, the optimization problem proposed by [Brião, S.L. et al. 2018] based on (3.52) is implemented according to the constraints (3.20), (3.22) , (3.23), (3.32) and (3.51) for the set of decision variables Γ'. In turn, the optimization strategy proposed by [Ernesto, J.G. et al. 2021] of (3.56), is defined based on the constraints (3.20), (3.22), (3.23), (3.32) and (3.55), according to Γ''. The Table 3.1 shows the number of variables, equations, inequalities, and bilinear terms common to the two strategies, according to the optimization constraints (3.20), (3.22), (3.23) and (3.32) with $N = \frac{r^2\times(r-1)}{2}$ and $N_o = r^2$.

The Table 3.2 presents the number of equations, inequalities, variables, and bilinear terms particular to each of the optimization strategies, according to the optimization constraints (3.51) and (3.55). From these results, it is possible to observe that the expansion via vertices presents a greater number of terms and bilinear constraints, which probably contributes to the increase in the complexity of the optimization problem. On the other hand, this strategy offers greater design flexibility compared to homogeneous expansion, as it is possible to independently expand the PI polyhedron in the direction of each vertex

Number of	
Equalities	$2n \times (g \times (N+N_o) + g_a + r \times g_u + 2n)$
Inequalities	$g \times (N+N_o) + g_a + r \times g_u$
Variables	$g \times (g \times (N+N_o) + 4n + g_a + r \times g_u) + 1$
Bilinear Terms	$g \times (g \times (2n \times (N+N_o) + (g_a + r \times g_u + 2n)))$

Table 3.1: Number of variables, equations, inequalities and bilinear terms common to both strategies.

v_j. This strategy is notably advantageous when considering asymmetric constraints, as it avoids stopping the expansion when the PI polyhedron reaches the limit defined by a given state constraint. In general, this flexibility implies solutions with larger PI polyhedra compared to homogeneous expansion.

Number of	Homogeneous Expansion	Expansion via Vertices
Equalities	$g \times 2n$	$--$
Inequalities	g	$g \times \bar{j}$
Variables	$g \times 4n + 1$	$\bar{j}$
Bilinear Terms	$--$	$\bar{j} \times g \times 2n$

Table 3.2: Number of variables, equations, inequalities and bilinear terms particular to each of the strategies.

For the symmetric case, the polyhedrons have a compact representation, with a reduction of the quantities presented in Tables 3.1 and 3.2. This implies a smaller number of computations to be performed and, consequently, a smaller optimization problem complexity than the general case.

Some works in the literature on control systems subject to constraints compute ellipsoidal regions of attraction associated with quadratic Lyapunov functions. For example, [de Souza, C. et al. 2022] presents a control approach for discrete-time LPV systems with input constraints based on the solution of LMIs. In general, constraints associated with state variables are naturally described by polyhedral sets. Thus, the polyhedral estimates of regions of attraction fit better to the shape of these constraints than those represented by ellipsoidal sets. Therefore, obtaining larger estimates of the regions of attraction using polyhedrons is more common than using ellipsoids.

The number of rows defining Y, g (g_s in the symmetrical case) is a tunable parameter. The larger g, the larger the number of variables in the optimization problem and, consequently, the larger the possibility of finding a solution. However, the larger tends to be the complexity of the invariant polyhedron. We point out, however, that the obtained matrix Y may contain redundant rows, which makes the actual polyhedral complexity smaller than that fixed *a priori*.

Finally, our approach presents itself as an easy-to-implement alternative compared to other strategies, such as those based on predictive control (see, e.g., [Ding, B. 2011, Ding, B., & Pan, H. 2016, Ping, X. et al. 2021]), which require the solution of a (possibly non-convex) constrained optimization problem at each time step, resulting, in general, in a large online computational load.

In our approach, most of the computational load concerns the computation of the controller and observer gains from the solutions of bilinear programming problems, which are executed offline. The control law in (3.26) is a function of the estimated state $\hat{x}(k)$, which is updated online through (3.24) from the measurement of the output $y(k)$. With $\hat{x}(k)$, the estimates of the premise variables $\hat{z}(k)$ can be calculated, and, consequently, the control signal is obtained by the combination $u(k) = -\sum_{i=1}^{r} \alpha(\hat{x}(k))K_i\hat{x}(k)$. All these online calculations are easy-to-implement and "cheap".

The comments made for the case **B** can be extended to the case **A** about presented optimization strategies. The term N_o in Table 3.1 is suppressed. It implies reducing the solutions' complexity compared to the case **B**.

3.5 Control Action Calculation

As highlighted in the previous section, the online computation of the control signal $u(k)$ requires only the measurement of the output $y(k)$. In the following, we provide the algorithm that describes a computational implementation of the dynamic output feedback controller based on the fuzzy T-S observer for the case **A** (Algorithm (1)).

For the case **B**, the algorithm remains unchanged; however, the estimated state is now updated using (3.24), and the control signal is obtained from (3.26). Furthermore, it is necessary to calculate the estimated membership functions from $\hat{x}(k)$.

Algoritmo 1: Case **A**

Input: $A_i, B_i, C, K_i, L_i, y(k)$
Output: $u(k)$

1: **Loop** For each output $y(k)$ do
2: Update $\hat{x}(k+1)$ from (3.12);
3: Calculate $\alpha_i(y(k))$;
4: Compute $u(k)$ from (3.13);
5: **return** $u(k)$;
6: Apply to the system the signal $u(k)$;
7: **end**

3.6 Numerical Examples

In this work, as in [Brião, S.L. et al. 2018], the KNITRO nonlinear optimization software [Byrd, R.H. et al. 2000, Byrd, R.H. et al. 2003, Nocedal, J. 2006] was used to solve the optimization problems defined in (3.52) and (3.56), due to its efficiency in dealing with problems involving bilinearities. KNITRO only guarantees convergence to a local optimal solution.

It is possible to determine the search region based on limits defined, a priori, for the variables of the optimization problem. In the following examples, all algorithms available in KNITRO are tested simultaneously. In general, if you are not sure which algorithm works best for your application, it is recommended to set the KNITRO options: *alg* = 5, *ma_terminate* = 1 [Waltz, R.A., & Nocedal, J. 2004]. The example 3.6.1 refers to the case **A**, while the examples 3.6.2 and 3.6.3 to the case **B**. For the Examples 3.6.1 and 3.6.3, we use the conditions related to the existence of a PI generic shaped polyhedron λ-contractive $R[Y,\mathbf{1}]$. In contrast, to the Example 3.6.2 we use the conditions for a symmetric shaped polyhedron $\mathcal{S}[Y_s,\mathbf{1}]$. The elements of the matrices M, H_{ij}, H_{ijc}, Q_i are contained in the interval $[0,1]$ (or $[-1,1]$, for symmetric polyhedra), $\lambda \in [0;\ 0.99999]$ and $\phi_j > 0$. To the Example 3.6.1, the elements of Y, K_i, L_i, P are defined in $[-10^1, 10^1]$, whereas for the Example 3.6.3 in $[-10^3, 10^3]$. For the example 3.6.2, the elements of Y_s, K_i, L_i and P_s are defined in set $[-10^3, 10^3]$.

The projections into the state-space and observation error space of the generic shaped polyhedron $R[Y,\mathbf{1}]$ are here denoted by $p_x(R[Y,\mathbf{1}])$ and $p_e(R[Y,\mathbf{1}])$, respectively. In the symmetric case, we consider the polyhedron $\mathcal{S}[Y_s,\mathbf{1}]$, whose projections in state and observation error spaces are, respectively, $p_x(\mathcal{S}[Y_s,\mathbf{1}])$ and $p_e(\mathcal{S}[Y_s,\mathbf{1}])$. This notation is used only to illustrate the solutions presented in this section. To calculate the vertices v_j, the hypervolume of the obtained PI polyhedra, as well as the projections on $p_x(\cdot)$ and $p_e(\cdot)$, the Multi-Parametric Toolbox 3.0 [Herceg, M. et al. 2013] (or MPT3 for abbreviate) is used.

Example 3.6.1 *Consider the following discrete-time nonlinear system [Huang, D., & Nguang, S.K. 2007]:*

$$\begin{aligned} x_1(k+1) &= \sigma x_1(k) + \beta x_2(k) + Tu(k) \\ x_2(k+1) &= Tx_1(k) + x_2(k) \\ y(k) &= x_2(k) \end{aligned} \tag{3.57}$$

where $\sigma = 1 - 0.1125T$, $\beta = -T(0.02 + 0.67x_2(k)^2)$ *and* $T = 0.01s$ *is the sampling period. The* (3.57) *system consists of the representation of a nonlinear mass-spring-damper mechanical system discretized based on the Euler's method and which can be represented as a fuzzy T-S model, by:*

$$\begin{gathered} A_1 = \begin{bmatrix} \sigma & -0.02T \\ T & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} \sigma & -1.53T \\ T & 1 \end{bmatrix}, \\ B_1 = \begin{bmatrix} T \\ 0 \end{bmatrix},\ B_2 = \begin{bmatrix} T \\ 0 \end{bmatrix},\ C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \end{gathered} \tag{3.58}$$

subjected to the state, error and control constraints, respectively: $x_1(k) \in [-3,\ 3] = [-\underline{x}_1,\ \overline{x}_1]$, $x_2(k) \in [-1,\ 1.5] = [-\underline{x}_2,\ \overline{x}_2]$, $e_1(k) \in [-3,\ 3] = [-\underline{e}_1,\ \overline{e}_1]$, $e_2(k) \in [-1,\ 1.5] = [-\underline{e}_2,\ \overline{e}_2]$ *e* $u(k) \in [-100,\ 100]$.

The T-S model described above is obtained through the so-called sector nonlinearity approach [Wang, H.O., & Tanaka, K. 2004], such that the membership functions can be expressed by:

$$\alpha_1(k) = 1 - \tfrac{x_2(k)^2}{2.25},\ \alpha_2(k) = \tfrac{x_2(k)^2}{2.25}, \tag{3.59}$$

and the premise variable is defined by $z(k) = 0.67x_2(k)^2$. *It falls clearly in the case* **A**, *since* $\alpha_i(z(k)) = \alpha_i(x_2(k)) = \alpha_i(y(k))$.

For the homogeneous expansion, the variable λ *assumes the maximum value, and the variable* γ *assumes the minimum value for* $a = 0.98$. *This effect is reported in [Brião, S.L. et al. 2018] and it is associated with the weighting of the variables* λ *and* γ*, respectively, by* a *and* $(1-a)$. *A choice* $a < 0.98$ *would not affect the solution since the value of* λ *reaches its upper bound and, due to the weighting effect,* γ *reaches it is lower bound.*

As seen in Table 3.3, the hypervolume associated with the polyhedron found using the expansion via vertices is larger than the one related to the homogeneous expansion, so that λ *reaches its maximum value* $\lambda = 0.99999$. *In contrast, the homogeneous expansion is characterized by establishing a trade-off between the convergence speed and the PI polyhedron size. Thus, it is possible to opt for a solution that favors the convergence speed over the size of a polyhedron and vice versa.*

#	Obj. Func.	a	γ / Φ	λ	*Vol*
1	$\mathcal{F}(\lambda,\gamma)$	1.00	2609.01	0.97865	0.375
2	$\mathcal{F}(\lambda,\gamma)$	0.99	2.16312	0.99998	7.054
3	$\mathcal{F}(\lambda,\gamma)$	0.98	1.99112	0.99999	10.726
4	$\Phi(\phi_j)$	$--$	7.58574	0.99999	27.852

Table 3.3: DOF results with KNITRO: Optimal solutions for case **A**.

In this example, the initial constraint polyhedron is asymmetric. Thus, the conditions for generic shaped polyhedra are used. The number of vertices that define the polyhedron $R[S,\boldsymbol{1}]$ *is* $\bar{j} = 16$. *Therefore, the optimization problem for homogeneous expansion contains* 36 *more equations and* 57 *variables, fewer* 135 *inequalities, and* 576 *bilinear terms than the one for the expansion via vertices.*

Table 3.4 shows the gain matrices K_i, $i = 1.2$ *and* L_i, $i = 1,2$ *obtained for this example with the vertices that define* $R[S,\boldsymbol{1}]$. *The results are associated with the corresponding matrix* Y*, shown in Table 3.6, for a polyhedron defined by* $n_y = 9$ *rows.*

Case	K_i	L_i^{T}
A	$\begin{bmatrix}10.000 & 9.870\end{bmatrix}_1$ $\begin{bmatrix}10.000 & 8.360\end{bmatrix}_2$	$\begin{bmatrix}3.857 & 1.647\end{bmatrix}_1$ $\begin{bmatrix}3.670 & 1.574\end{bmatrix}_2$

Table 3.4: DOF design: K_i and L_i for case **A**.

To justify the use of the estimated state feedback control structure, we have tested *the Static Output Feedback (SOF) controller for both optimization strategies described in Section 3.4.*

Note that if we consider a SOF control law $u(k) = \sum_{i=1}^{r} \alpha_i(y(k))K_i y(k)$, *where* $K_i \in \mathbb{R}^{m\times p}$, *such that the closed-loop system is given by* (2.66), *the optimization problems presented in Section 3.4 can be described similarly. This can be achieved by replacing the conditions in* (3.19), (3.22), *and* (3.23) *with* (2.67), (2.20), *and* (2.70), *respectively. Table 3.5 presents the optimal solutions for the SOF case.*

#	Obj. Func.	a	γ / Φ	λ	*Vol*
1	$\mathcal{F}(\lambda,\gamma)$	1.00	1003.03	0.99949	0.020
2	$\mathcal{F}(\lambda,\gamma)$	0.99	22.9985	0.99999	0.189
3	$\Phi(\phi_j)$	$--$	0.10711	0.99999	0.221

Table 3.5: SOF results with KNITRO: Optimal solutions for case **A**.

When $a = 1.00$, *the homogeneous expansion yields* $\lambda = 0.99949$. *Upon comparing this result with solution* #1 *in Table* (3.3), *it is evident that the proposed approach yields a faster transient response than the SOF.*

Regarding the admissible states, it is possible to estimate the volume of the polyhedron $R[Y,\mathbf{1}]$ *in the state-space by using the projection* $p_x(R[Y,\mathbf{1}])$. *For solution* #4 *presented in Table 3.3, the estimated volume obtained is* $Vol(p_x(R[Y,\mathbf{1}])) = 12.109$. *By comparing this result with solution* #3 *in the Table 3.5, it can be concluded that the proposed approach leads to a larger region of admissible states compared to the SOF.*

Case	n_y	Y
A	9	$\begin{bmatrix} -0.0450 & -0.1851 & 0.2962 & 0.1710 \\ 0.3548 & -0.0041 & -0.0161 & 0.1167 \\ 0.0210 & 0.0363 & 0.3359 & 0.0022 \\ 0.0171 & 0.0600 & 0.2364 & -1.7181 \\ 0.3578 & 0.1596 & -0.1921 & 0.4904 \\ 0.0782 & 1.0364 & 0.6017 & -1.4151 \\ -0.1679 & -1.6157 & 0.2364 & -0.5973 \\ -0.097 & -0.1731 & -1.7656 & 4.1788 \\ -0.3530 & 0.0126 & -0.0202 & -0.0117 \end{bmatrix}$

Table 3.6: KNITRO results: Polyhedron matrix $R[Y,\mathbf{1}]$.

Figures 3.1 and 3.2 show, respectively, the state and error trajectories plotted on the projections $p_{x_1,x_2}(R[Y,\mathbf{1}])$ *and* $p_{e_1,e_2}(R[Y,\mathbf{1}])$. *Four vertices of the polyhedron* $R[Y,\mathbf{1}]$ *are used as initial conditions. Here,* $x(0)$, $e(0)$ *and* $x(\infty)$, $e(\infty)$, *in* ***black***, *represent the initial conditions and the steady-states.*

Note that the trajectories of the state and error variables converge asymptotically to the origin without violating the constraints, once it is guaranteed that the polyhedral set $R[Y,\mathbf{1}]$ *is PI* λ*-contractive, such that* $R[Y,\mathbf{1}] \subseteq R[S,\mathbf{1}]$, *as shown Figures 3.1 and 3.2.*

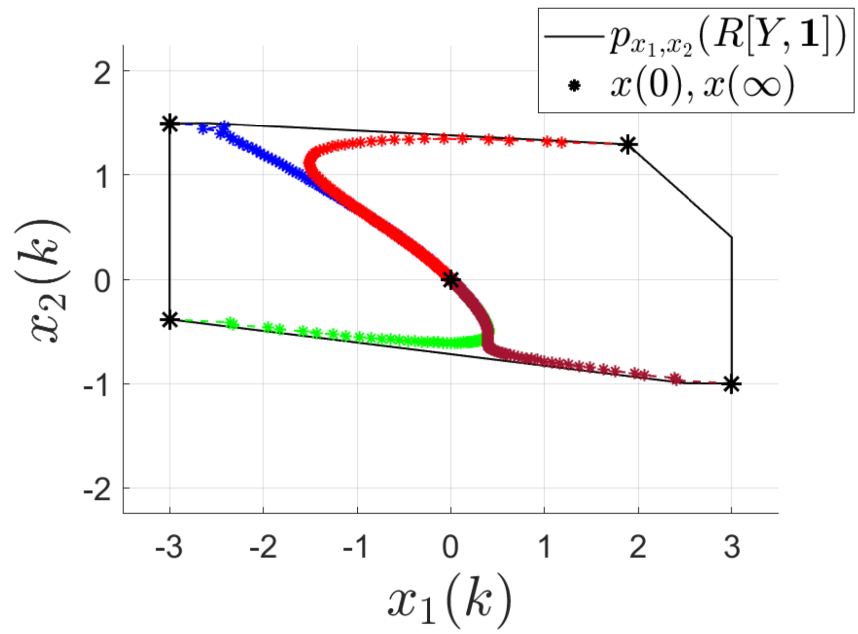


Figure 3.1: State Trajectories - Case **A**.

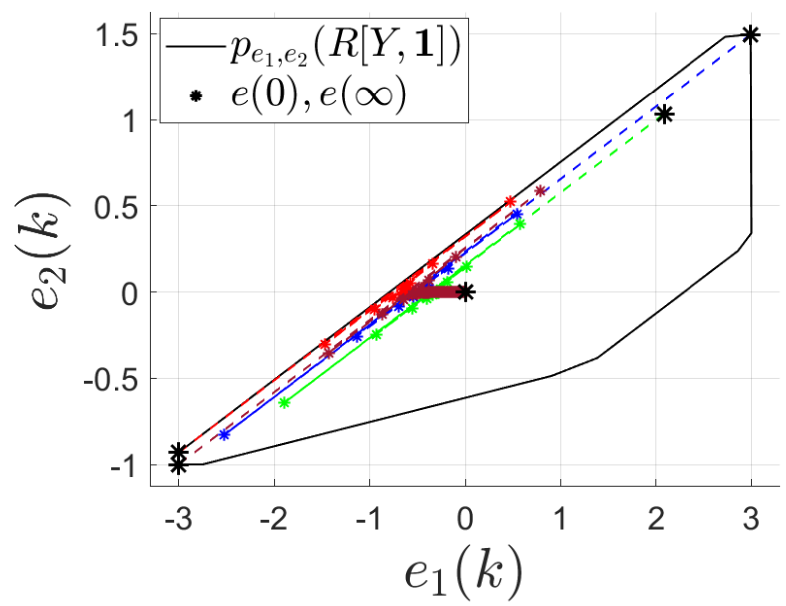


Figure 3.2: Error Trajectories - Case **A**.

Figure 3.3 represents the respective control signals $u(k)$. The control inputs $u(k) \in [-100,\ 100]$, $\forall k \geq 0$, such that the control constraints are satisfied.

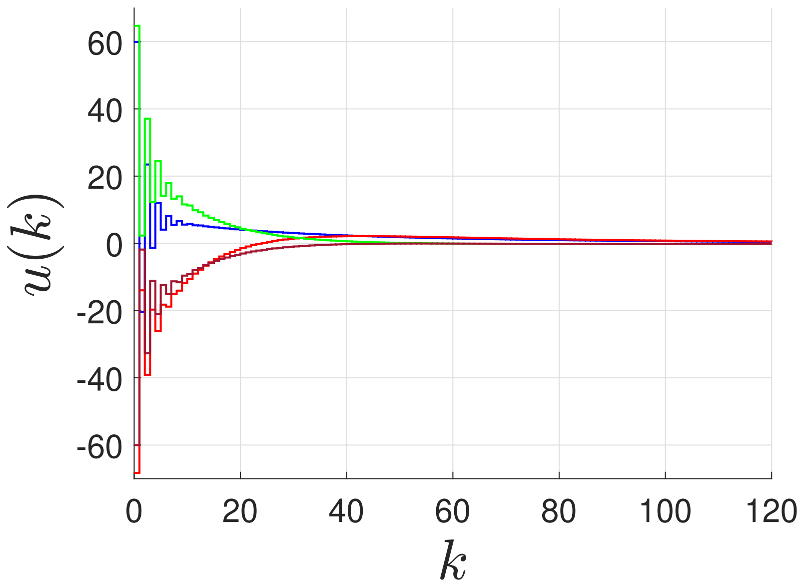


Figure 3.3: Control Trajectories - Case **A**.

Example 3.6.2 *Consider the nonlinear model of a stirred tank reactor (CSTR) studied in, e.g., [Ding, B. 2011, Ping, X. et al. 2021], where a discrete-time fuzzy T-S model was derived using a sample time $T = 0.05$ min. The matrices defining the model are given by:*

$$
\begin{aligned}
A_1 &= \begin{bmatrix} 0.8227 & -0.00168 \\ 6.1233 & 0.9367 \end{bmatrix}, A_2 = \begin{bmatrix} 0.9654 & -0.00182 \\ -0.6759 & 0.9433 \end{bmatrix}, \\
A_3 &= \begin{bmatrix} 0.8895 & -0.00294 \\ 2.9447 & 0.9968 \end{bmatrix}, A_4 = \begin{bmatrix} 0.8930 & -0.00062 \\ 2.7738 & 0.8864 \end{bmatrix}, \\
B_1 &= \begin{bmatrix} -0.000092 \\ 0.1014 \end{bmatrix}, B_2 = \begin{bmatrix} -0.000097 \\ 0.1016 \end{bmatrix}, \\
B_3 &= \begin{bmatrix} -0.000157 \\ 0.1045 \end{bmatrix}, B_4 = \begin{bmatrix} -0.000034 \\ 0.0986 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix},
\end{aligned}
\tag{3.60}
$$

subject to the state, error and control constraints: $x_1(k) \in [-0.5,\ 0.5] = [-\underline{x}_1,\ \overline{x}_1]$, $x_2(k) \in [-5,\ 5] = [-\underline{x}_2,\ \overline{x}_2]$, $e_1(k) \in [-0.5,\ 0.5] = [-\underline{e}_1,\ \overline{e}_1]$, $e_2(k) \in [-5,\ 5] = [-\underline{e}_2,\ \overline{e}_2]$ and $u(k) \in [-10,\ 10]$.

In [Ding, B. 2011, Ping, X. et al. 2021], the output matrix is $C = [0\ \ 1]$. Here, we use $C = [1\ \ 0]$ to exploit the more complicated case where the membership functions are expressed in terms of variables that must be estimated.

As in [Ping, X. et al. 2021], the premise variables are defined by:

$$
\begin{aligned}
z_1(k) &= 7.2 \times 10^{10} e^{\frac{-8750}{x_2(k)+350}}, \\
z_2(k) &= \begin{cases} \frac{3.6\times 10^{10}\left(e^{\frac{-8750}{x_2(k)+350}} - e^{\frac{-8750}{350}}\right)}{x_2(k)}, & x_2(k) \neq 0, \\ 0.0357, & x_2(k) = 0, \end{cases}
\end{aligned}
\tag{3.61}
$$

which are not measured directly because $y(k) = x_1(k)$.

The universe of discourse used to obtain this model is defined by:

$$
\begin{aligned}
\underline{\hat{x}}_1 &= -\underline{x}_1 - \overline{e}_1 = -0.5 - 0.5 = -1, \\
\overline{\hat{x}}_1 &= \overline{x}_1 + \underline{e}_1 = 0.5 + 0.5 = 1, \\
\underline{\hat{x}}_2 &= -\underline{x}_2 - \overline{e}_2 = -5 - 5 = -10, \\
\overline{\hat{x}}_2 &= \overline{x}_2 + \underline{e}_2 = 5 + 5 = 10,
\end{aligned}
\tag{3.62}
$$

where $\underline{\hat{x}}_i$ and $\overline{\hat{x}}_i$ are, respectively, the lower and upper bound of $\hat{x}_i$. By doing so, we guarantee that $\alpha(\hat{x}(k)) \in \Delta$, provided that the constraints on $x(k)$ and $e(k)$ are satisfied, which will be the case if a PI polyhedron is obtained by the proposed optimization approach. This choice implies a tightening of the constraints associated with the augmented space. A possible way to avoid this problem is to obtain the fuzzy T-S model-based on a larger region of validity.

In our approach, the impossibility of measuring all state variables that make up the membership functions is the reason that motivates the use of the T-S fuzzy observer. On the *other hand, in [Ping, X. et al. 2021], disturbances and bounded noise are also considered a source of uncertainty in the measurement of states.*

In this example, the conditions that characterize the existence of a symmetric PI λ-contractive polyhedron are used. In general, when the set of initial constraints is symmetric, it is natural that the polyhedron PI is also. In this case, the optimization problem for the expansion via vertices contains 63 *inequalities,* 7 *variables, and* 256 *bilinear terms more than for the homogeneous expansion.*

The results summarized in Table 3.7 show the values of variables γ and λ of the objective function $\mathcal{F}(\lambda,\gamma)$, as well as the solution found for $\Phi(\phi_j)$ with variables $\Phi(\phi_j) = \sum_{j=1}^{\bar{j}} \phi_j$ and λ associated.

Similar results to those presented in Example 3.6.1 are observed here. For the homogeneous expansion, the variable λ assumes the maximum value, and the variable γ assumes the minimum value when $a = 0.99$. The hypervolume associated with the polyhedron found by using the expansion via vertices is greater than that associated with homogeneous expansion, such that λ reaching its maximum value $\lambda = 0.99999$, as can be seen in Table 3.7.

Obj. Func.	a	γ / Φ	λ	Volume
$\mathcal{F}(\lambda,\gamma)$	1.00	4.73×10^4	0.89538	1.17×10^{-5}
$\mathcal{F}(\lambda,\gamma)$	0.99	13.65850	0.99999	0.830
$\Phi(\phi_j)$	$--$	1.198280	0.99999	1.465

Table 3.7: DOF results with KNITRO: Optimal solutions for case **B**.

Table 3.8 shows the gain matrices K_i and L_i, $i = 1,2,3,4$, obtained for this example using the second strategy. The results are associated with the corresponding matrix Y_s, shown in Table 3.9, defined by $n_y = 4$ rows.

Case	K_i	L_i^T
B	$\begin{bmatrix}2.760 & 1.148\end{bmatrix}_1$	$\begin{bmatrix}1.010 & -39.538\end{bmatrix}_1$
	$\begin{bmatrix}10.221 & 1.101\end{bmatrix}_2$	$\begin{bmatrix}1.009 & -40.017\end{bmatrix}_2$
	$\begin{bmatrix}7.964 & 1.066\end{bmatrix}_3$	$\begin{bmatrix}0.978 & -38.361\end{bmatrix}_3$
	$\begin{bmatrix}7.545 & 1.131\end{bmatrix}_4$	$\begin{bmatrix}0.929 & -35.769\end{bmatrix}_4$

Table 3.8: DOF design: K_i and L_i for case **B**.

*Figures 3.4 and 3.5 show, respectively, the state and error trajectories plotted on the projections $p_{x_1,x_2}(\mathcal{S}[Y_s,\mathbf{1}])$ and $p_{e_1,e_2}(\mathcal{S}[Y_s,\mathbf{1}])$. Four vertices of the polyhedron $\mathcal{S}[Y_s,\mathbf{1}]$ are used as initial conditions. The initial conditions and the steady-states are represented in **black**.*

*Figure 3.6 depicts the respective control signals $u(k)$. Dashed **black** lines represent the upper and lower bounds.*

Case	n_y	Y_s
B	4	$\begin{bmatrix} 1.165 & -0.078 & -8.541 & 0.145 \\ 1.333 & -0.274 & 6.304 & 0.115 \\ -2.300 & 0.103 & 11.978 & 0.288 \\ -6.474 & -0.113 & -4.754 & -0.123 \end{bmatrix}$

Table 3.9: KNITRO results: Polyhedron matrix $\mathcal{S}[Y_s, \mathbf{1}]$.

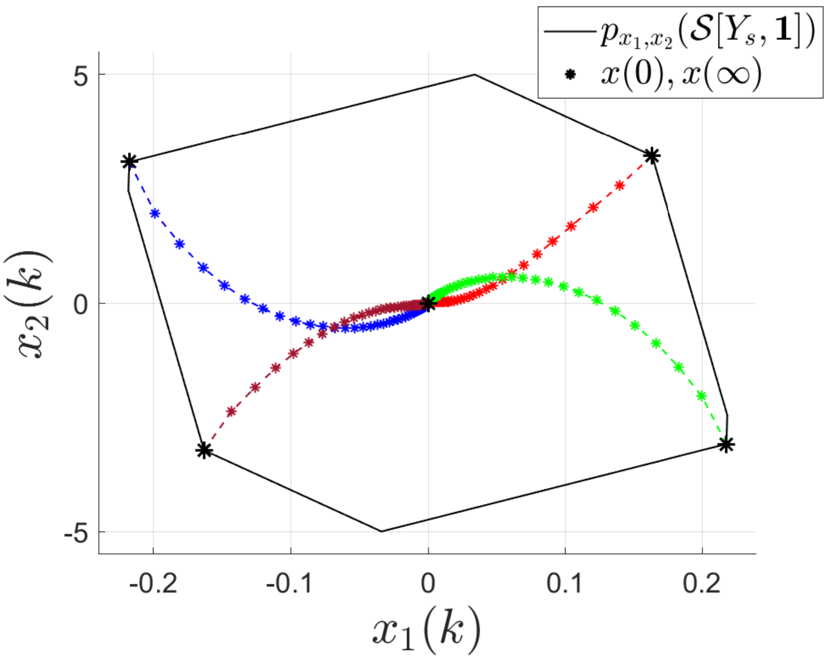


Figure 3.4: State trajectories - Case **B**.

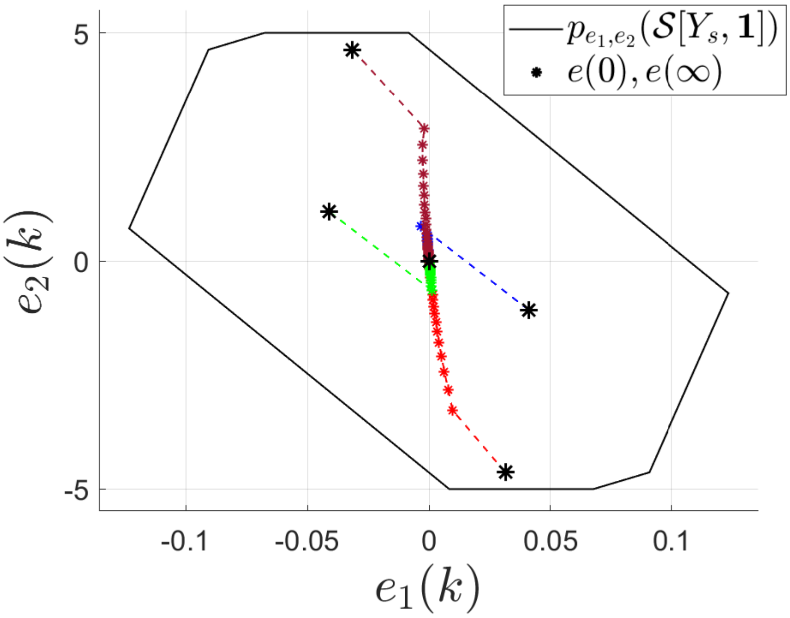


Figure 3.5: Error trajectories - Case **B**.

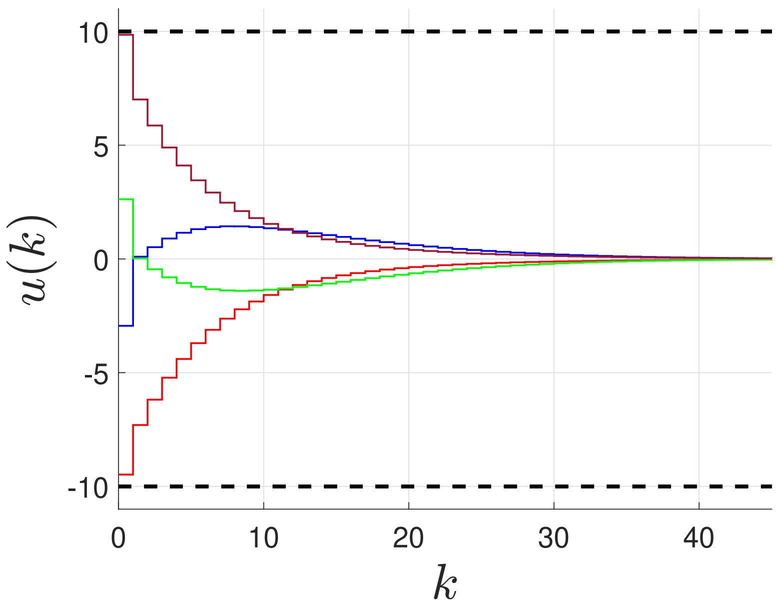


Figure 3.6: Control trajectories - Case **B**.

Figure 3.7 depicts the difference between $\alpha_i(z)$ *and* $\alpha_i(\hat{z})$*, associated with the trajectory in* ***red*** *of the Figures 3.4 and 3.5. Given that* $\alpha_i(z) - \alpha(\hat{z}) \to 0$*, it is possible to guarantee that the convergence condition for the fuzzy observer is satisfied.*

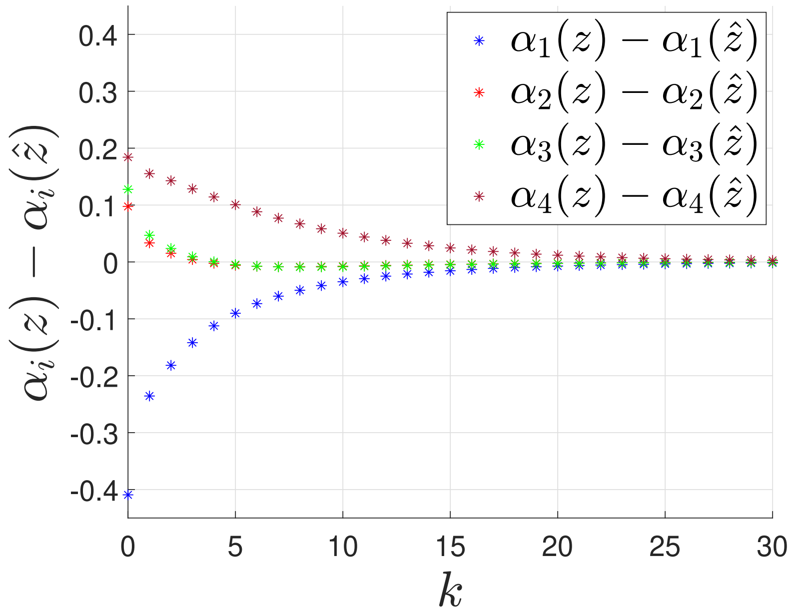


Figure 3.7: Difference between membership functions and their estimates - Case **B**.

In [Ping, X. et al. 2021], an invariant set is computed in the first stage, and then the control action is generated through the online solution of an optimization problem. The advantage of our approach is that we compute offline and simultaneously the invariant set and the corresponding PDC controller, which requires much less online computing effort.

Example 3.6.3 *Consider the following discrete-time nonlinear system [Tanaka, K. et al. 1997, Wang, H.O., & Tanaka, K. 2004, Song, W., & Liang, J. 2013]:*

$$\begin{aligned}
x_1(k+1) &= (1-T\sigma)x_1(k) + T\sigma x_2(k) + Tu(k) \\
x_2(k+1) &= T\rho x_1(k) + (1-T)x_2(k) - Tx_1(k)x_3(k) \\
x_3(k+1) &= Tx_1(k)x_2(k) + (1-T\beta)x_3(k) \\
y(k) &= x_2(k)
\end{aligned} \tag{3.63}$$

where $T = 0.01s$, $\sigma = 10$, $\rho = 28$ *and* $\beta = 8/3$, *subjected to the state, error and control constraints:* $x_1(k) \in [-2,\ 2]$, $x_2(k) \in [-2,\ 2]$, $x_3(k) \in [-1,\ 2]$, $e_1(k) \in [-2,\ 2]$, $e_2(k) \in [-2,\ 2]$, $e_3(k) \in [-2,\ 2]$ *and* $u(k) \in [-250,\ 250]$.

The nonlinear system (3.63) *is the discrete-time representation of the Lorenz system based on Euler's method and it can be represented as a fuzzy T-S model with the following matrices:*

$$A_1 = \begin{bmatrix} 1-T\sigma & T\sigma & 0 \\ T\rho & 1-T & -4T \\ 0 & 4T & 1-T\beta \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1-T\sigma & T\sigma & 0 \\ T\rho & 1-T & 4T \\ 0 & -4T & 1-T\beta \end{bmatrix}, \tag{3.64}$$

$$B_1 = B_2 = \begin{bmatrix} T \\ 0 \\ 0 \end{bmatrix},\ C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.$$

The premise variable and its estimate are given by:

$$z_1(k) = x_1(k),\ \hat{z}_1(k) = \hat{x}_1(k),$$

which are not directly measured because $y(k) = x_2(k)$.

The universe of discourse used to obtain this model is defined by:

$$\begin{aligned}
\underline{\hat{x}}_1 &= -\underline{x}_1 - \overline{e}_1 = -2-2 = -4, \\
\overline{\hat{x}}_1 &= \overline{x}_1 + \underline{e}_1 = 2+2 = 4 \\
\underline{\hat{x}}_2 &= -\underline{x}_2 - \overline{e}_2 = -2-2 = -4, \\
\overline{\hat{x}}_2 &= \overline{x}_2 + \underline{e}_2 = 2+2 = 4, \\
\underline{\hat{x}}_3 &= -\underline{x}_3 - \overline{e}_3 = -1-2 = -3, \\
\overline{\hat{x}}_3 &= \overline{x}_3 + \underline{e}_2 = 2+2 = 4.
\end{aligned} \tag{3.65}$$

In this example, the initial constraint polyhedron is asymmetric. Thus, the conditions for the general case are used. The number of vertices that define the polyhedron $R[S,\mathbf{1}]$ *is* $\overline{j} = 64$. *Therefore, the optimization problem for homogeneous expansion contains more* 72 *equations and* 81 *variables, fewer* 756 *inequalities, and* 4608 *bilinear terms than the one for the expansion via vertices.*

The results summarized in Table 3.10 show the values of variables γ *and* λ *of the* *objective function $\mathcal{F}(\lambda,\gamma)$, as well as the solution found for $\Phi(\phi_j)$ with variables $\Phi(\phi_j) = \sum_{j=1}^{\bar{j}} \phi_j$ and λ associated.*

Similar results to those presented in Examples 3.6.1 and 3.6.2 are observed here. As seen in Table 3.10, the hypervolume associated with the polyhedron found by using the expansion via vertex is greater than that associated with homogeneous expansion.

Obj. Func.	a	γ / Φ	λ	Volume
$\mathcal{F}(\lambda,\gamma)$	1.00	5208.65	0.98465	0.023
$\mathcal{F}(\lambda,\gamma)$	0.99	1.02773	0.99999	1.132
$\Phi(\phi_j)$	$--$	9.76696	0.99999	9.921

Table 3.10: DOF results with KNITRO: Optimal solutions for case **B**.

Table 3.11 shows the gain matrices K_i and L_i, $i = 1,2$, obtained for this example using the second strategy. The results are associated with the corresponding matrix Y in Table 3.12. The results were obtained with a symmetric polyhedron with $n_y = 12$ rows.

Case	K_i	L_i^T
B	$\begin{bmatrix}32.209 & 20.519 & -4.751\end{bmatrix}_1$ $\begin{bmatrix}32.212 & 20.717 & 4.591\end{bmatrix}_2$	$\begin{bmatrix}1.751 & 1.616 & 0.033\end{bmatrix}_1$ $\begin{bmatrix}1.749 & 1.613 & -0.047\end{bmatrix}_2$

Table 3.11: DOF design: K_i and L_i for case **B**.

Case	n_y	Y
B	12	$\begin{bmatrix} 0.491 & 0.573 & 0.128 & 0.597 & -0.575 & 0.567 \\ 0.488 & 0.571 & -1.256 & 0.590 & -0.570 & 0.012 \\ -1.255 & 0.278 & 0.021 & -1.727 & 1.459 & -0.013 \\ -2.429 & -2.819 & 0.133 & 0.088 & 1.457 & -0.086 \\ -2.615 & -3.021 & 0.134 & -0.431 & -1.718 & -0.067 \\ 0.554 & -0.157 & -0.005 & -0.094 & -0.548 & -0.002 \\ -0.128 & 0.538 & 0.094 & -1.744 & 2.377 & -0.014 \\ 0.380 & 0.598 & -0.045 & 0.853 & -1.005 & 0.028 \\ -12.245 & -14.007 & 0.265 & -14.606 & 14.173 & 0.390 \\ 0.487 & 0.568 & 1.229 & 0.585 & -0.571 & -1.089 \\ 0.414 & 0.254 & 0.008 & 0.151 & 0.883 & 0.003 \\ -0.972 & -3.186 & 0.002 & -1.400 & 1.136 & -0.024 \end{bmatrix}$

Table 3.12: KNITRO results: Polyhedron matrix $\mathcal{S}[Y_s, \mathbf{1}]$.

*The Figures 3.8 e 3.9 show state and estimation error trajectories plotted on the projections $p_{x_1,x_2,x_3}(R[Y,\mathbf{1}])$ and $p_{e_1,e_2,e_3}(R[Y,\mathbf{1}])$. Four vertices of the polyhedron $R[Y,\mathbf{1}]$ are chosen as initial conditions. The initial conditions and the steady-states are represented in **black**.*

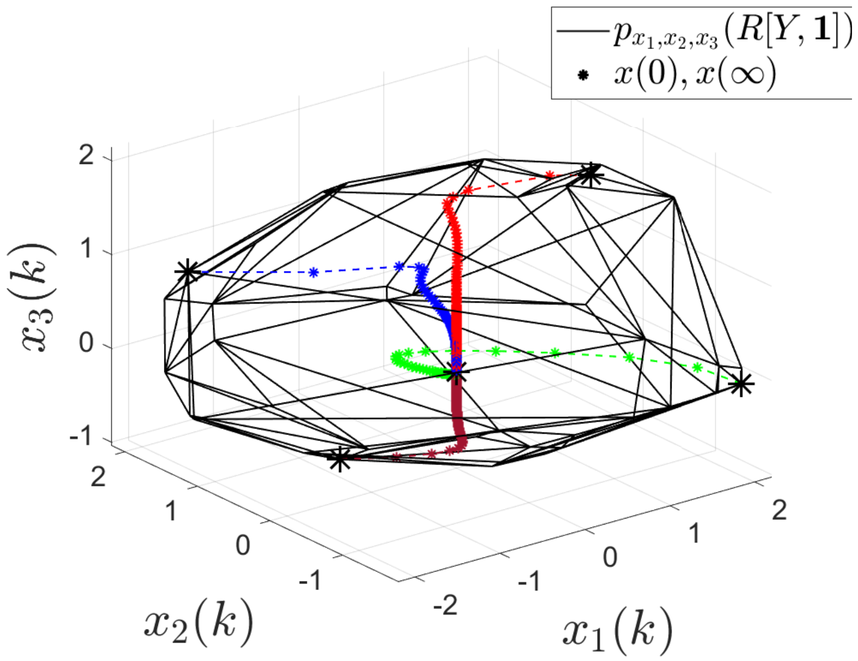


Figure 3.8: State trajectories - Case **B**.

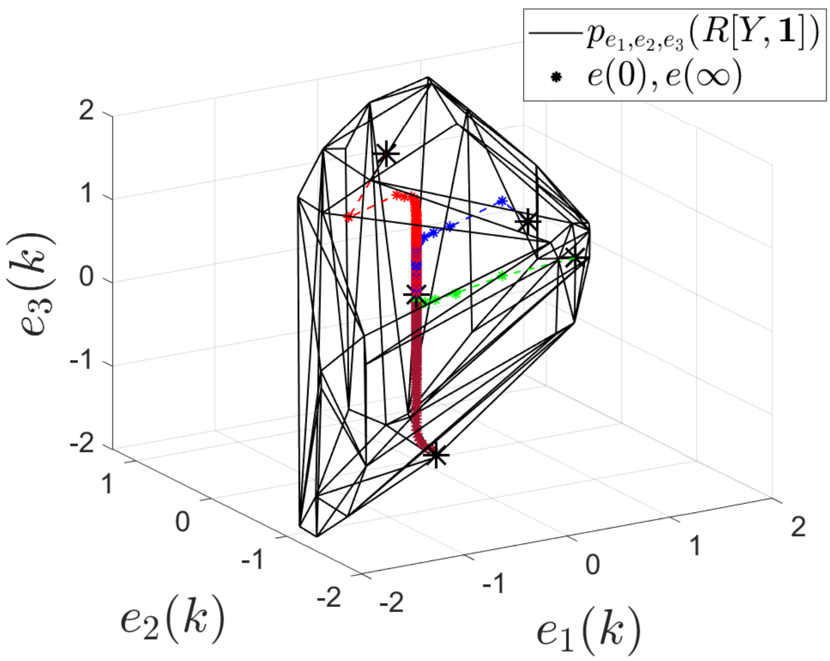


Figure 3.9: Error trajectories - Case **B**.

Figure 3.10 is depicted the respective control signals $u(k)$*. The control inputs* $u(k) \in [-250,\ 250]$*,* $\forall k \geq 0$*, such that the control constraints are satisfied.*

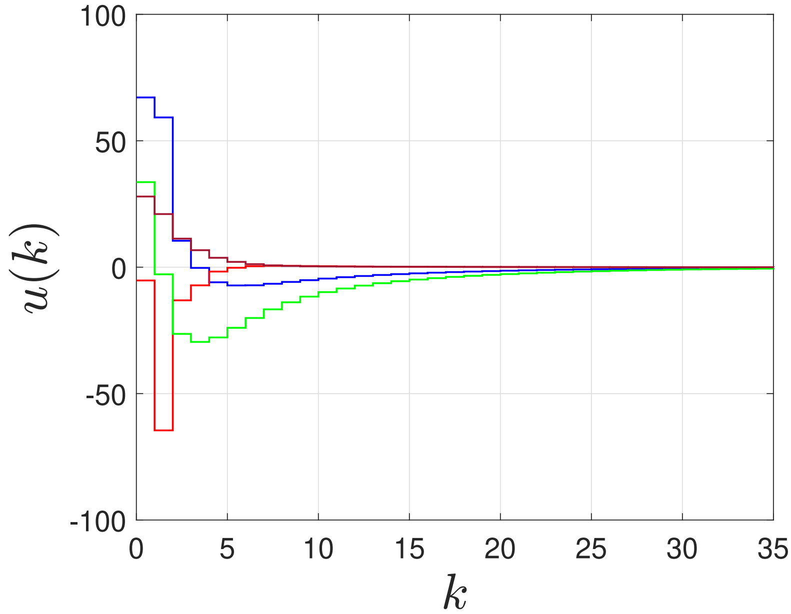


Figure 3.10: Control trajectories - Case **B**.

Figure 3.11 depicts the difference between $\alpha_i(x)$ *and* $\alpha_i(\hat{x})$ *associated with the trajectory in* ***blue*** *of the Figures 3.8 and 3.9. As in Example 3.6.2, the convergence condition for the fuzzy observer is satisfied.*

Given that the solution to the optimization problem is feasible, the conditions presented in Sections 3.2 and 3.3 are sufficient to guarantee, respectively, the existence of a PI λ*-contractive polyhedron and the state and control constraints in the augmented state-space. These conditions do not use information from the membership functions or their estimates to calculate gains and the PI set. Therefore, it is possible to guarantee that the presented solutions are valid as long as* $\alpha(x) \in \Delta$ *and* $\alpha(\hat{x}) \in \Delta$*.*

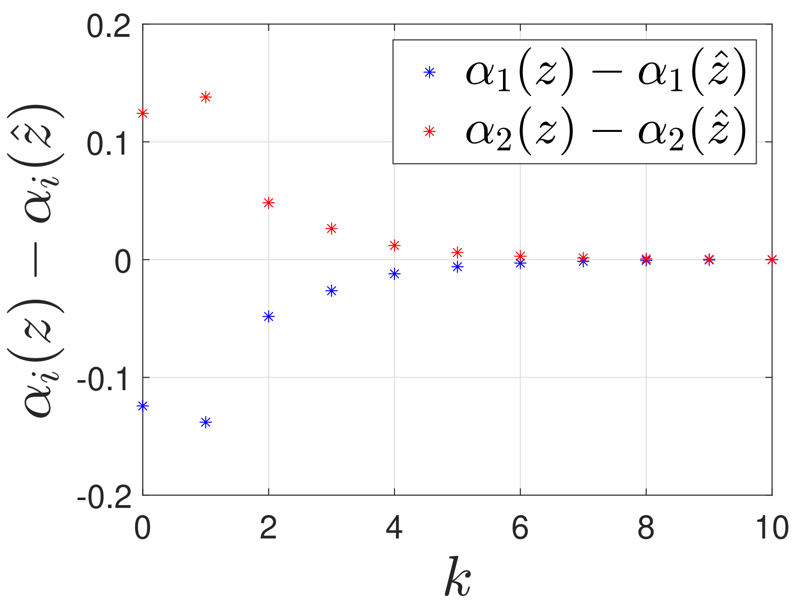


Figure 3.11: Difference between membership functions and their estimates - Case **B**.

3.7 Conclusions

In this chapter, a systematic design method of observer-based output feedback controllers for fuzzy T-S systems was proposed. Sufficient conditions were established for a polyhedron to be positively invariant w.r.t. to an augmented state-space. These conditions were translated into a bilinear optimization problem whose solution delivers the controller gains and a PI λ-contractive polyhedron guaranteeing the satisfaction of the constraints. Then, the constrained regulator problem is solved because as $R[Y,\mathbf{1}]$ is PI λ-contractive (or $\mathcal{S}[Y_s,\mathbf{1}]$, in the symmetric case), the trajectory of the augmented state remain inside $R[Y,\mathbf{1}]$ and $x_a(k) \to 0$ as $k \to \infty$.

Two types of observers found in the literature were presented: the first considers the membership functions dependent only on the system output (case **A**); the second, in turn, refers to the case **B**, where these functions can be associated with any state variables. In case **B**, for the invariance conditions to be valid, it is necessary to reduce the universe of discourse associated with the state variables or associate the local models to the universe of discourse of the estimated state variables.

Two optimization strategies were used here. The design strategy called homogeneous expansion presents a lower numerical complexity compared to the expansion via vertices, in addition to allowing a larger set of different solutions that allow establishing a compromise between the speed of convergence of the augmented state trajectories and the size of the set $R[Y,\mathbf{1}]$ (or $\mathcal{S}[Y_s,\mathbf{1}]$, for symmetric shaped polyhedra). On the other hand, the expansion via vertices offers greater design flexibility, which implies solutions with larger PI polyhedra compared to homogeneous expansion.

Conditions were established for general and symmetric polyhedra. When the validity region Ω is characterized through a symmetric polyhedral set, it is natural that the PI polyhedron to be obtained is also symmetric. Otherwise, the invariance conditions for general polyhedra are used.

Chapter 4

Output Feedback Constant Reference Tracking

In this chapter, the design of an Integral-Proportional (I-P)-Like controller for fuzzy T-S systems based on dynamic output feedback is presented. The control objective is eliminating the error between a piecewise constant reference and the system output. This approach draws inspiration from the work of [Figueiredo, L.S. et al. 2020], who dealt with the tracking problem for LPV systems under saturating actuators. Linear parameter-varying (LPV) models describe complex systems' dynamics with high fidelity. If the time-varying parameters of the LPV system are computed from measurable system variables, then the system is called a quasi-LPV system. An example of such a system can be found in the Takagi-Sugeno approach, where membership functions can be analytically computed from the states or output of the system. In [Lopes, A.N. et al. 2020], an extension of the technique proposed by [Figueiredo, L.S. et al. 2020] to fuzzy T-S systems based on output feedback was presented. Nevertheless, [Lopes, A.N. et al. 2020] focuses solely on the scenario where the membership functions depend on measured variables, thereby leaving the challenge of tracking fuzzy T-S systems with membership functions that rely on non-accessible states unresolved.

As presented in Chapter 3, the design methodology involves solving a bilinear optimization problem to calculate the controller and observer gains and the corresponding positive invariant set. For the designed controller, a PDC (I-P)-like control law with a feedforward term is used by its capability to eliminate the tracking error and enhance the system's tracking response to the desired set-point, when compared to the standard I-P controller [Åström, K. J., & Hägglund, T. 2006].

The proposed approach explores the concept of robust positive invariance to design a local stabilizing I-P tracking controller for fuzzy T-S systems. Analogous to the conditions for robust invariance of polyhedral sets, sufficient conditions are established for the invariance of polyhedral sets in presence of a piecewise constant reference $r(k)$, which can be interpreted as a bounded disturbance. A similar technique for constrained continuous-time linear systems is presented in [dos Santos, G.F. et al. 2023].

For the purpose of tracking, two optimization strategies proposed by [dos Santos, G.F. et al. 2023] are employed here. The first strategy aims to maximize the set of admissible reference signals, which means allowing the system to effectively track a wide range of set-points. This approach enhances the controller's flexibility to adapt to different operating conditions. The second strategy focuses on minimizing the bounds of the integral of the tracking error. By reducing these limits, the controller aims to achieve a faster transient response, as smaller integral state bounds indicate a more efficient control action. This strategy promotes improved tracking performance by minimizing the accumulated error during the system's operation.

We consider two situations: in the first one (case **A**), the membership functions depend only on the output, while in the second one (case **B**), they depend on unmeasured states [Tanaka, K. et al. 1997, Wang, H.O., & Tanaka, K. 2004]. Similar to the presentation in Chapter 3, the invariance conditions for both general and symmetric polyhedra are presented.

The control design methodology involves formulating the constrained tracking problem as an optimization problem subject to constraints. The conditions that guarantee local asymptotic tracking for a piecewise constant reference and the augmented state (state + estimation error + integral of the tracking error) and control constraints are incorporated as constraints of the optimization problem.

Similarly to that presented in Chapter 3, the solution to the optimization problem is obtained offline to the control implementation and provides the gains for the controller and the observer, as well as the associated PI λ-contractive polyhedron. Moreover, the dynamics of the state, the estimation error and the integral of the tracking error are represented from a model described in the augmented state-space and the constraints that compose the optimization problem are formulated according to this representation.

4.1 Preliminaries

Consider a discrete-time nonlinear system given by:

$$\begin{aligned} x(k+1) &= f(x(k),u(k)) \\ y(k) &= Cx(k), \end{aligned} \tag{4.1}$$

where $f(\cdot):\mathbb{R}^{n+m}\to\mathbb{R}^n$, $x(k)\in\mathbb{R}^n$ is the state vector, $u(k)\in\mathbb{R}^m$ is the control input, $y(k)\in\mathbb{R}^p$ the measured output and $C\in\mathbb{R}^{p\times n}$ is the output matrix.

In a compact region of the state-space, the system (4.1) can be expressed locally from the fuzzy T-S model with r rules in the form (see, e.g., [Wang, H.O., & Tanaka, K. 2004, Wang, H.O. et al. 1996]):

$$\begin{aligned} x(k+1) &= \sum_{i=1}^{r} \alpha_i(x(k))(A_i x(k) + B_i u(k)) \\ y(k) &= Cx(k), \end{aligned} \tag{4.2}$$

where $\alpha_i(x(k))$ represent the membership functions such that the vector $\alpha(x(k))$ belongs to the standard simplex $\Delta\in\mathbb{R}^r$, defined as:

$$\Delta = \{\alpha(x(k))\in\mathbb{R}^r : \sum_{i=1}^{r}\alpha_i(x(k)) = 1,\ \alpha_i(x(k))\geq 0\}. \tag{4.3}$$

The system (4.2) is subject to state and control constraints, which are represented by closed polyhedral sets containing the origin as follows:

$$\mathcal{X} = R[S_x, \mathbf{1}] = \{x(k) \in \mathbb{R}^n : S_x x(k) \leq \mathbf{1}\},\ S_x \in \mathbb{R}^{g_x \times n}, \tag{4.4}$$

$$\mathfrak{U} = R[S_u, \mathbf{1}] = \{u(k) \in \mathbb{R}^m : S_u u(k) \leq \mathbf{1}\},\ S_u \in \mathbb{R}^{g_u \times m}. \tag{4.5}$$

According to the formulation presented in the following, due to the need to use the observer in the estimation process and for tracking purposes, we define a set of constraints for the estimation error $e(k) \in \mathbb{R}^n$ and an additional set of constraints for the integral of the tracking error $v(k) \in \mathbb{R}^p$:

$$\mathcal{E} = R[S_e, \mathbf{1}] = \{e(k) \in \mathbb{R}^n : S_e e(k) \leq \mathbf{1}\}, \tag{4.6}$$

$$\mathcal{V} = R[S_v, \mathbf{1}] = \{v(k) \in \mathrm{R}^p : S_v v(k) \leq \mathbf{1}\}, \tag{4.7}$$

where $S_e \in \mathbb{R}^{g_e \times n}$ and $S_v = \begin{bmatrix} X_{I_1} \\ -X_{I_2} \end{bmatrix} \in \mathbb{R}^{g_v \times p}$, $X_{I_i} = diag\{\varepsilon_{ij}^{-1}\} \in \mathbb{R}^{p \times p}$, with $\varepsilon_{ij}^{-1} > 0$ for $i = 1,2,\ j = 1,\cdots,p$.

Therefore, in the augmented state-space, the constraint polyhedron can be defined as follows:

$$R[S, \mathbf{1}] = \{x_a(k) \in \mathrm{R}^{n_a} : S x_a(k) \leq \mathbf{1}\},\ S = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_e & 0 \\ 0 & 0 & S_v \end{bmatrix}, \tag{4.8}$$

where $x_a(k)^{\mathrm{T}} = [x(k)\ e(k)\ v(k)] \in \mathbb{R}^{2n+p}$ is the augmented state, $S \in \mathrm{R}^{g_a \times (2n+p)}$, with $g_a = g_x + g_e + g_v$.

Moreover, we assume that $y(k)$ must track a set-point reference signal $r(k) \in \mathbb{R}^p$, where $r(k)$ is bounded by an asymmetric hyperrectangle described by the set

$$\mathfrak{R} = R[S_r, \mathbf{1}] = \{r(k) \in \mathrm{R}^p : S_r r(k) \leq \mathbf{1}\}, \tag{4.9}$$

where $S_r = \begin{bmatrix} X_{R_1} \\ -X_{R_2} \end{bmatrix} \in \mathbb{R}^{g_r \times p}$, $X_{R_i} = diag\{\rho_{ij}^{-1}\} \in \mathbb{R}^{p \times p}$, with $\rho_{ij}^{-1} > 0$ for $i = 1,2,\ j = 1,\cdots,p$.

If symmetric polyhedra are considered, the set of constraints in the augmented state-space can be defined as

$$\mathcal{S}[S_s, \mathbf{1}] = \{x_a(k) \in \mathbb{R}^{2n+p} : |S_s x_a(k)| \leq \mathbf{1}\},\ S_s = \begin{bmatrix} S_x^s & 0 & 0 \\ 0 & S_e^s & 0 \\ 0 & 0 & S_v^s \end{bmatrix}, \tag{4.10}$$

where $S_s \in \mathrm{R}^{g_a^s \times (2n+p)}$, with $g_a^s = g_x^s + g_e^s + g_v^s$, such that

$$\begin{aligned} \mathcal{X}_s &= \mathcal{S}[S_x^s, \mathbf{1}] = \{x(k) \in \mathbb{R}^n : |S_x^s x(k)| \leq \mathbf{1}\},\ S_x^s \in \mathbb{R}^{g_x^s \times n}, && (4.11)\\ \mathfrak{U}_s &= \mathcal{S}[S_u^s, \mathbf{1}] = \{u(k) \in \mathbb{R}^m : |S_u^s u(k)| \leq \mathbf{1}\},\ S_u^s \in \mathbb{R}^{g_u^s \times p}, && (4.12)\\ \mathcal{E}_s &= \mathcal{S}[S_e^s, \mathbf{1}] = \{e(k) \in \mathbb{R}^n : |S_e^s e(k)| \leq \mathbf{1}\},\ S_e^s \in \mathbb{R}^{g_e^s \times n}, && (4.13)\\ & && (4.14) \end{aligned}$$

and

$$\begin{aligned} \mathcal{V}_s &= \mathcal{S}[S_v^s, \mathbf{1}] = \{v(k) \in \mathbb{R}^p : |S_v^s v(k)| \leq \mathbf{1}\},\ S_v^s \in \mathbb{R}^{g_v^s \times p}, && (4.15)\\ \mathfrak{R}_s &= \mathcal{S}[S_r^s, \mathbf{1}] = \{r(k) \in \mathrm{R}^p : |S_r^s r(k)| \leq \mathbf{1}\},\ S_r^s \in \mathbb{R}^{g_r^s \times p}, && (4.16) \end{aligned}$$

where $S_v^s = X_I$, $X_I = diag\{\varepsilon_i^{-1}\} \in \mathbb{R}^{p \times p}$, such that $g_v^s = p$, $\varepsilon_i^{-1} > 0$, $i = 1, \cdots, p$ and $S_r^s = X_R$, $X_R = diag\{\rho_i^{-1}\} \in \mathbb{R}^{p \times p}$, with $g_r^s = p$, $\rho_i^{-1} > 0$, $i = 1, \cdots, p$.

4.2 Positive Invariance of General Polyhedra

4.2.1 Case A

In this case, we consider that $\alpha_i(x(k))$ is, by hypothesis, given by $\alpha_i(y(k))$. A fuzzy T-S observer can be expressed with r rules in the form:

$$\begin{aligned} \hat{x}(k+1) &= \sum_{i=1}^{r} \alpha_i(y(k))[A_i \hat{x}(k) + B_i u(k) + L_i(y(k) - \hat{y}(k))],\\ \hat{y}(k) &= C\hat{x}(k). \end{aligned} \tag{4.17}$$

where $\hat{x}(k) \in \mathbb{R}^n$ is the estimated state vector, $y(k) \in \mathbb{R}^p$ is the measured output, $\hat{y}(k) \in \mathbb{R}^p$ $L_i \in \mathbb{R}^{n \times p}$ is the estimated output and $\alpha_i(\cdot)$ are the same membership functions that define the model (4.2).

The estimation error is defined by $e(k) = x(k) - \hat{x}(k)$ and its dynamics is:

$$e(k+1) = \sum_{i=1}^{r} \alpha_i(y(k))(A_i - L_i C)e(k), \tag{4.18}$$

obtained from Equations (4.2) and (4.17).

Moreover, the integral of the tracking error $v(k) \in \mathbb{R}^p$ is defined as

$$v(k+1) = r(k) - Cx(k+1) + v(k), \tag{4.19}$$

based on the topology of an integral action presented in [Figueiredo, L.S. et al. 2020], where $y(k+1) = Cx(k+1) \in \mathbb{R}^p$ and the reference sign is considered constant during the transient response, with $r(k) \equiv r(k+1)$. The dynamics of the tracking error integral can be obtained by substituting (4.2) in (4.19).

Now, consider the following (I-P)-Like control law, according to the PDC scheme given by:

$$u(k) = -\sum_{i=1}^{r} \alpha_i(y(k)) \Big(K_{P_i}\hat{x}(k) - K_{I_i}v(k) - K_{R_i}r(k) \Big), \tag{4.20}$$

where $K_{P_i} \in \mathbb{R}^{m\times n}$, $K_{I_i} \in \mathbb{R}^{m\times p}$ and $K_{R_i} \in \mathbb{R}^{m\times p}$, such that the corresponding augmented closed-loop system is:

$$\begin{aligned} x_a(k+1) &= \sum_{i=1}^{r} \alpha_i^2(y(k))(G_{ii}^a x_a(k) + R_{ii}r(k)) + \\ 2\sum_{i=1}^{r}\sum_{i<j}^{r} &\alpha_i(y(k))\alpha_j(y(k)) \left[\left(\frac{G_{ij}^a + G_{ji}^a}{2}\right) x_a(k) + \left(\frac{Rij+Rji}{2}\right) r(k) \right], \end{aligned} \tag{4.21}$$

with

$$G_{ij}^a = \begin{bmatrix} A_i - B_iK_{P_j} & B_iK_{P_j} & B_iK_{I_j} \\ 0 & A_i - L_iC & 0 \\ -C(A_i - B_iK_{P_j}) & -CB_iK_{P_j} & I - CB_iK_{I_j} \end{bmatrix} \text{ and } R_{ij} = \begin{bmatrix} B_iK_{R_j} \\ 0 \\ I - CB_iK_{R_j} \end{bmatrix}, \tag{4.22}$$

and $x_a(k)^{\mathrm{T}} = [x(k)\ e(k)\ v(k)] \in \mathrm{R}^{2n+p}$ is the augumented state vector, $x(k) \in \mathbb{R}^n$ is the state vector, $e(k) \in \mathbb{R}^n$ is the estimation error, $r(k) \in \mathbb{R}^p$ is the reference, and the integral of the tracking error $v(k) \in \mathbb{R}^p$.

Note that a standard I-P controller has the structure $u(k) = -\sum_{i=1}^{r} \alpha_i(y(k)) \Big(K_{P_i}(y(k) - r(k)) - K_{I_i}v(k) \Big)$, which implies that $K_{P_i} = -K_{R_i} \in \mathbb{R}^{m\times p}$. On the other hand, the controller proposed in (4.20) contains a feedforward term with $K_{P_i} \neq -K_{R_i}$.

According to [Åström, K. J., & Hägglund, T. 2006], the feedforward term has a anticipatory effect since it allows the control actions to be executed before any errors are generated due to set-point changes. As a result, controllers that incorporate feedforward tend to exhibit better transient performance. On the other hand, feedback of the estimated state results generally in larger sets of admissible states associated with them than static output feedback control, as discussed in Chapter 3. Therefore, the choice of the proposed control structure is supported by the above facts and results presented in [dos Santos, G.F. et al. 2023].

The linear control constraints w.r.t. system (4.21)-(4.22) can be expressed from (4.5) and (4.20) by:

$$\begin{aligned} S_u u(k) = -S_u &\left[\sum_{i=1}^{r} \alpha_i(y(k))(K_{P_i}x(k) - K_{P_i}e(k) - K_{I_i}v(k) - K_{R_i}r(k)) \right] = \\ &\sum_{i=1}^{r} \alpha_i(y(k)) S_u(K_i^a x_a(k)\ +\ K_{R_i}r(k)) \leq \mathbf{1}, \end{aligned} \tag{4.23}$$

such that

$$R[S_u K_i^a, \mathbf{1}] = \{x_a(k) \in \mathbb{R}^{2n+p} : S_u K_i^a x_a(k) \leq\ \mathbf{1}\} \tag{4.24}$$

and

$$R[S_u K_{R_i}, \mathbf{1}] = \{r(k) \in \mathbb{R}^p : S_u K_{R_i} r(k) \leq \mathbf{1}\}, \tag{4.25}$$

where $S_u \in \mathbb{R}^{g_u \times m}$, $K_i^a = [-K_{P_i}\ K_{P_i}\ K_{I_i}]$, $K_i^a \in \mathbb{R}^{m\times(2n+p)}$ and $K_{R_i} \in \mathbb{R}^{m\times p}$.

Let $\Omega \subseteq \mathbb{R}^{2n+p}$ denote the region of validity of the fuzzy T-S model in the augmented state-space, and $R[S,\mathbf{1}]$, $R[S_u K_i^a,\mathbf{1}]$, $R[S_u K_{R_i},\mathbf{1}]$ be polyhedral sets. The intersection $\Omega \cap R[S,\mathbf{1}] \cap (R[S_u K_i^a,\mathbf{1}] \oplus R[S_u K_{R_i},\mathbf{1}])$ forms a compact polyhedron containing the origin. The possibility of imposing trajectories of $x_a(k)$ within a given polyhedral set $R[Y,\mathbf{1}] \subset \Omega$ w.r.t. closed-loop system (4.21)-(4.22) can be achieved from the Definition (2.4.1).

Then, our objective becomes to obtain the matrices $K_{P_i},\ K_{I_i},\ K_{R_i},\ L_i,\ i = 1,\cdots,r$ and a polyhedral set

$$R[Y,\mathbf{1}] = \{x_a(k) \in \mathbb{R}^{2n+p} : Y x_a(k) \leq \mathbf{1}\}, \tag{4.26}$$

contained in the set of constraints $\Omega \cap R[S,\mathbf{1}] \cap (R[S_u K_i^a,\mathbf{1}] \oplus R[S_u K_{R_i},\mathbf{1}])$, which is PI λ-contractive under the control law (4.20), such that for any trajectory starting at $R[Y,\mathbf{1}]$, the output $y(k)$ converges asymptotically to the desired set-point, without violating the constraints.

In the following, we present sufficient conditions that guarantee that a polyhedral set $R[Y,\mathbf{1}]$ is PI λ-contractive concerning the fuzzy system T-S under a PDC (I-P)-Like control law. To simplify the notation, from this point on, we drop the explicit dependence of x_a on the variable k in the membership functions.

Theorem 4.2.1 *The polyhedron $R[Y,\mathbf{1}]$ is PI λ-contractive with respect to the closed-loop system* (4.21)-(4.22)*, if matrices exist $H_{ii} \in \mathrm{R}^{g\times g}$, $Z_{ii} \in \mathrm{R}^{g\times g_r}$, $i = 1,2,\cdots,r$, $j = 1,2,\cdots,r$ and $H_{ij} \in \mathrm{R}^{g\times g}$, $Z_{ij} \in \mathrm{R}^{g\times g_r}$, $i = 1,2,\cdots,r$ and $j = i+1,\cdots,r$, such that:*

$$\begin{gathered} H_{ii}Y = YG_{ii}^a,\ H_{ii} \geq 0, \\ Z_{ii}S_r = YR_{ii},\ Z_{ii} \geq 0, \\ H_{ii}\mathbf{1} + Z_{ii}\mathbf{1} \leq \lambda\mathbf{1}, \\ H_{ij}Y = Y\left(\frac{G_{ij}^a + G_{ji}^a}{2}\right),\ H_{ij} \geq 0, \\ Z_{ij}S_r = Y\left(\frac{R_{ij}+R_{ji}}{2}\right),\ Z_{ij} \geq 0, \\ H_{ij}\mathbf{1} + Z_{ij}\mathbf{1} \leq \lambda\mathbf{1}. \end{gathered} \tag{4.27}$$

Proof 4.2.1 *Consider* $Yx_a(k) \leq \boldsymbol{1}$ *and* $S_r r(k) \leq \boldsymbol{1}$. *Then, from* (4.21) *and* (4.27):

$$\begin{aligned}
& Yx_a(k+1) = \sum_{i=1}^{r} \alpha_i^2(y) Y(G_{ii}^a x_a(k) + R_{ii} r(k)) \; + \\
& 2\sum_{i=1}^{r}\sum_{i<j}^{r} \alpha_i(y)\alpha_j(y) Y \left[\left(\frac{G_{ij}^a + G_{ji}^a}{2} \right) x_a(k) + \left(\frac{R_{ij} + R_{ji}}{2} \right) r(k) \right] = \\
& \sum_{i=1}^{r} \alpha_i^2(y)(H_{ii} Y x_a(k) + Z_{ii} S_r r(k)) + 2\sum_{i=1}^{r}\sum_{i<j}^{r} \alpha_i(y)\alpha_j(y)(H_{ij} Y x_a(k) + Z_{ij} S_r r(k)) \leq \\
& \sum_{i=1}^{r} \alpha_i^2(y)((H_{ii}\boldsymbol{1} + Z_{ii}\boldsymbol{1}) + 2\sum_{i=1}^{r}\sum_{i<j}^{r} \alpha_i(y)\alpha_j(y)(H_{ij}\boldsymbol{1} + Z_{ij}\boldsymbol{1}) \leq \\
& \sum_{i=1}^{r} \alpha_i^2(y)\lambda\boldsymbol{1} + 2\sum_{i<j}^{r} \alpha_i(y)\alpha_j(y)\lambda\boldsymbol{1} = \lambda \left(\sum_{i=1}^{r} \alpha_i(y) \right)^2 \boldsymbol{1} = \lambda\boldsymbol{1} < \boldsymbol{1}.
\end{aligned}$$

This proves that the polyhedron $R[Y, \boldsymbol{1}]$ *is PI and* λ*-contractive.* □

To ensure the solution does not result in a degenerate polyhedron, the matrix $Y \in \mathbb{R}^{g\times(2n+p)}$ must have full column-rank. This is equivalent to the existence of a pseudo-inverse matrix $P \in \mathbb{R}^{(2n+p)\times g}$, such that:

$$PY = \mathbf{I} \tag{4.28}$$

must hold, where $\mathbf{I} \in \mathbb{R}^{(2n+p)\times(2n+p)}$.

Theorem 4.2.1 is related to the existence of a PI polyhedron, guaranteeing that if $x_a(0) \in R[Y, \mathbf{1}]$, then $x_a(k) \in R[Y, \mathbf{1}]$, for $k = 1, 2, \cdots$. Furthermore, $0 < \lambda < 1$ guarantees the contraction of the polyhedron $R[Y, \mathbf{1}]$ in the augmented state-space, so that if $x_a(k) \in R[Y, \mathbf{1}]$, then $x_a(k+1) \in \lambda R[Y, \mathbf{1}]$.

It is important to observe that when $r(k) = 0$, the terms in Equation (4.21) associated with the reference also become zero, consequently, can be eliminated. Hence, the tracking problem for $r(k) = 0$ can be interpreted as the regulator problem. It can be demonstrated that if $x_a(k) \in R_\lambda[Y, \mathbf{1}]$, $R_\lambda[Y, \mathbf{1}] \equiv \lambda R[Y, \mathbf{1}]$, then $x_a(k+1) \in \lambda R_\lambda[Y, \mathbf{1}]$ for all $k \geq 0$. Therefore, if $x_a(0) \in R[Y, \mathbf{1}]$, it follows that $x_a(k) \in \lambda^k R[Y, \mathbf{1}]$. If the conditions presented in Theorem 4.2.1 are satisfied, then $x_a(k)$ will converge to zero as k approaches infinity. The proof for this statement is analogous to the one presented in Subsection 3.2.1.

In the linear case, it is possible to apply the *Internal Model Principle* (IMP), which states that any (I-P)-Like stabilizing controller ensures local asymptotic tracking to a constant desired reference $r(k)$ [Chen, B.M. et al. 2004, Section 9.2.2]. As discussed in Section 2.8, positive invariance of $R[Y, \mathbf{1}]$ ensures, at least, closed-loop local stability. Positive invariance with a contraction rate $\lambda < 1$, implies local asymptotic stability for $r(k) = 0$. For $r(k)$ constant and admissible, the equilibrium point of (4.21)-(4.22) is characterized by $v(k+1) = v(k) = \bar{v}$. From (4.19), this last expression implies for the tracking error in the equilibrium: $\bar{e}_t = 0$.

In [Lopes, A.N. et al. 2020, Figueiredo, L.S. et al. 2020], control strategies based on an (I-P)-Like control law are presented, where local stability of the model within an ellipsoidal region of attraction is guaranteed. As the model is stable, the integral control action is capable of eliminating the tracking error.

In our approach, the estimate of the region of attraction is obtained in the form of the polyhedron $R[Y,\mathbf{1}]$. Once the conditions established in Theorem 4.2.1 are satisfied, $R[Y,\mathbf{1}]$ is PI λ-contractive, and the system (4.21)-(4.22) is locally stable into the validity region of the T-S model. In turn, the integral control action is continuously adjusted until the tracking error becomes null, and the condition $v(k+1) = v(k)$ is satisfied. The results presented in this chapter demonstrate the effectiveness of the proposed approach.

Tracking Target Calculation

Introducing integral action to the controller has the basic principle of eliminating the tracking error between the output and a constant reference. For a given constant reference r, the trajectory of $x_a(k)$ tends asymptotically to a single point (equilibrium) in the augmented state space.

In steady-state, $x_a(k+1) - x_a(k) = 0$ when $k \to \infty$, given that

$$x_a(k+1) - x_a(k)\Big|_{k\to\infty} = \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i(y(\infty))\alpha_j(y(\infty))\left((G_{ij}^a - \mathbf{I})x_a(\infty) + R_{ij}r\right), \qquad (4.29)$$

with $y(\infty) \to r$.

Assumption 4.2.1 *The following square matrix is non-singular:*

$$\overline{M} = \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i(r)\alpha_j(r)(G_{ij}^a - \boldsymbol{I}). \qquad (4.30)$$

The equilibrium state $x_a(\infty)$ is determined from the solution of the steady-state equation:

$$\overline{M}x(\infty) = -\sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i(r)\alpha_j(r)R_{ij}r. \qquad (4.31)$$

Assumption 4.2.1 implies that the target calculation has a unique solution for r, given that the matrix $\overline{M}$ has rank $2n+p$ and allows for the existence of its inverse. The desired set-point r for the controlled variables must be such that $r \in \Re$ and the state $x_a(\infty)$ does not violate the constraints in $x_a(k) \in R[Y,\mathbf{1}]$. The reference signal satisfying the above requirement is called admissible.

State and Control Constraints

Conditions under which the constrained constant reference tracking problem can be solved are based on the matrix version of Farkas' lemma [Hennet, J.C. 1989, Schrijver, A. 1998]. Constraints in the augmented state can be satisfied if the PI polyhedron is contained within the constraint polyhedron, that is if $R[Y,\mathbf{1}] \subseteq R[S,\mathbf{1}]$. This is guaranteed, if and only if there exists a matrix $M \geq 0$, with $M \in \mathbb{R}^{g_a \times g}$, such that

$$\begin{aligned} MY &= S, \\ M\mathbf{1} &\leq \mathbf{1}. \end{aligned} \tag{4.32}$$

The characterization of the polyhedral inclusion $R[Y,\mathbf{1}] \subseteq R[S,\mathbf{1}]$ is analogous to the linear case [Hennet, J.C. 1989, Schrijver, A. 1998].

In turn, the control constraints can be established from the following theorem:

Theorem 4.2.2 *The polyhedron defined by $R[S_u K_i^a, \mathbf{1}]$ characterize the control constraints defined in the augmented state-space. The inclusion of the polyhedral domain given by $R[Y,\mathbf{1}] \subseteq R[S_u K_i^a, \mathbf{1}]$ is guaranteed, with $K_i^a = [-K_{P_i}\ K_{P_i}\ K_{I_i}]$, $K_i^a \in \mathbb{R}^{m\times(2n+p)}$, $K_{R_i} \in \mathbb{R}^p$ and $S_u \in \mathbb{R}^{g_u \times m}$, if there are $Q_i \geq 0$, $Q_i \in \mathbb{R}^{g_u \times g}$ and $Q_i^r \geq 0$, $Q_i^r \in \mathbb{R}^{g_u \times g_r}$, such that*

$$\begin{aligned} Q_i Y = S_u K_i^a,\ Q_i \geq 0, \\ Q_i^r S_r = S_u K_{R_i},\ Q_i^r \geq 0, \\ Q_i \mathbf{1} + Q_i^r \mathbf{1} \leq \mathbf{1}. \end{aligned} \tag{4.33}$$

Proof 4.2.2 *Consider $Yx_a(k) \leq \mathbf{1}$ and $S_r r(k) \leq \mathbf{1}$. Then, from* (4.20), (4.24) *and* (4.25)*:*

$$S_u u(k) = \sum_{i=1}^{r} \alpha(y) S_u (K_i^a x_a(k)\ +\ K_{R_i} r(k)) \leq \sum_{i=1}^{r} \alpha(y)(Q_i Y x_a(k) + Q_i^r S_r r(k)) \leq$$
$$\sum_{i=1}^{r} \alpha(y)(Q_i \mathbf{1} + Q_i^r \mathbf{1}) \leq \sum_{i=1}^{r} \alpha(y)\mathbf{1} = \mathbf{1}.$$

This proves the inclusion of the polyhedral domain given by $R[Y,\mathbf{1}] \subseteq R[S_u K_i^a, \mathbf{1}]$, such that $K_i^a R[Y,\mathbf{1}] \oplus K_{R_i}\mathfrak{R} \subseteq \mathcal{U}$, where $\oplus$ denotes the Minkowski sum operator. □

Note that, due to the fact that $\alpha_i(y(k)) \in \Delta$ (4.3), we have that if $x_a(k) \in R[S_u K_i^a,\ \mathbf{1}]$ and $r(k) \in R[S_u K_{R_i}, \mathbf{1}]$, $i = 1, \cdots, r$, then $S_u u(k) \leq \mathbf{1}$ and the control constraints are satisfied.

4.2.2 Case B

In the general case, the membership functions in fuzzy systems may depend on non-accessible states. Therefore, an estimation mechanism is required to calculate these variables.

The fuzzy T-S observer can be represented with r rules in the following form:

$$\begin{aligned} \hat{x}(k+1) &= \sum_{i=1}^{r} \alpha_i(\hat{x}(k))[A_i \hat{x}(k) + B_i u(k) + L_i(y(k) - \hat{y}(k))], \\ \hat{y}(k) &= C\hat{x}(k). \end{aligned} \tag{4.34}$$

where $\alpha_i(\hat{x}(k))$ are membership functions applied to the vector of estimated premise variables $\hat{x}(k)$, which must belong to the simplex:

$$\Delta = \{\alpha(\hat{x}(k)) \in \mathbb{R}^r : \sum_{i=1}^{r} \alpha_i(\hat{x}(k)) = 1,\ \alpha_i(\hat{x}(k)) \geq 0\}. \tag{4.35}$$

For definition, the estimation error $e(k)$ is given by (4.18) and the integral of the tracking error $v(k)$ by (4.19). The dynamic of the estimation error can be obtained from Equations (4.2) and (4.34), whereas the integral of the tracking error can be derived by substituting (4.2) in (4.19).

The PDC control law is given by:

$$u(k) = -\sum_{i=1}^{r} \alpha_i(\hat{x}(k)) \Big(K_{P_i}\hat{x}(k) - K_{I_i}v(k) - K_{R_i}r(k) \Big), \tag{4.36}$$

with $K_{P_i} \in \mathbb{R}^{m\times n}$, $K_{I_i} \in \mathbb{R}^{m\times p}$ and $K_{R_i} \in \mathbb{R}^{m\times p}$, such that the corresponding augmented closed-loop system is:

$$\begin{aligned} &x_a(k+1) = \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i(x(k))\alpha_j^2(\hat{x}(k))(G^a_{ijj}x_a(k) \ + \ R_{ijj}r(k)) \ + \\ &2\sum_{i=1}^{r}\sum_{j<c}^{r} \alpha_i(x(k))\alpha_j(\hat{x}(k))\alpha_c(\hat{x}(k)) \left[\left(\tfrac{G^a_{ijc}+G^a_{icj}}{2} \right) x_a(k) + \left(\tfrac{R_{ijc}+R_{icj}}{2} \right) r(k) \right], \end{aligned} \tag{4.37}$$

where

$$G^a_{ijc} = \begin{bmatrix} A_i - B_iK_{P_c} & B_iK_{P_c} & B_iK_{I_c} \\ S^1_{ijc} & S^2_{ijc} & (B_i - B_j)K_{I_c} \\ -C(A_i - B_iK_{P_c}) & -CB_iK_{P_c} & I - CB_iK_{I_c} \end{bmatrix} \text{ and } R_{ijc} = \begin{bmatrix} B_iK_{R_c} \\ (B_i - B_j)K_{R_c} \\ I - CB_iK_{R_c} \end{bmatrix} \tag{4.38}$$

$$S^1_{ijc} = (A_i - A_j) - (B_i - B_j)K_{P_c}, \tag{4.39}$$

$$S^2_{ijc} = A_j - L_jC + (B_i - B_j)K_{P_c}, \tag{4.40}$$

and $x_a(k)^{\mathrm{T}} = [x(k)\ e(k)\ v(k)] \in \mathrm{R}^{2n+p}$ is the augumented state vector, $e(k) \in \mathbb{R}^n$ is the estimation error, $r(k) \in \mathbb{R}^p$ is the reference and $v(k) \in \mathbb{R}^p$ is the integral of the tracking error.

The linear control constraints w.r.t. system (4.37)-(4.40) can be represented by (4.5) and (4.36) as:

$$\begin{aligned} &S_u u(k) = -S_u \left[\sum_{i=1}^{r} \alpha_i(\hat{x}(k))(K_{P_i}x(k) - K_{P_i}e(k) - K_{I_i}v(k) - K_{R_i}r(k)) \right] = \\ &\sum_{i=1}^{r} \alpha_i(\hat{x}(k))S_u(K^a_i x_a(k) \ + \ K_{R_i}r(k)) \leq \mathbf{1}, \end{aligned} \tag{4.41}$$

where

$$R[S_uK^a_i, \mathbf{1}] = \{x_a(k) \in \mathbb{R}^{2n} : S_uK^a_i x_a(k) \leq \mathbf{1}\} \tag{4.42}$$

and

$$R[S_u K_{R_i}, \mathbf{1}] = \{r(k) \in \mathbb{R}^p : S_u K_{R_i} r(k) \leq \mathbf{1}\} \tag{4.43}$$

denote the constraints in the augmented state-space and the space associated with references, respectively, with $S_u \in \mathbb{R}^{g_u \times m}$, $K_i^a = [-K_{P_i}\ K_{P_i}\ K_{I_i}]$, $K_i^a \in \mathbb{R}^{m\times(2n+p)}$ and $K_{R_i} \in \mathbb{R}^{m\times p}$.

The possibility of imposing trajectories of $x_a(k)$ within a given polyhedral set $R[Y,\mathbf{1}] \subset \Omega$ w.r.t. closed-loop system (4.37)-(4.40) can be achieved from the Definition (2.4.1). Moreover, the control objectives under constraints are the same as those defined for the case **A**, but considering the control law defined in (4.36).

In the following, we present sufficient conditions that guarantee that a polyhedral set $R[Y,\mathbf{1}]$ is PI λ-contractive with respect to the fuzzy T-S system under a PDC (I-P)-Like control law. Once again, we will simplify the notation, discarding the explicit dependence of x_a on the variable k in the membership functions.

Theorem 4.2.3 *The polyhedron $R[Y,\mathbf{1}]$ is PI λ-contractive w.r.t. closed-loop system* (4.37)-(4.40), *if there are matrices $H_{ijj} \in \mathrm{R}^{g\times g}$, $Z_{ijj} \in \mathrm{R}^{g\times g_r}$, $i = 1,2,\cdots,r$, $j = 1,2,\cdots,r$ and $H_{ijc} \in \mathrm{R}^{g\times g}$, $Z_{ijc} \in \mathrm{R}^{g\times g_r}$, $i = 1,2,\cdots,r$, $j = 1,\cdots,(r-1)$ and $c = j+1,\cdots,r$, such that:*

$$\begin{gathered} H_{ijj}Y = YG^a_{ijj},\ H_{ijj} \geq 0,\\ Z_{ijj}S_r = YR_{ijj},\ Z_{ijj} \geq 0,\\ H_{ijj}\mathbf{1} + Z_{ijj}\mathbf{1} \leq \lambda\mathbf{1},\\ H_{ijc}Y = Y\left(\frac{G^a_{ijc}+G^a_{icj}}{2}\right),\ H_{ijc} \geq 0,\\ Z_{ijc}S_r = Y\left(\frac{R_{ijc}+R_{icj}}{2}\right),\ Z_{ijc} \geq 0,\\ H_{ijc}\mathbf{1} + Z_{ijc}\mathbf{1} \leq \lambda\mathbf{1}. \end{gathered} \tag{4.44}$$

Proof 4.2.3 *Consider $Yx_a(k) \leq \mathbf{1}$ and $S_r r(k) \leq \mathbf{1}$. Then, from* (4.37) *and* (4.44)*:*

$$\begin{aligned} &Yx_a(k+1) = \sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i(x)\alpha_j^2(\hat{x})Y(G^a_{ijj}x_a(k) + R_{ijj}r(k)) +\\ &2\sum_{i=1}^{r}\sum_{j<c}^{r}\alpha_i(x)\alpha_j(\hat{x})\alpha_c(\hat{x})Y\left[\left(\frac{G^a_{ijc}+G^a_{icj}}{2}\right)x_a(k) + \left(\frac{R_{ijc}+R_{icj}}{2}\right)r(k)\right] =\\ &\sum_{i=1}^{r}\alpha_i(x)\left[\sum_{j=1}^{r}\alpha_j^2(\hat{x})(H_{ijj}Yx_a(k) + Z_{ijj}S_r r(k)) + 2\sum_{j<c}^{r}\alpha_j(\hat{x})\alpha_c(\hat{x})(H_{ijc}Yx_a(k) +\right.\\ &\left.Z_{ijc}S_r r(k))\right] \leq \sum_{i=1}^{r}\alpha_i(x)\left[\sum_{j=1}^{r}\alpha_j^2(\hat{x})(H_{ijj}\mathbf{1} + Z_{ijj}\mathbf{1}) + 2\sum_{j<c}^{r}\alpha_j(\hat{x})\alpha_c(\hat{x})(H_{ijc}\mathbf{1} + Z_{ijc}\mathbf{1})\right] \leq\\ &\sum_{i=1}^{r}\alpha_i(x)\left[\sum_{j=1}^{r}\alpha_j^2(\hat{x})\lambda\mathbf{1} + 2\sum_{j<c}^{r}\alpha_j(\hat{x})\alpha_c(\hat{x})\lambda\mathbf{1}\right] = \sum_{i=1}^{r}\alpha_i(x)\left[\lambda\left(\sum_{j=1}^{r}\alpha_j(\hat{x})\right)^2\mathbf{1}\right] = \lambda\mathbf{1} < \mathbf{1}. \end{aligned}$$

This proves that the polyhedron $R[Y,\mathbf{1}]$ is PI and λ-contractive. □

The Theorem 4.2.3 concerns the existence of a polyhedron PI and λ-contractive, such that $x_a(k) \rightarrow 0$ when $k \rightarrow \infty$, if $r(k) = 0$. The proof of this theorem is similar to that presented in Subsection 3.2.2. For the case **B**, the same comments made in the case **A** can be extended regarding the invariance and contraction effect of the sets $R[Y,\mathbf{1}]$.

It is essential to point out that, for the case **B**, the conditions mentioned above are only valid if $\alpha(\hat{x}(k))$ is within the validity domain Ω of the fuzzy T-S model. As a result, the universe of discourse for $x(k)$ and $e(k)$ is reduced. Thus, the set $R[S,\mathbf{1}]$ must be decreased, or the local models must be redefined.

When the conditions of Theorem 4.2.3 are satisfied, $R[Y,\mathbf{1}]$ is PI λ-contractive, and the dynamics of the system (4.37)-(4.40) is considered locally stable. Therefore, since $x_a(0) \in R[Y,\mathbf{1}]$, the integral control action can eliminate the tracking error for a constant reference $r(k)$.

Tracking Target Calculation

For a given constant reference r, the trajectory of $x_a(k)$ tends asymptotically to the equilibrium point in the augmented state-space. Therefore, $x_a(k+1) - x_a(k) = 0$ when $k \rightarrow \infty$, given that

$$x_a(k+1) - x_a(k)\Big|_{k\rightarrow\infty} = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{c=1}^{r} \alpha_i(x(\infty))\alpha_j(\hat{x}(\infty))\alpha_c(\hat{x}(\infty))\left((G^a_{ijc} - \mathbf{I})x_a(\infty) + R_{ijc}r\right). \tag{4.45}$$

Assuming that $x_a(k+1) - x_a(k) = 0$, Equation (4.45) can be reformulated as follows:

$$\overline{M}x_a(\infty) = -\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{c=1}^{r} \alpha_i(x(\infty))\alpha_j(\hat{x}(\infty))\alpha_c(\hat{x}(\infty))R_{ijc}r, \tag{4.46}$$

where

$$\overline{M} = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{c=1}^{r} \alpha_i(x(\infty))\alpha_j(\hat{x}(\infty))\alpha_c(\hat{x}(\infty))(G^a_{ijc} - \mathbf{I}). \tag{4.47}$$

The lack of uniqueness in the solution of Equation (4.46) stems from the dependence of the membership functions on $x(\infty)$ and its estimate. Any $x(k) \in \mathcal{X}$ that satisfies the equation $y(\infty) = Cx(\infty) = r$ can be used to determine the output variable. This observation applies to the estimated output variable $\hat{y}(\infty)$. Therefore, any point $x(\infty)$ satisfying (4.46) for a given reference r is an equilibrium point of the system (4.37)-(4.40).

The set-point for the controlled variables must be such that $r \in \Re$ and the augmented state $x_a(\infty)$ do not violate the constraints in $x_a(k) \in R[Y,\mathbf{1}]$. The reference signal satisfying the above requirement is called admissible.

State and Control Constraints

Similar to the case **A**, the satisfaction of constraints in the augmented state can be ensured if the PI polyhedron is enclosed within the constraint polyhedron, which can be established if the conditions presented in Equation (4.32) are fulfilled.

The satisfaction of the control constraints can be ensured by verifying the conditions stated in (4.33), in accordance with Theorem 4.2.2 and the control law expressed in (4.20).

Proof 4.2.4 *Consider* $Yx_a(k) \leq \mathbf{1}$ *and* $S_r r(k) \leq \mathbf{1}$*. Then, from* (4.36), (4.42) *and* (4.43)*:*

$$S_u u(k) = \sum_{i=1}^{r} \alpha(\hat{x}) S_u (K_i^a x_a(k) \; + \; K_{R_i} r(k)) \leq \sum_{i=1}^{r} \alpha(\hat{x})(Q_i Y x_a(k) + Q_i^r S_r r(k)) \leq$$
$$\sum_{i=1}^{r} \alpha(\hat{x})(Q_i \mathbf{1} + Q_i^r \mathbf{1}) \leq \sum_{i=1}^{r} \alpha(\hat{x}) \mathbf{1} = \mathbf{1}.$$

This proves the inclusion of the polyhedral domain given by $R[Y,\mathbf{1}] \subseteq R[S_u K_i^a, \mathbf{1}]$*, such that* $K_i^a R[Y,\mathbf{1}] \oplus K_{R_i} \mathfrak{R} \subseteq \mathcal{U}$*, where* $\oplus$ *denotes the Minkowski sum operator.* □

It is important to point out that, for the case **B**, the conditions above are valid if $\alpha(\hat{x}(k)) \in \Delta$.

4.3 Positive Invariance of Symmetric Polyhedra

4.3.1 Case A

Consider the case where the membership functions depend only on the output of the system, such that the closed-loop fuzzy T-S system can be expressed by:

$$\begin{aligned} x_a(k+1) = & \sum_{i=1}^{r} \alpha_i^2(y(k))(G_{ii}^a x_a(k) + R_{ii} r(k)) + \\ & 2 \sum_{i=1}^{r} \sum_{i<j}^{r} \alpha_i(y(k))\alpha_j(y(k)) \left[\left(\frac{G_{ij}^a + G_{ji}^a}{2} \right) x_a(k) + \left(\frac{Rij + Rji}{2} \right) r(k) \right], \end{aligned} \tag{4.48}$$

where

$$G_{ij}^a = \begin{bmatrix} A_i - B_i K_{P_j} & B_i K_{P_j} & B_i K_{I_j} \\ 0 & A_i - L_i C & 0 \\ -C(A_i - B_i K_{P_j}) & -C B_i K_{P_j} & I - C B_i K_{I_j} \end{bmatrix} \text{ e } R_{ij} = \begin{bmatrix} B_i K_{R_j} \\ 0 \\ I - C B_i K_{R_j} \end{bmatrix}, \tag{4.49}$$

and $x_a(k)^{\mathrm{T}} = [x(k) \; e(k) \; v(k)] \in \mathrm{R}^{2n+p}$ is the augumented state vector, $r(k) \in \mathbb{R}^p$ is the reference, $e(k) \in \mathbb{R}^n$ is the estimation error and $v(k) \in \mathbb{R}^p$ is the integral of the tracking error.

The linear control constraints w.r.t. system (4.48)-(4.49) can be expressed from (4.12) and (4.20) by:

$$|S_u^s u(k)| = \left| \sum_{i=1}^{r} \alpha_i(y(k)) S_u^s (K_i^a x_a(k) \; + \; K_{R_i} r(k)) \right| \leq \mathbf{1}, \tag{4.50}$$

where

$$\mathcal{S}[S_u^s K_i^a, \mathbf{1}] = \{x_a(k) \in \mathbb{R}^{2n} : |S_u^s K_i^a x_a(k)| \leq \mathbf{1}\} \tag{4.51}$$

and

$$\mathcal{S}[S_u^s K_{R_i}, \mathbf{1}] = \{r(k) \in \mathbb{R}^p : |S_u^s K_{R_i} r(k)| \leq \mathbf{1}\} \tag{4.52}$$

denote the constraints in the augmented state-space and the space associated with references, respectively, with $S_u^s \in \mathbb{R}^{g_u^s \times m}$, $K_i^a = [K_{P_i}\ K_{P_i}\ K_{I_i}]$, $K_i^a \in \mathbb{R}^{m\times(2n+p)}$ e $K_{R_i} \in \mathbb{R}^{m\times p}$.

The intersection of the sets Ω, $\mathcal{S}[S_s, \mathbf{1}]$ and $\mathcal{S}[S_u^s K_i^a, \mathbf{1}] \oplus \mathcal{S}[S_u^s K_{R_i}, \mathbf{1}]$ forms a symmetrical and compact polyhedron containing the origin. The possibility of imposing trajectories of $x_a(k)$ within a given polyhedral set $\mathcal{S}[Y_s, \mathbf{1}] \subset \Omega$ w.r.t. closed-loop system (4.48)-(4.49) can be achieved from the Definition (2.4.1), where $u(k) \in \mathcal{U}_s$, $\forall k \in \mathbb{N}$.

Then, our objective becomes to obtain the matrices K_{P_i}, K_{I_i}, K_{R_i}, L_i, $i = 1, \cdots, r$ and a polyhedral set

$$\mathcal{S}[Y_s, \mathbf{1}] = \{x_a(k) \in \mathbb{R}^{2n+p} : |Y_s x_a(k)| \leq \mathbf{1}\ \}, \tag{4.53}$$

contained in the constraint set $\Omega \cap \mathcal{S}[S_s, \mathbf{1}] \cap (\mathcal{S}[S_u^s K_i^a, \mathbf{1}] \oplus \mathcal{S}[S_u^s K_{R_i}, \mathbf{1}])$, which is PI λ-contractive under the control law (4.20), such that for any trajectory starting at $\mathcal{S}[Y_s, \mathbf{1}]$, with $Y_s \in \mathbb{R}^{g_s \times (2n+p)}$, the output $y(k)$ converges asymptotically to the desired set-point, without violating the constraints.

In the following, we present sufficient conditions that guarantee that a symmetric polyhedral set $\mathcal{S}[Y_s, \mathbf{1}]$ is PI λ-contractive concerning the fuzzy T-S system under a PDC (I-P)-Like control law. Once again, we will simplify the notation, discarding the explicit dependence of x_a on the variable k in the membership functions.

Theorem 4.3.1 *The polyhedron $\mathcal{S}[Y_s, \mathbf{1}]$ is PI λ-contractive w.r.t. closed-loop system* (4.21)-(4.22), *if there are matrices $H_{ii} \in \mathrm{R}^{g_s \times g_s}$, $Z_{ii} \in \mathrm{R}^{g_s \times g_r^s}$, $i = 1, 2, \cdots, r$, $j = 1, 2, \cdots, r$ and $H_{ij} \in \mathrm{R}^{g_s \times g_s}$, $Z_{ij} \in \mathrm{R}^{g_s \times g_r^s}$, $i = 1, 2, \cdots, r$ and $j = i+1, \cdots, r$, such that:*

$$\begin{aligned}
H_{ii} Y_s &= Y_s G_{ii}^a, \\
Z_{ii} S_r^s &= Y_s R_{ii}, \\
||H_{ii}||_\infty + ||Z_{ii}||_\infty &\leq \lambda, \\
H_{ij} Y_s &= Y_s \left(\frac{G_{ij}^a + G_{ji}^a}{2} \right), \\
Z_{ij} S_r^s &= Y_s \left(\frac{R_{ij} + R_{ji}}{2} \right), \\
||H_{ij}||_\infty + ||Z_{ij}||_\infty &\leq \lambda.
\end{aligned} \tag{4.54}$$

Proof 4.3.1 *Consider* $|Y_s x_a(k)| \leq \mathbf{1}$ *and* $|S_r^s r(k)| \leq \mathbf{1}$. *Then, from* (4.21) *and* (4.54)*:*

$$|Y_s x_a(k+1)| = \left| \sum_{i=1}^{r} \alpha_i^2(y) Y_s (G_{ii}^a x_a(k) + R_{ii} r(k)) \right. +$$

$$\left. 2\sum_{i=1}^{r}\sum_{i<j}^{r} \alpha_i(y)\alpha_j(y) Y_s \left[\left(\frac{G_{ij}^a + G_{ji}^a}{2} \right) x_a(k) + \left(\frac{R_{ij} + R_{ji}}{2} \right) r(k) \right] \right| \leq$$

$$\sum_{i=1}^{r} \alpha_i^2(y)(|H_{ii} Y_s x_a(k) + Z_{ii} S_r^s r(k)|) + 2\sum_{i=1}^{r}\sum_{i<j}^{r} \alpha_i(y)\alpha_j(y)(|H_{ij} Y_s x_a(k) + Z_{ij} S_r^s r(k)|) \leq$$

$$\sum_{i=1}^{r} \alpha_i^2(y)((|H_{ii}|\mathbf{1} + |Z_{ii}|\mathbf{1}) + 2\sum_{i=1}^{r}\sum_{i<j}^{r} \alpha_i(y)\alpha_j(y)(|H_{ij}|\mathbf{1} + |Z_{ij}|\mathbf{1}) \leq$$

$$\sum_{j=1}^{r} \alpha_i^2(y)\lambda\mathbf{1} + 2\sum_{i<j}^{r} \alpha_i(y)\alpha_j(y)\lambda\mathbf{1} = \lambda\left(\sum_{j=1}^{r} \alpha_i(y) \right)^2 \mathbf{1} = \lambda\mathbf{1} < \mathbf{1},$$

given that $||H_{ii}||_\infty + ||Z_{ii}||_\infty \leq \lambda$ *and* $||H_{ij}||_\infty + ||Z_{ij}||_\infty \leq \lambda$ *are equivalent to* $|H_{ii}|\mathbf{1} + |Z_{ii}|\mathbf{1}$ $\leq \lambda\mathbf{1}$ *and* $|H_{ij}|\mathbf{1} + |Z_{ij}|\mathbf{1} \leq \lambda\mathbf{1}$.

To ensure the solution does not result in a degenerate polyhedron, the matrix $Y_s \in \mathbb{R}^{g_s \times (2n+p)}$ must have full column-rank. This is equivalent to the existence of a pseudo-inverse matrix $P_s \in \mathbb{R}^{(2n+p)\times g_s}$, such that:

$$P_s Y_s = \mathbf{I}, \tag{4.55}$$

must hold, where $\mathbf{I} \in \mathbb{R}^{(2n+p)\times(2n+p)}$.

Theorem 4.3.1 is related to the existence of a symmetric polyhedron PI, guaranteeing that if $x_a(0) \in \mathcal{S}[Y_s, \mathbf{1}]$, then $x_a(k) \in \mathcal{S}[Y_s, \mathbf{1}]$, for $k = 1, 2, \cdots$. Furthermore, $0 < \lambda < 1$ guarantees the contraction of the polyhedron $\mathcal{S}[Y_s, \mathbf{1}]$ in the augmented state-space, so that if $x_a(k) \in \mathcal{S}[Y_s, \mathbf{1}]$, then $x_a(k+1) \in \lambda\mathcal{S}[Y_s, \mathbf{1}]$.

If $r(k) = 0$, the terms in Equation (4.48) associated with the reference become zero and, consequently, can be eliminated. Hence, the tracking problem for $r(k) = 0$ can be interpreted as the regulator problem. It can be demonstrated that if $x_a(k) \in \mathcal{S}_\lambda[Y_s, \mathbf{1}]$, $\mathcal{S}_\lambda[Y_s, \mathbf{1}] \equiv \lambda\mathcal{S}[Y_s, \mathbf{1}]$, $x_a(k+1) \in \lambda\mathcal{S}_\lambda[Y_s, \mathbf{1}], \forall k \geq 0$. Therefore, if $x_a(0) \in \mathcal{S}[Y_s, \mathbf{1}]$, it follows that $x_a(k) \in \lambda^k\mathcal{S}[Y_s, \mathbf{1}]$. If the conditions presented in Theorem 4.3.1 are satisfied, then $x_a(k)$ will converge to zero as k approaches infinity. The proof for this statement is analogous to the one presented in Subsection 3.3.1.

The estimate of the region of attraction is obtained in the form of the symmetrical polyhedron $\mathcal{S}[Y_s, \mathbf{1}]$. Once the conditions established in Theorem 4.3.1 are satisfied, $\mathcal{S}[Y_s, \mathbf{1}]$ is PI λ-contractive and the system (4.48)-(4.49) is locally stable. The integral control action varies until the tracking error becomes null and the condition $v(k+1) = v(k)$ is achieved.

Tracking Target Calculation

Equation (4.31) determines the equilibrium state $x_a(\infty)$, and its solution is unique if Assumption 4.2.1 holds true, implying that $\overline{M}$ is invertible and can be calculated using Equation (4.29).

The desired set-point for the controlled variables should be such that $r \in \mathfrak{R}_s$ and the augmented state $x_a(\infty)$ do not violate the constraints in $x_a(k) \in \mathcal{S}[Y_s, \mathbf{1}]$. The reference signal satisfying the above requirement is called admissible.

State and Control Constraints

The conditions for solving the problem of tracking constant references under symmetric constraints are based on the matrix version of Farkas's lemma [Hennet, J.C. 1989, Schrijver, A. 1998]. The constraints in the augmented state can be satisfied if the PI polyhedron is contained within the constraint polyhedron, i.e., $\mathcal{S}[Y_s, \mathbf{1}] \subseteq \mathcal{S}[S_s, \mathbf{1}]$. This is guaranteed if and only if there exists a matrix $M \in \mathbb{R}^{g_a^s \times g_s}$ that satisfies:

$$\begin{aligned} MY_s &= S_s, \\ ||M||_\infty &\leq \mathbf{1}. \end{aligned} \tag{4.56}$$

The characterization of the polyhedral inclusion $\mathcal{S}[Y_s, \mathbf{1}] \subseteq \mathcal{S}[S_s, \mathbf{1}]$ is analogous to the linear case [Hennet, J.C. 1989, Schrijver, A. 1998].

Control constraints can be established from the following theorem:

Theorem 4.3.2 *The polyhedron defined by $\mathcal{S}[S_u^s K_i^a, \mathbf{1}]$ characterize the control constraints in the augmented state-space. The inclusion of the polyhedral domain given by $\mathcal{S}[Y_s, \mathbf{1}] \subseteq \mathcal{S}[S_u^s K_i^a, \mathbf{1}]$ is guaranteed, with $K_i^a = [-K_{P_i}\ K_{P_i}\ K_{I_i}]$, $K_i^a \in \mathbb{R}^{m \times (2n+p)}$, $K_{R_i} \in \mathbb{R}^p$ and $S_u^s \in \mathbb{R}^{g_u^s \times m}$, if there are $Q_i \geq 0$, $Q_i \in \mathbb{R}^{g_u^s \times g_s}$ and $Q_i^r \geq 0$, $Q_i^r \in \mathbb{R}^{g_u^s \times g_r^s}$, such that*

$$\begin{aligned} Q_i Y_s &= S_u^s K_i^a, \\ Q_i^r S_r^s &= S_u^s K_{R_i}, \\ ||Q_i||_\infty + ||Q_i^r||_\infty &\leq 1. \end{aligned} \tag{4.57}$$

Proof 4.3.2 *Consider $|Y_s x_a(k)| \leq \mathbf{1}$ and $|S_r^s r(k)| \leq \mathbf{1}$. Then, from* (4.20), (4.51) *and* (4.52)*:*

$$|S_u^s u(k)| = \left| \sum_{i=1}^{r} \alpha(y) S_u^s (K_i^a x_a(k) \ + \ K_{R_i} r(k)) \right| \leq \sum_{i=1}^{r} \alpha(y)(|Q_i Y_s x_a(k) + Q_i^r S_r^s r(k)|) \leq$$

$$\sum_{i=1}^{r} \alpha(y)(|Q_i|\mathbf{1} + |Q_i^r|\mathbf{1}) \leq \sum_{i=1}^{r} \alpha(y)\mathbf{1} = \mathbf{1}.$$

given that $||Q_i||_\infty + ||Q_i^r||_\infty \leq 1$ is equivalent to $|Q_i|\mathbf{1} + |Q_i^r|\mathbf{1} \leq \mathbf{1}$.

This proves the inclusion of the polyhedral domain given by $\mathcal{S}[Y_s, \mathbf{1}] \subseteq \mathcal{S}[S_u^s K_i^a, \mathbf{1}]$, such that $K_i^a \mathcal{S}[Y_s, \mathbf{1}] \oplus K_{R_i} \mathfrak{R}_s \subseteq \mathcal{U}_s$, where $\oplus$ denotes the Minkowski sum operator. □

Therefore, if $x_a(k) \in \mathcal{S}[S_u^s K_i^a, \mathbf{1}]$ and $r(k) \in \mathcal{S}[S_u^s K_{R_i}, \mathbf{1}]$ for $i = 1, \cdots, r$, then $S_u^s u(k) \leq \mathbf{1}$ and the control constraints are satisfied, such that $\alpha_i(y(k)) \in \Delta$.

4.3.2 Case B

Consider the case where the membership functions depend on non-accessible states, such that the closed-loop fuzzy T-S system can be expressed by:

$$\begin{aligned} x_a(k+1) = & \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i(x(k))\alpha_j^2(\hat{x}(k))(G_{ijj}^a x_a(k) \ + \ R_{ijj} r(k)) \ + \\ & 2\sum_{i=1}^{r}\sum_{j<c} \alpha_i(x(k))\alpha_j(\hat{x}(k))\alpha_c(\hat{x}(k))\left[\left(\frac{G_{ijc}^a + G_{icj}^a}{2}\right)x_a(k) + \left(\frac{R_{ijc}+R_{icj}}{2}\right)r(k)\right], \end{aligned} \tag{4.58}$$

where

$$G_{ijc}^a = \begin{bmatrix} A_i - B_i K_{P_c} & B_i K_{P_c} & B_i K_{I_c} \\ S_{ijc}^1 & S_{ijc}^2 & (B_i - B_j) K_{I_c} \\ -C(A_i - B_i K_{P_c}) & -C B_i K_{P_c} & I - C B_i K_{I_c} \end{bmatrix} \text{ e } R_{ijc} = \begin{bmatrix} B_i K_{R_c} \\ (B_i - B_j) K_{R_c} \\ I - C B_i K_{R_c} \end{bmatrix} \tag{4.59}$$

$$S_{ijc}^1 = (A_i - A_j) - (B_i - B_j) K_{P_c}, \tag{4.60}$$

$$S_{ijc}^2 = A_j - L_j C + (B_i - B_j) K_{P_c}, \tag{4.61}$$

and $x_a(k)^{\mathrm{T}} = [x(k) \ e(k) \ v(k)] \in \mathrm{R}^{2n+p}$ is the augumented state vector, $r(k) \in \mathbb{R}^p$ is the reference, $e(k) \in \mathbb{R}^n$ is the estimation error and $v(k) \in \mathbb{R}^p$ is the integral of the tracking error.

The linear control constraints w.r.t. system (4.58)-(4.61) can be expressed from (4.12) and (4.36) by:

$$|S_u^s u(k)| = \left| \sum_{i=1}^{r} \alpha_i(y(k)) S_u^s (K_i^a x_a(k) \ + \ K_{R_i} r(k)) \right| \leq \mathbf{1}, \tag{4.62}$$

where

$$\mathcal{S}[S_u^s K_i^a, \mathbf{1}] = \{x_a(k) \in \mathbb{R}^{2n} : |S_u^s K_i^a x_a(k)| \leq \mathbf{1}\} \tag{4.63}$$

and

$$\mathcal{S}[S_u^s K_{R_i}, \mathbf{1}] = \{r(k) \in \mathbb{R}^p : |S_u^s K_{R_i} r(k)| \leq \mathbf{1}\} \tag{4.64}$$

denote the constraints in the augmented state-space and the space associated with references, respectively, with $S_u^s \in \mathbb{R}^{g_u^s \times m}$, $K_i^a = [K_{P_i} \ K_{P_i} \ K_{I_i}]$, $K_i^a \in \mathbb{R}^{m \times (2n+p)}$ and $K_{R_i} \in \mathbb{R}^{m \times p}$.

The intersection of the sets Ω, $\mathcal{S}[S_s, \mathbf{1}]$ and $\mathcal{S}[S_u^s K_i^a, \mathbf{1}] \oplus \mathcal{S}[S_u^s K_{R_i}, \mathbf{1}]$ forms a symmetrical and compact polyhedron containing the origin. The possibility of imposing trajectories of $x_a(k)$ within a given polyhedral set $\mathcal{S}[Y_s, \mathbf{1}] \subset \Omega$ w.r.t. closed-loop system (4.58)-(4.61) can be achieved from the Definition (2.4.1), where $u(k) \in \mathcal{U}_s$, $\forall k \in \mathbb{N}$. Moreover, the control objectives under constraints are the same as those defined for the case **A**, but considering the control law defined in (4.36).

In the following, we present sufficient conditions that guarantee that a symmetric polyhedral set $\mathcal{S}[Y_s,\mathbf{1}]$ is PI λ-contractive with respect to the fuzzy system T-S under a PDC control law. Once again, we will simplify the notation, discarding the explicit dependence of x_a on the variable k in the membership functions.

Theorem 4.3.3 *The polyhedron $\mathcal{S}[Y_s,\mathbf{1}]$ is PI λ-contractive w.r.t. closed-loop system* (4.58)-(4.61)*, if there are matrices $H_{ijj} \in \mathrm{R}^{g_s \times g_s}$, $Z_{ijj} \in \mathrm{R}^{g_s \times g_r^s}$, $i = 1,2,\cdots,r$, $j = 1,2,\cdots,r$ and $H_{ijc} \in \mathrm{R}^{g_s \times g_s}$, $Z_{ijc} \in \mathrm{R}^{g_s \times g_r^s}$, $i = 1,2,\cdots,r$, $j = 1,\cdots,(r-1)$ and $c = j+1,\cdots,r$, such that:*

$$\begin{aligned} H_{ijj}Y_s &= Y_s G^a_{ijj}, \\ Z_{ijj}S^s_r &= Y_s R_{ijj}, \\ ||H_{ijj}||_\infty + ||Z_{ijj}||_\infty &\leq \lambda, \\ H_{ijc}Y_s &= Y_s\left(\frac{G^a_{ijc}+G^a_{icj}}{2}\right), \\ Z_{ijc}S^s_r &= Y_s\left(\frac{R_{ijc}+R_{icj}}{2}\right), \\ ||H_{ijc}||_\infty + ||Z_{ijc}||_\infty &\leq \lambda. \end{aligned} \tag{4.65}$$

Proof 4.3.3 *Consider $|Y_s x_a(k)| \leq \mathbf{1}$ and $|S^s_r r(k)| \leq \mathbf{1}$. Then, from* (4.58) *and* (4.65)*:*

$$\begin{aligned} &|Y_s x_a(k+1)| = \Bigg| \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i(x)\alpha_j^2(\hat{x})Y_s(G^a_{ijj}x_a(k) + R_{ijj}r(k)) + \\ &2\sum_{i=1}^{r}\sum_{j<c}^{r} \alpha_i(x)\alpha_j(\hat{x})\alpha_c(\hat{x})Y_s\left[\left(\frac{G^a_{ijc}+G^a_{icj}}{2}\right)x_a(k) + \left(\frac{R_{ijc}+R_{icj}}{2}\right)r(k)\right]\Bigg| \leq \\ &\sum_{i=1}^{r}\alpha_i(x)\left[\sum_{j=1}^{r}\alpha_j^2(\hat{x})(|H_{ijj}Y_s x_a(k) + Z_{ijj}S^s_r r(k)|) + 2\sum_{j<c}^{r}\alpha_j(\hat{x})\alpha_c(\hat{x})(|H_{ijc}Y_s x_a(k) + \right. \\ &\left. Z_{ijc}S^s_r r(k)|)\right] \leq \sum_{i=1}^{r}\alpha_i(x)\left[\sum_{j=1}^{r}\alpha_j^2(\hat{x})(|H_{ijj}|\mathbf{1} + |Z_{ijj}|\mathbf{1}) + 2\sum_{j<c}^{r}\alpha_j(\hat{x})\alpha_c(\hat{x})(|H_{ijc}|\mathbf{1} + \right. \\ &\left. |Z_{ijc}|\mathbf{1})\right] \leq \sum_{i=1}^{r}\alpha_i(x)\left[\sum_{j=1}^{r}\alpha_j^2(\hat{x})\lambda\mathbf{1} + 2\sum_{j<c}^{r}\alpha_j(\hat{x})\alpha_c(\hat{x})\lambda\mathbf{1}\right] = \\ &\sum_{i=1}^{r}\alpha_i(x)\left[\lambda\left(\sum_{j=1}^{r}\alpha_j(\hat{x})\right)^2\mathbf{1}\right] = \lambda\mathbf{1} < \mathbf{1}, \end{aligned}$$

given that $||H_{ijj}||_\infty + ||Z_{ijj}||_\infty \leq \lambda$ and $||H_{ijc}||_\infty + ||Z_{ijc}||_\infty \leq \lambda$ are equivalent to $|H_{ijj}|\mathbf{1} + |Z_{ijj}|\mathbf{1} \leq \lambda\mathbf{1}$ and $|H_{ijc}|\mathbf{1} + |Z_{ijc}|\mathbf{1} \leq \lambda\mathbf{1}$.

This proves that the polyhedron $\mathcal{S}[Y_s,\mathbf{1}]$ is PI and λ-contractive. □

Theorem 4.3.3 is related to the existence of a symmetric polyhedron PI and λ-contractive, so that if $r(k) = 0$, $x_a(k) \to 0$ when $k \to \infty$. The proof is analogous to the one presented in Subsection 3.3.2. The same comments made regarding the case **A** can be extended to the case **B**, regarding the invariance and contraction effect of sets $\mathcal{S}[Y_s, \mathbf{1}]$. It is important to point out that, for the case **B**, the conditions above are valid if $\alpha(\hat{x}(k)) \in \Delta$.

Once the conditions established in Theorem 4.3.3 are satisfied, $\mathcal{S}[Y_s, \mathbf{1}]$ is PI λ-contractive and the the system (4.58)-(4.61) is locally stable. Given that $x_a(0) \in \mathcal{S}[Y_s, \mathbf{1}]$, the integral control action is able to eliminate the tracking error for a constant reference $r(k)$.

Tracking Target Calculation

Equation (4.46) may not have a unique solution due to the membership functions being dependent on $x(\infty)$ and its estimate. Therefore, any $x(\infty)$ that satisfies (4.46) for a given reference r can be considered an equilibrium point of the system (4.58)-(4.61).

The desired set-point for the controlled variables must be such that $r \in \mathfrak{R}_s$ and the state $x_a(\infty)$ do not violate the constraints in $x_a(k) \in \mathcal{S}[Y_s, \mathbf{1}]$. The reference signal satisfying the above requirement is called admissible.

State and Control Constraints

The satisfaction of constraints in the augmented state is ensured when the PI polyhedron is included into the constraints polyhedron. This inclusion can be guaranteed by verifying the conditions in (4.56).

The control constraints are satisfied if the conditions in (4.57) are guaranteed, according to Theorem 4.3.3 and the control law (4.36).

Proof 4.3.4 *Consider* $Yx_a(k) \leq \mathbf{1}$ *and* $S_r r(k) \leq \mathbf{1}$. *Then, from* (4.36), (4.63) *and* (4.64)*:*

$$|S_u^s u(k)| = \left| \sum_{i=1}^{r} \alpha(\hat{x}) S_u^s (K_i^a x_a(k) \;+\; K_{R_i} r(k)) \right| \leq \sum_{i=1}^{r} \alpha(\hat{x})(|Q_i Y_s x_a(k) + Q_i^r S_r^s r(k)|) \leq$$

$$\sum_{i=1}^{r} \alpha(\hat{x})(|Q_i|\mathbf{1} + |Q_i^r|\mathbf{1}) \leq \sum_{i=1}^{r} \alpha(\hat{x})\mathbf{1} = \mathbf{1},$$

given that $||Q_i||_\infty + ||Q_i^r||_\infty \leq 1$ *is equivalent to* $|Q_i|\mathbf{1} + |Q_i^r|\mathbf{1} \leq \mathbf{1}$.

This proves the inclusion of the polyhedral domain given by $\mathcal{S}[Y_s, \mathbf{1}] \subseteq \mathcal{S}[S_u^s K_i^a, \mathbf{1}]$, *such that* $K_i^a \mathcal{S}[Y_s, \mathbf{1}] \oplus K_{R_i} \mathfrak{R}_s \subseteq \mathcal{U}_s$, *where* $\oplus$ *denotes the Minkowski sum operator.* □

In the case **B**, the conditions above are valid if $\alpha(\hat{x}(k)) \in \Delta$.

4.4 Design Strategy Using Bilinear Optimization

The conditions of positive invariance and respect for the constraints proposed above involve linear and bilinear relationships between matrices. These bilinear products can be considered design constraints and the problem discussed in this section can be treated as a nonlinear optimization problem.

For the purpose of tracking, two optimization strategies proposed by [dos Santos, G.F. et al. 2023] are employed here. The first aims to maximize the set of admissible reference signals and the other to minimize the limits of the integral of the tracking error. Maximizing the reference signal limits means that the system can accommodate a wider range of reference signals without violating any constraints. This strategy can be particularly useful when the system needs to accommodate set-point changes in its operation. On the other hand, minimizing the limits of the integral of the tracking error results in an improvement in the speed of the transient response. In numerous control applications, a faster transient response is highly desirable as it enables the system to promptly respond to environmental changes.

Thus, depending on the requirements of the user and the specific characteristics of the system, a choice can be made between obtaining a larger set of operating points or enhancing the convergence speed towards the desired set-point. Further elaboration on the strategies will be provided in the following sections.

4.4.1 Optimization Strategies: General Polyhedra

As long as the cost function $\Phi_i(\cdot)$ is concerned, the choice is not unique, and it depends on the user's needs. Two possible choices are:

i) $\Phi_1 = trace(X_{R_1} + X_{R_2})$. This choice allows to maximize the volume of an asymmetric hyperrectangle $\mathfrak{R}$ of all admissible reference signals, defined in Equation (4.9), by minimizing $\rho_{ij}^{-1} > 0$, for $i = 1, 2$ and $j = 1, \cdots, p$.

ii) $\Phi_2 = \frac{1}{trace(X_{I_1} + X_{I_2})}$. This choice allows to minimize the bounds on the integral of the tracking errors $\varepsilon_{ij}^{-1} > 0$, for $i = 1, 2$ and $j = 1, \cdots, p$.

For the case **A**, the bilinear optimization problem can be formulated as follows:

$$\begin{aligned} &\min_{\Gamma'} \ \Phi_i(\cdot) \\ &\text{subject to (4.27), (4.28), (4.32), (4.33),} \\ &\qquad \psi_l' \leq f_l \leq \psi_l'',\ l = 1, \cdots, \bar{l}, \end{aligned} \tag{4.66}$$

with $\Gamma' = (G_{ij}^a,\ M,\ Y,\ H_{ij},\ Z_{ij},\ Q_i,\ Q_i^r,\ X_{R_1},\ X_{R_2},\ X_{I_1},\ X_{I_2},\ K_{P_i},\ K_{I_i},\ K_{R_i},\ L_i,\ P,\ \lambda)$.

Additional constraints $\psi_l' \leq f_l \leq \psi_l''$, with $\bar{l} \in \mathbb{N}^*$, are imposed to the elements of matrices in Γ' to limit the search space of the optimizer.

When considering the case **B**, the optimization problem can be described similarly by substituting (4.27), G_{ij}^a, Z_{ij}, and H_{ij} with (4.44), G_{ijc}^a, Z_{ijc}, and H_{ijc}, respectively.

Given that $R[Y, \mathbf{1}]$ is represented in the augmented state-space, there is generally no direct proportionality between the sizes of the sets $\mathfrak{R}$ and $R[Y, \mathbf{1}]$. Consequently, admissible regions can be characterized by very tight constraints, particularly in the estimation error space. Thus, if necessary, an additional constraint needs to be incorporated into the optimization problem (4.66) to ensure at least a specified set of initial conditions.

Remark 4.4.1 *If the polyhedral set* $R[Y,\boldsymbol{1}]$ *obtained from* $\Phi_i(\cdot)$ *is characterized by very tight constraints, an additional constraint can be added to* (4.66) *to impose that* $R[S^0,\boldsymbol{1}] \subseteq R[Y,\boldsymbol{1}]$*, where*

$$R[S^0,\boldsymbol{1}] = \{x_a(k) : S^0 x_a(k) \le \boldsymbol{1}\}, S^0 \in \mathbb{R}^{g\times(2n+p)} \tag{4.67}$$

is the minimum set of admissible initial states. From the Lemma 2.1.2, such a set inclusion condition is satisfied if and only if there exists a non-negative matrix $G \in \mathbb{R}^{g\times g}$*, such that:*

$$GS^0 = Y,\ G\boldsymbol{1} \le \boldsymbol{1}. \tag{4.68}$$

Furthermore, the minimizing of the objective function Φ_2 can significantly reduce the set of admissible references by minimizing the bounds of the integral of the tracking error. To avoid extreme solutions, a guaranteed set of admissible references must be associated to the optimization problem.

Remark 4.4.2 *To address the problem of obtaining a solution with an excessively small set of admissible references through the optimization* Φ_2*, it is recommended to define a guaranteed set* $\mathfrak{R}$*, such that* $0 < \rho_{ij}^{-1} \le \overline{\rho}_{ij}^{-1}$*, for* $i = 1,2$ *and* $j = 1,\cdots,p$*.*

4.4.2 Optimization Strategies: Symmetric Polyhedra

i) $\Phi_1 = trace(X_R)$. This choice allows to maximize the volume of a symmetric hyperrectangle $\mathfrak{R}_s$ of all admissible reference signals, defined in Equation (4.16), by minimizing $\rho_i^{-1} > 0$, for $i = 1,\cdots,p$.

ii) $\Phi_2 = \frac{1}{trace(X_I)}$. This choice allows to minimize the bounds on the integral of the tracking errors $\varepsilon_i^{-1} > 0$, for $i = 1,\cdots,p$.

For the case **A**, the bilinear optimization problem can be formulated as follows:

$$\begin{array}{l} \min\limits_{\Gamma''} \ \Phi(\cdot) \\ \text{subject to (4.54), (4.55), (4.56), (4.57),} \\ \qquad \psi_l' \le f_l \le \psi_l'',\ l = 1,\cdots,\bar{l}, \end{array} \tag{4.69}$$

with $\Gamma'' = (G_{ij}^a,\ M,\ Y,\ H_{ij},\ Z_{ij},\ Q_i,\ Q_i^r,\ X_R,\ X_I,\ K_{P_i},\ K_{I_i},\ K_{R_i},\ L_i,\ P,\ \lambda)$.

Additional constraints $\psi_l' \le f_l \le \psi_l''$, with $\bar{l} \in \mathbb{N}^*$, are imposed to the elements of matrices in Γ'' to limit the search space of the optimizer.

When considering the case **B**, the optimization problem can be described similarly by substituting (4.54), G_{ij}^a, Z_{ij}, and H_{ij} with (4.65), G_{ijc}^a, Z_{ijc}, and H_{ijc}, respectively.

Remark 4.4.3 *If the polyhedral set* $\mathcal{S}[Y_s,\boldsymbol{1}]$ *obtained from* $\Phi_i(\cdot)$ *is is characterized by very tight constraints, an additional constraint can be added to* (4.69) *to impose that* $\mathcal{S}[S_s^0,\boldsymbol{1}] \subseteq \mathcal{S}[Y_s,\boldsymbol{1}]$*, where*

$$\mathcal{S}[S_s^0,\boldsymbol{1}] = \{x_a(k) : S_s^0 x_a(k) \le \boldsymbol{1}\},\ S_s^0 \in \mathbb{R}^{g_s\times(2n+p)} \tag{4.70}$$

is the minimum set of admissible initial states. A set inclusion condition is satisfied if and only if there exists a matrix $G \in \mathbb{R}^{g_s \times g_s}$:

$$GS_s^0 = Y_s,\ ||G||_\infty \leq 1. \tag{4.71}$$

To avoid solutions with an excessively small set of admissible references from the optimization Φ_2, it is recommended to define a guaranteed set $\mathfrak{R}_s$, such that $0 < \rho_i^{-1} \leq \overline{\rho}_i^{-1}$, for $i = 1, \cdots, p$. (see Remark 4.4.2)

4.5 Control Action Calculation

The online computation of the control signal $u(k)$ requires only the measurement of the output $y(k)$ and the reference signal $r(k)$. In the following, we provide the algorithm that describes a computational implementation of the stabilizing I-P tracking controller with feedforward term for the case **A** (Algorithm (2)).

For the case **B**, the algorithm remains unchanged; however, the estimated state is now updated using (4.34), and the control signal is obtained from (4.36). Furthermore, it is necessary to calculate the estimated membership functions from $\hat{x}(k)$.

Algoritmo 2: Case **A**

Input: $A_i, B_i, C, K_{P_i}, K_{I_i}, K_{R_i}, L_i, y(k)$, $r(k)$
Output: $u(k)$
Loop For each output $y(k)$ do
  Update $\hat{x}(k+1)$ from (4.17);
  Calculate $\alpha_i(y(k))$;
  Compute $u(k)$ from (4.20);
  return $u(k)$;
  Apply to the system the signal $u(k)$;
end

4.6 Numerical Examples

The KNITRO nonlinear optimization software was used to solve the optimization problem defined in Equation (4.66). To achieve optimal results, the same KNITRO options used in Chapter 3 are employed here.

In Example 4.6.1, symmetric shaped polyhedra are considered, and the optimization variables have been bounded (element by element):

$$\begin{aligned} M,\ Z_{ij},\ H_{ij},\ Z_{ijc},\ H_{ijc},\ Q_i^r,\ Q_i &\in [-1, 1], \\ Y_s,\ K_{P_i},\ K_{I_i},\ K_{R_i},\ L_i,\ P_s &\in [-10^3, 10^3], \\ \lambda &\in [0, 0.99999]. \end{aligned}$$

In Example 4.6.2, general polyhedra are considered, and the optimization variables have been bounded (element by element):

$$\begin{aligned} M,\ Z_{ij},\ H_{ij},\ Z_{ijc},\ H_{ijc},\ Q_i^r,\ Q_i &\in [0,1], \\ Y,\ K_{P_i},\ K_{I_i},\ K_{R_i},\ L_i,\ P &\in [-10^3, 10^3], \\ X_{R_1},\ X_{R_2} &\in [10^{-3}, \overline{\rho}_{i1}^{-1}], \\ X_{I_1},\ X_{I_2} &\in [10^{-3}, 10^3], \\ \lambda &\in [0, 0.99999]. \end{aligned}$$

Let the projection of a polyhedron $R[Y,\mathbf{1}]$ into the state-space and estimation error space be denoted as $p_x(R[Y,\mathbf{1}])$ and $p_e(R[Y,\mathbf{1}])$, respectively. For the symmetric case, we consider the polyhedron $\mathcal{S}[Y_s,\mathbf{1}]$, and denote its projections onto the state and estimation error spaces as $p_x(\mathcal{S}[Y_s,\mathbf{1}])$ and $p_e(\mathcal{S}[Y_s,\mathbf{1}])$. To compute the volume of the PI polyhedra, as well as their projections onto $p_x(\cdot)$ and $p_e(\cdot)$, we employed the multi-parametric toolbox 3.0 (MPT3).

Example 4.6.1 *Consider a discrete-time fuzzy T-S model obtained from a nonlinear model (see, e.g., [Lendek, Z. et al. 2011]) with a sampling period of $T = 0.1s$. The matrices that define the model are given by:*

$$A_1 = \begin{bmatrix} 1.1 & 0.1 \\ -0.1 & 0.8 \end{bmatrix},\ A_2 = \begin{bmatrix} 1.1 & 0.1 \\ -0.3 & 0.8 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1.1 & 0.2 \\ -0.1 & 0.9 \end{bmatrix},\ A_4 = \begin{bmatrix} 1.1 & 0.2 \\ -0.3 & 0.9 \end{bmatrix}, \tag{4.72}$$

$$B_1 = B_2 = B_3 = B_4 = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The system is subject to the following constrains on the variables of state, estimation error and control, respectively: $x_1(k) \in [-1,\ 1] = [-\underline{x}_1,\ \overline{x}_1]$, $x_2(k) \in [-1,\ 1] = [-\underline{x}_2,\ \overline{x}_2]$, $e_1(k) \in [-1,\ 1] = [-\underline{e}_1,\ \overline{e}_1]$, $e_2(k) \in [-1,\ 1] = [-\underline{e}_2,\ \overline{e}_2]$ and $u(k) \in [-10,\ 10]$.

As in [Lendek, Z. et al. 2011], the premise variables are defined by $z_1(k) = x_1(k)$, $z_2(k) = x_1(k)^2$. The membership functions associated with these variables are as follows:

$$\begin{aligned} \mathcal{N}_1(k) &= \tfrac{z_1(k)-z_1(\underline{x_2})}{z_1(\overline{x_2})-z_1(\underline{x_2})},\ 1-\mathcal{N}_1(k) = \tfrac{z_1(\overline{x_2})-z_1(k)}{z_1(\overline{x_2})-z_1(\underline{x_2})}, \\ \mathcal{N}_2(k) &= \tfrac{z_2(k)-z_2(\underline{x_2})}{z_2(\overline{x_2})-z_2(\underline{x_2})},\ 1-\mathcal{N}_2(k) = \tfrac{z_2(\overline{x_2})-z_2(k)}{z_2(\overline{x_2})-z_2(\underline{x_2})}, \end{aligned} \tag{4.73}$$

where

$$\begin{aligned} &\mathcal{M}_{11}(k) = \mathcal{N}_1(k),\ \mathcal{M}_{12} = \mathcal{N}_2(k), \\ &\mathcal{M}_{21}(k) = \mathcal{N}_1(k),\ \mathcal{M}_{22} = 1-\mathcal{N}_2(k), \\ &\mathcal{M}_{31}(k) = 1-\mathcal{N}_1(k),\ \mathcal{M}_{32}(k) = \mathcal{N}_2(k), \\ &\mathcal{M}_{41}(k) = 1-\mathcal{N}_1(k),\ \mathcal{M}_{42}(k) = 1-\mathcal{N}_2(k), \end{aligned} \tag{4.74}$$

and

$$\begin{aligned} \alpha_1(k) = \mathcal{M}_{11}(k)\mathcal{M}_{12}(k),\ \alpha_2(k) = \mathcal{M}_{21}(k)\mathcal{M}_{22}(k), \\ \alpha_3(k) = \mathcal{M}_{31}(k)\mathcal{M}_{32}(k),\ \alpha_4(k) = \mathcal{M}_{41}(k)\mathcal{M}_{42}(k), \end{aligned} \tag{4.75}$$

with

$$z_1(k) := z_1(x_1(k)), \quad z_2(k) := z_2(x_1(k)). \tag{4.76}$$

This is evidently the case **A**, *as the membership functions* $\alpha_i(z(k)) = \alpha_i(x_1(k)) = \alpha_i(y(k))$.

[Lendek, Z. et al. 2011] present a LMI control approach based on a state observer to solve the regulator problem for fuzzy T-S systems. This approach does not take into account constraints for the state and the estimation error and, consequently, does not define the validity region for the proposed controller. Here, we adapt the case presented in [Lendek, Z. et al. 2011] to the constrained tracking problem, and, to limit the control effort, we assume $|u(k)| \leq 10$.

The optimization (4.69) *was tackled using two different setups. In the first setup, the objective function* Φ_1 *and the constraints* (4.54)-(4.57) *were employed. The second setup involved the use of the same objective function and constraints as the first setup, along with additional constraints in* (4.71), *with* $\mathcal{S}[S_s^0, \boldsymbol{1}]$, *such that* $|x_a(k)| \leq 0.2$. *The optimal solutions for both configurations are presented in Table 4.1.*

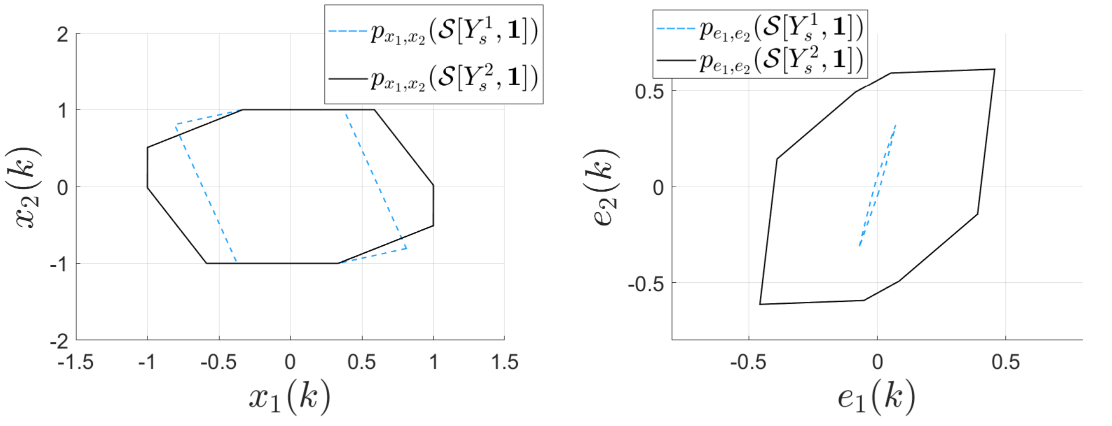
#	X_R	X_I	$Vol(p_x(\mathcal{S}[Y_s,\mathbf{1}]))$	$Vol(p_e(\mathcal{S}[Y_s,\mathbf{1}]))$	$Vol(\mathcal{S}[Y_s,\mathbf{1}])$
1	3.00046	0.01	2.359	0.008	0.001
2	3.00041	0.01	3.265	0.799	0.630

Table 4.1: DOF results with KNITRO: Optimal solutions for case **A**.

As showed in Table 4.1, when adopting the basic configuration, it was verified that the resulting PI polyhedron presented an excessively reduced size. To achieve a solution with wider admissible region, it was necessary to impose the conditions in (4.71). *Once these conditions are satisfied, the polyhedral inclusion relation* $\mathcal{S}[S_s^0, \boldsymbol{1}] \subseteq \mathcal{S}[Y_s, \boldsymbol{1}]$ *is guaranteed. The set* $\mathcal{S}[S_s^0, \boldsymbol{1}]$ *was obtained via trial and error, to preserve the characteristics of the basic solution, while expanding the PI polyhedron.*

Figure 4.1 depicts the projections of the polyhedra $\mathcal{S}[Y_s^1, \boldsymbol{1}]$ *and* $\mathcal{S}[Y_s^2, \boldsymbol{1}]$ *onto the state and error space. It is evident that the set* $\mathcal{S}[Y_s^2, \boldsymbol{1}]$, *obtained for the setup incorporating the additional constraint* (4.71), *exhibits a larger set of admissible states.*

The gain matrices K_{P_i}, K_{I_i}, K_{R_i} *and* L_i, $i = 1,2,3,4$, *obtained for both configurations, are presented in Table 4.2. The corresponding matrices* Y_s^1 *and* Y_s^2, *with* $n_y = 5$ *lines, associated with symmetric polyhedra are shown in Table 4.3.*



(a) Projections onto state-space. (b) Projections onto estimation error space.

Figure 4.1: Basic setup and with additional constraint.

#	K_{P_i}	K_{I_i}	K_{R_i}	L_i^T
1	$[10.9625 \;\; -2.9805]_1$ $[12.4556 \;\; -3.3586]_2$ $[11.0993 \;\; -2.7939]_3$ $[12.8158 \;\; -3.3855]_4$	$[4.5363]_1$ $[4.8178]_2$ $[4.6534]_3$ $[5.2822]_4$	$[6.8613]_1$ $[7.1057]_2$ $[7.4966]_3$ $[7.8138]_4$	$[1.4425 \;\; 2.8260]_1$ $[1.4559 \;\; 2.6913]_2$ $[1.8868 \;\; 3.4177]_3$ $[1.8926 \;\; 3.2414]_4$
2	$[10.6742 \;\; 1.0154]_1$ $[10.0367 \;\; 1.0825]_2$ $[10.7534 \;\; 2.0182]_3$ $[10.2316 \;\; 1.9265]_4$	$[2.6356]_1$ $[2.6473]_2$ $[2.6412]_3$ $[2.6489]_4$	$[2.5887]_1$ $[2.8018]_2$ $[2.5451]_3$ $[2.5410]_4$	$[1.3452 \;\; 0.1145]_1$ $[1.3892 \;\; -0.0368]_2$ $[1.4797 \;\; 0.2027]_3$ $[1.4332 \;\; -0.0096]_4$

Table 4.2: DOF design: K_i and L_i for case **A**.

Y_s^1	-0.2843	1.0029	92.5353	-17.3372	0.8001
	4.7532	-6.6528	-64.5381	17.0489	-5.6890
	2.2533	-10.4940	-49.3919	13.0478	-8.0536
	-3.2921	9.2874	-99.3615	26.2481	7.3772
	-0.2418	0.0188	5.3000	-1.4001	0.8642
Y_s^2	-2.2202	-1.0823	-0.0186	0.9193	0.7563
	-1.0899	0.9596	-1.9527	0.6225	0.3713
	-0.0353	0.9987	3.6411	-0.3060	0.0120
	0.0107	-0.2030	-1.5199	2.3452	0.9125
	0.2643	-0.7704	0.8506	-2.9457	0.1627

Table 4.3: KNITRO results: Polyhedra matrices $\mathcal{S}[Y_s^i,\mathbf{1}]$.

In the following, we seek to verify the system's ability to follow a piecewise constant reference. The controller is defined by the parameters K_{P_i}, K_{I_i}, K_{R_i} and L_i, $i = 1,2$ for the second setup (as indicated in Table 4.5) and $\mathcal{S}[Y_s^2, \boldsymbol{1}]$ is characterized by the matrix Y_s^2 (as specified in Table 4.6).

The simulation scenario considered involves the system initially following a constant reference $r(k) = 0$, which is subsequently changed to $r(k) = 0.3$ after $k = 30$, and finally, at $k = 60$, the reference is set to $r(k) = -0.3$. The initial state and the estimation error were $x(0)^{\mathrm{T}} = [0.1 \;\; 0.3]$ and $e(0)^{\mathrm{T}} = [-0.2 \;\; 0.1]$, as in [Lendek, Z. et al. 2011], such that $x_a(0)^{\mathrm{T}} = [0.1 \;\; 0.3 \; -0.2 \;\; 0.1 \;\; 0.1]$.

Figure 4.2 illustrates the output trajectory, clearly demonstrating that the system output tracks the reference's trajectory, since $x_a(0) \in \mathcal{S}[Y_s^2, \boldsymbol{1}]$ and $r(k) \in \mathfrak{R}_s$. Furthermore, Figure 4.3 depicts the trajectories of the estimation error variables, which converge to zero.

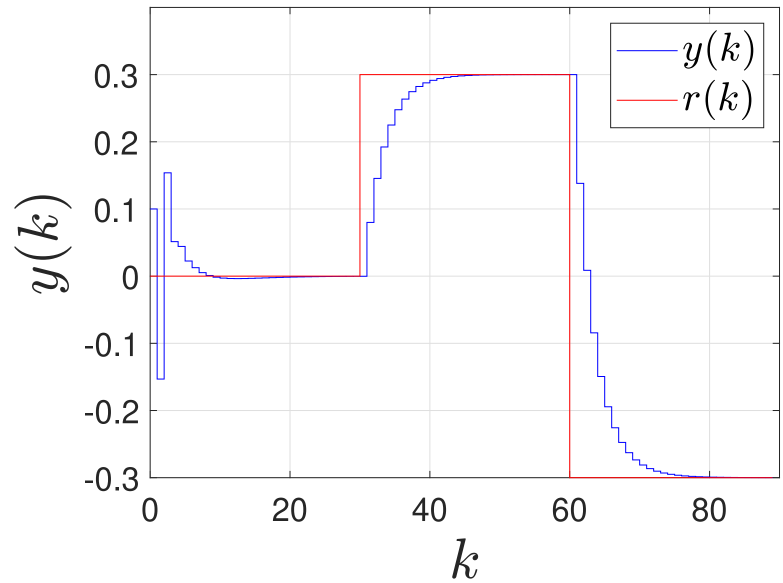


Figure 4.2: Output trajectory - Case **A**.

Figure 4.4 represents the control signal $u(k)$ that corresponds to the output trajectory shown in Figure 4.2. The figure presented provides evidence that the constraints on the control variable are satisfied.

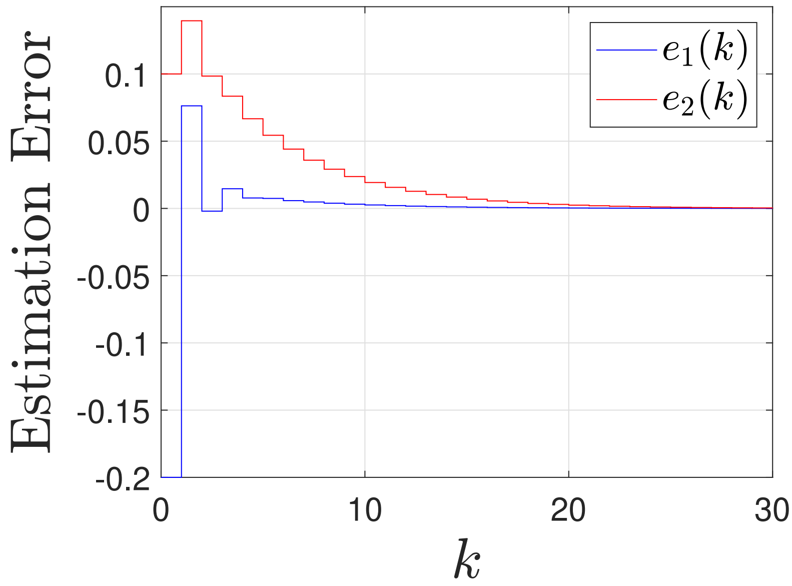


Figure 4.3: Estimation error trajectories - Case **A**.

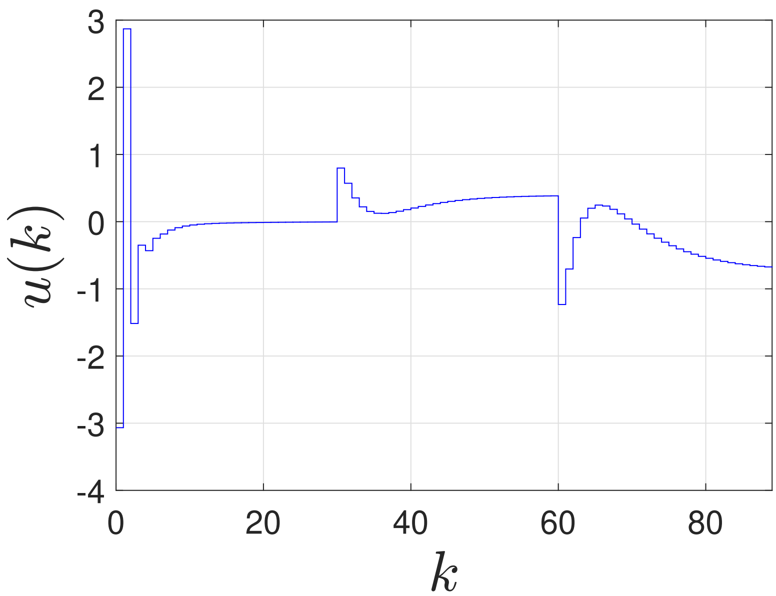


Figure 4.4: Control trajectory - Case **A**.

Example 4.6.2 *Consider the discrete-time fuzzy T-S model [Ueno, N. et al. 2011], defined by the following matrices:*

$$A_1 = \begin{bmatrix} 1.2 & -0.2 \\ 0.1 & -0.4 \end{bmatrix},\ A_2 = \begin{bmatrix} 1.2 & -0.2 \\ 0.2 & 0 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 1.3 \\ 0.7 \end{bmatrix},\ B_2 = \begin{bmatrix} 1.5 \\ 1.0 \end{bmatrix},\ C = \begin{bmatrix} 10 & -2 \end{bmatrix}. \tag{4.77}$$

The system is subject to the following constraints on the state, estimation error, integral of the tracking error, and control, respectively: $x_1(k) \in [-10,\ 20] = [-\underline{x}_1,\ \overline{x}_1]$, $x_2(k) \in [-10,\ 20] = [-\underline{x}_2,\ \overline{x}_2]$, $e_1(k) \in [-10,\ 20] = [-\underline{e}_1,\ \overline{e}_1]$, $e_2(k) \in [-10,\ 20] = [-\underline{e}_2,\ \overline{e}_2]$ *and* $u(k) \in [-5,\ 5]$.

The membership functions and their estimates are given by:

$$\alpha_1(x(k)) = \tfrac{1}{1+e^{-x_1(k)}},\ \alpha_2(k) = \tfrac{e^{-x_1(k)}}{1+e^{-x_1(k)}},$$
$$\alpha_1(\hat{x}(k)) = \tfrac{1}{1+e^{-\hat{x}_1(k)}},\ \alpha_2(\hat{x}(k)) = \tfrac{e^{-\hat{x}_1(k)}}{1+e^{-\hat{x}_1(k)}}. \tag{4.78}$$

*This is evidently the case **B**, as the membership functions* $\alpha_i(x(k)) = \alpha_i(x_1(k)) \neq \alpha_i(y(k))$.

The universe of discourse used to obtain this model is defined by:

$$\begin{array}{l} \underline{\hat{x}}_1 = -\underline{x}_1 - \overline{e}_1 = -10-20 = -30, \\ \overline{\hat{x}}_1 = \overline{x}_1 + \underline{e}_1 = 20+10 = 30, \\ \underline{\hat{x}}_2 = -\underline{x}_2 - \overline{e}_2 = -10-20 = -30, \\ \overline{\hat{x}}_2 = \overline{x}_2 + \underline{e}_2 = 20+10 = 30, \end{array} \tag{4.79}$$

where $\underline{\hat{x}}_i$ *and* $\overline{\hat{x}}_i$ *are, respectively, the lower and upper bound of* $\hat{x}_i$.

When respected, the conditions in (4.32) *guarantee the polyhedral inclusion* $R[Y,\mathbf{1}] \subseteq R[S,\mathbf{1}]$, *hence the state* $x_1(k) \in [-10,\ 20]$. *Since* $\hat{x}_1(k) = x_1(k) - e_1(k)$, *the universe of discourse for the estimated state variable is* $\hat{x}_1(k) \in [-30,\ 30]$. *As the membership functions and their estimates in* (4.79) *are naturally bounded to the interval* $[0,\ 1]$, *it is not necessary to arrange the model and the universe of discourse defined by* $\hat{x}(k)$.

The optimization problem (4.66) *was solved for both* $\Phi_i(\cdot) = \Phi_1$ *and* $\Phi_i(\cdot) = \Phi_2$. *The optimal solutions are presented in Table 4.4.*

$\Phi_i(\cdot)$	$\overline{\rho}_{i1}^{-1}$	$[X_{R_1}\ X_{R_2}]^T$	$[X_{I_1}\ X_{I_2}]^T$	$Vol(R[Y,\mathbf{1}])$
Φ_1	10^3	$[0.01491\ -0.01733]^T$	$[0.00130\ -0.00132]^T$	5.89×10^5
Φ_2	10^3	$[998.378\ -998.527]^T$	$[121.198\ -132.802]^T$	4.43×10^{-12}
Φ_2	$1/10$	$[0.10000\ -0.10000]^T$	$[0.03631\ -0.04787]^T$	1.47×10^3
Φ_2	$1/30$	$[0.03333\ -0.03333]^T$	$[0.01514\ -0.01371]^T$	2.33×10^4
Φ_2	$1/50$	$[0.02000\ -0.02000]^T$	$[0.00587\ -0.00750]^T$	2.11×10^5

Table 4.4: DOF results with KNITRO: Optimal solutions for case **B**.

In [dos Santos, G.F. et al. 2023], a suitable trade-off between convergence speed and the size of admissible references set was achieved experimentally (through trial and error) by modifying the parameter n_y, corresponding to the number of rows of Y_s that characterizes the PI polyhedron. Here, the same procedure was adopted, but there no clear improvement has been achieved. The adjustment made in from Φ_2 was considered inadequate since the minimization of the admissible integral state bounds implied an excessive reduction of the sets $\mathfrak{R}$ and $R[Y,\mathbf{1}]$, for $10^{-3} \leq \rho_{i1}^{-1} \leq \overline{\rho}_{i1}^{-1}$, $i = 1,2$ and $\overline{\rho}_{i1}^{-1} = 10^3$.

As observed in Table 4.4, a reduction in $\overline{\rho}_{i1}^{-1}$ resulted in an expansion of the set $\mathfrak{R}$ and a reduction in the values of X_I, leading to a slower transient response. The opposite effect was observed when $\overline{\rho}_{i1}^{-1}$ was increased. This enhanced flexibility in the design enables the user to make adjustments that better suit their requirements. Choosing Φ_1 as cost function provided a solution with the largest set of admissible references while adopting Φ_2 resulted in a faster transient response, according to a chosen guaranteed set of admissible references.

Table 4.5 presents the gain matrices K_{P_i}, K_{I_i}, K_{R_i} and L_i, $i = 1,2$, obtained from ϕ_2 with $\overline{\rho}_{i1}^{-1} = 1/30$. The results are associated with the corresponding matrix Y, shown in Table 4.6, for a polyhedron defined by $n_y = 11$ lines.

Case	K_{P_i}	K_{I_i}	K_{R_i}	L_i^{T}
B	$\begin{bmatrix} 0.6583 & -0.2348 \end{bmatrix}_1$ $\begin{bmatrix} 0.7191 & -0.1696 \end{bmatrix}_2$	$\begin{bmatrix} 0.0192 \end{bmatrix}_1$ $\begin{bmatrix} 0.0199 \end{bmatrix}_2$	$\begin{bmatrix} 0.0253 \end{bmatrix}_1$ $\begin{bmatrix} 0.0290 \end{bmatrix}_2$	$\begin{bmatrix} 0.1173 & -0.0641 \end{bmatrix}_1$ $\begin{bmatrix} 0.1312 & -0.0560 \end{bmatrix}_2$

Table 4.5: DOF design: K_i and L_i for case **B**.

Case	n_y	Y
B	11	$\begin{bmatrix} 0.4373 & 0.1036 & -0.4860 & -0.2452 & -0.0222 \\ -0.0194 & -0.2264 & 0.4140 & -0.1525 & -0.0002 \\ -0.1221 & -0.2782 & -1.2184 & 0.4029 & 0.0045 \\ 0.4810 & 0.0850 & 0.8296 & -0.1472 & -0.0246 \\ 0.1479 & 0.1691 & -0.4055 & 0.1488 & -0.0060 \\ 0.0259 & 0.2056 & 1.1403 & -0.3825 & 0.0018 \\ -0.4622 & -0.1819 & 0.2943 & 0.3970 & 0.0248 \\ -0.2766 & -0.0381 & -0.4709 & 0.0710 & 0.0259 \\ -0.0023 & -0.0183 & -0.1016 & 0.0341 & 0.0163 \\ 0.0079 & 0.0203 & 0.0912 & -0.0291 & -0.0150 \\ -0.2495 & 0.0602 & -0.4712 & -0.0518 & 0.0067 \end{bmatrix}$

Table 4.6: KNITRO results: Polyhedron matrix $R[Y,\mathbf{1}]$.

In the following, we present two different situations. First, we seek to illustrate the invariance of the set $R[Y,\mathbf{1}]$ and compliance with the constraints established in the project. In the second situation, we aim to verify the system's ability to follow a piecewise constant *reference. In both cases, the controller is defined by the parameters* K_{P_i}, K_{I_i}, K_{R_i} *and* L_i, $i = 1,2$ *(as indicated in Table 4.5) and* $R[Y, \mathbf{1}]$ *is characterized by the matrix* Y *(as specified in Table 4.6).*

Situation 4.1 *Consider a constant reference* $r(k) = 30$, $\forall k \geq 0$, *and initial conditions defined from four vertices of the polyhedron* $R[Y, \mathbf{1}]$. *Here,* $x(0)$, $e(0)$, $v(0)$, $x(\infty)$, $e(\infty)$, *and* $v(\infty)$ *represent the initial conditions and steady-state. The constraints on the integral of the tracking error, admissible references, and control inputs are expressed as follows:* $-72.94 \leq v(k) \leq 66.05$; $|r(k)| \leq 30$; *and* $|u(k)| \leq 5$.

Figures 4.5 and 4.6 present the state and estimation error trajectories plotted on the projections $p_{x_1,x_2}(R[Y, \mathbf{1}])$ *and* $p_{e_1,e_2}(R[Y, \mathbf{1}])$, *respectively. The initial conditions and the steady-state are denoted by* $x(0)$, $e(0)$ *and* $x(\infty)$, $e(\infty)$.

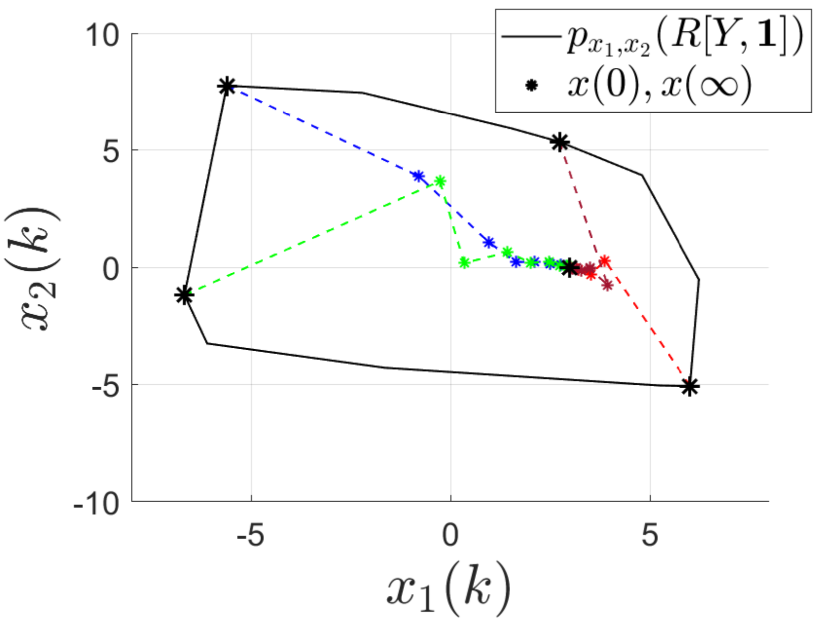


Figure 4.5: State trajectories - Case **B**.

Figures 4.7, 4.8, and 4.9 depict the trajectories of the integral of the tracking error, the control input, and the output, respectively. The presented figures provide evidence that the constraints on the control and the integral of the tracking error are satisfied, and the output converges to the reference.

The conditions in (4.44) *are satisfied, therefore the polyhedron* $R[Y, \mathbf{1}]$ *is PI and* λ*-contractive. Assuming that* $r(k) \in \mathfrak{R}$, *it follows that the output* $y(k) \rightarrow r(k)$, $k \rightarrow \infty$. *This is due to the integral control action, which is designed to eliminate any steady-state error between the output signal and the reference. Therefore, the system is able to track the desired reference within the specified limits.*

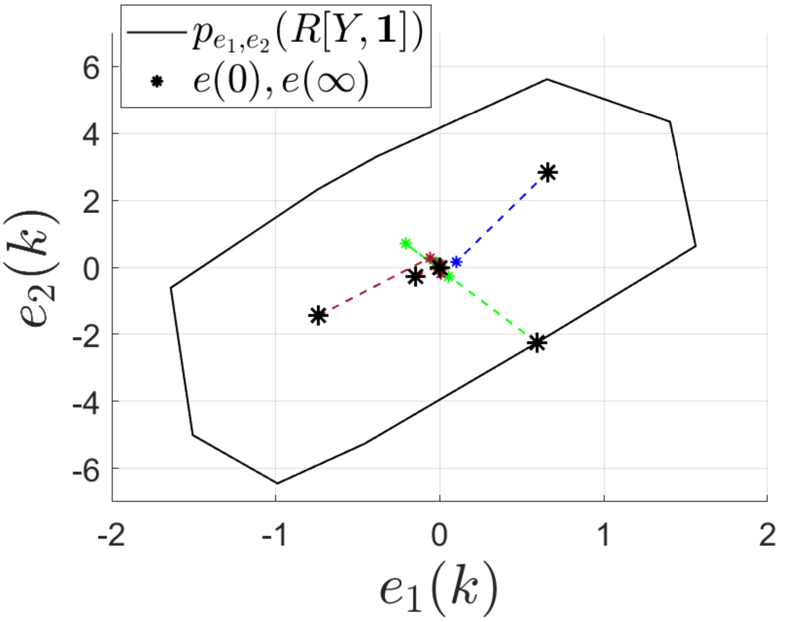


Figure 4.6: Estimation error trajectories - Case **B**.

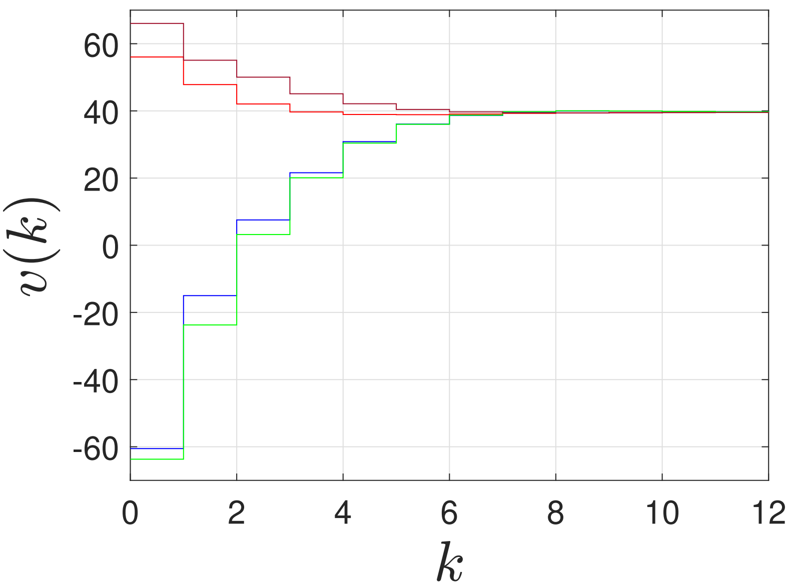


Figure 4.7: Integral state trajectories - Case **B**.

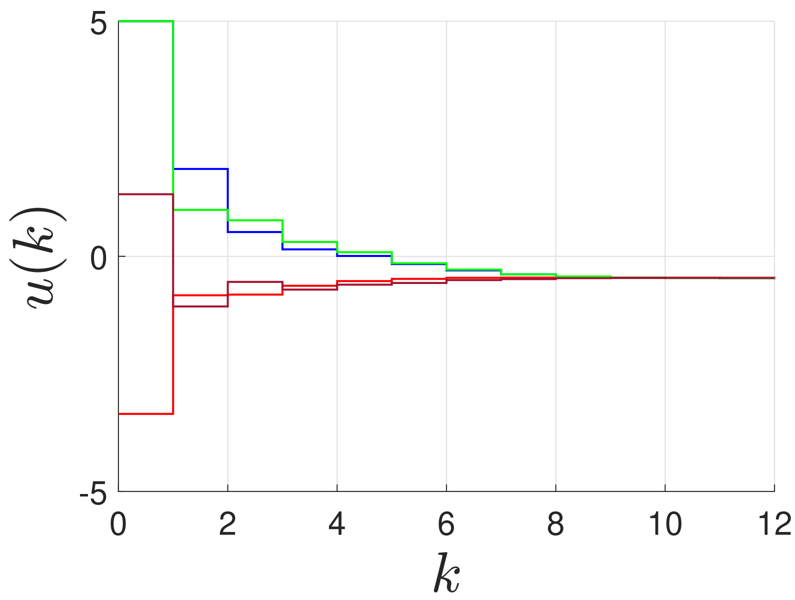


Figure 4.8: Control trajectories - Case **B**.

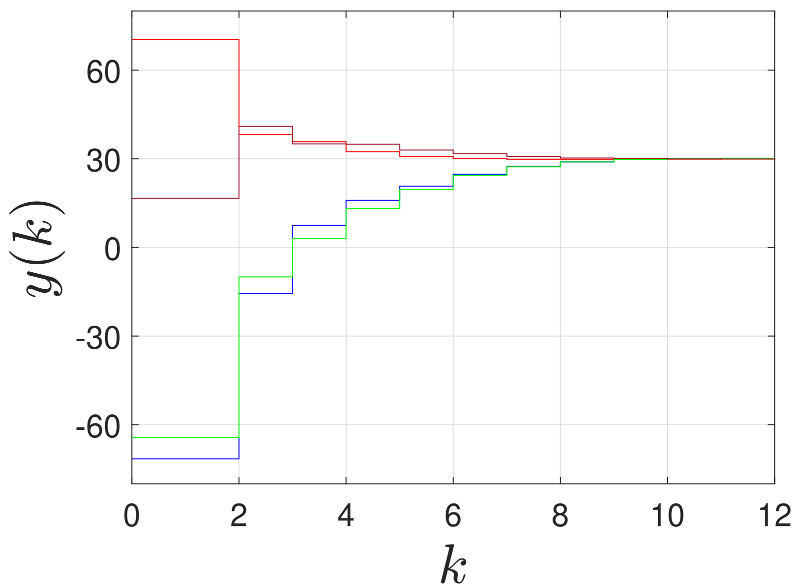


Figure 4.9: Output trajectories - Case **B**.

Situation 4.2 *Now, consider the scenario in which the system is initially required to track a constant reference* $r(k) = 30$*, followed by a reference change to* $r(k) = -30$ *after* $k = 15$ *and finally, at* 30 *time steps, the reference changes to* $r(k) = 0$*. We assume* $x(k)^{\mathrm{T}} = [0.8 \;\; 2]$ *and* $\hat{x}(k)^{\mathrm{T}} = [0 \;\; 0]$*, such that the initial state of the system is set to* $x_a(0)^{\mathrm{T}} = [0.8 \;\; 2 \;\; 0.8 \;\; 2 \;\; 16]$.

Figure 4.10 illustrates the tracking performance of the system for the aforementioned reference. The output, $y(k)$*, and the reference,* $r(k)$*, are depicted, and it can be observed that the system output asymptotically tracks the reference trajectory. Furthermore, Figure 4.11 presents the estimation error trajectories. As can be seen, the estimation error variables converge to zero.*

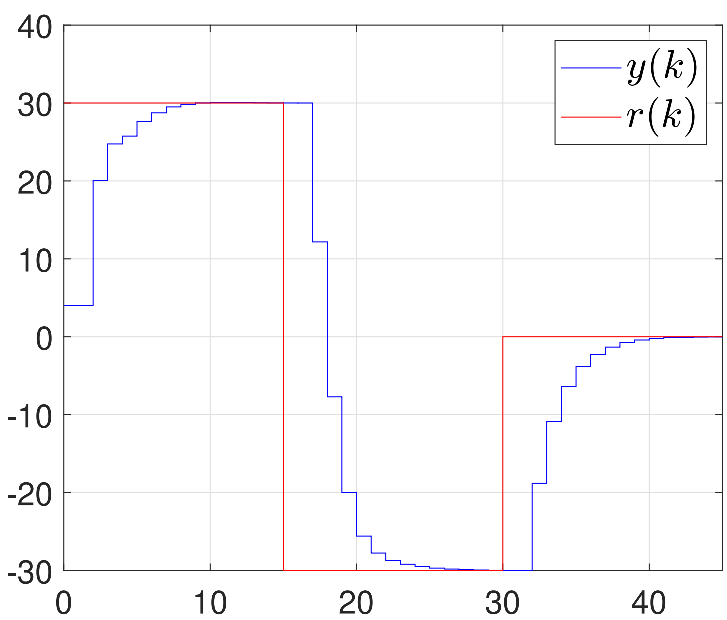


Figure 4.10: Output trajectory - Case **B**.

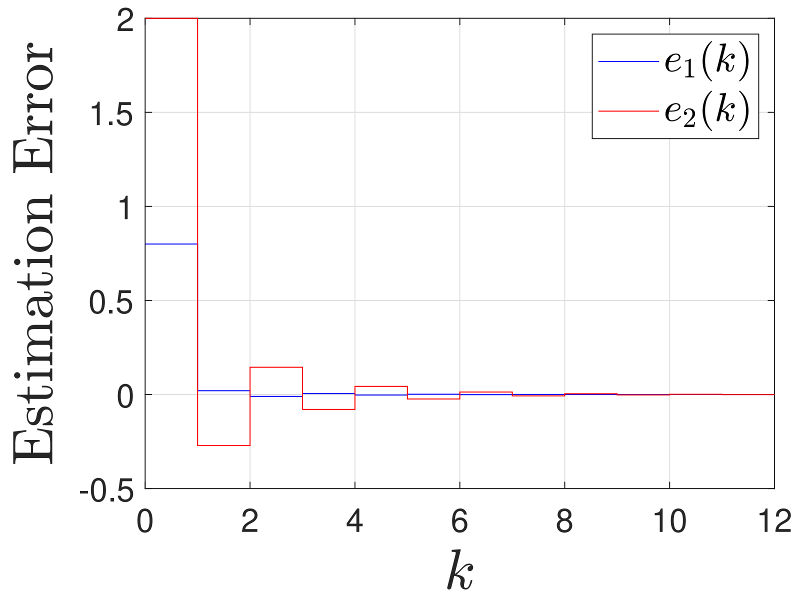


Figure 4.11: Estimation error trajectories - Case **B**.

Figure 4.12 represents the control signal $u(k)$ that corresponds to the output trajectory shown in Figure 4.10. The control input $u(k) \in [-5,\ 5]$, $\forall k \geq 0$, then the control constraints remain satisfied.

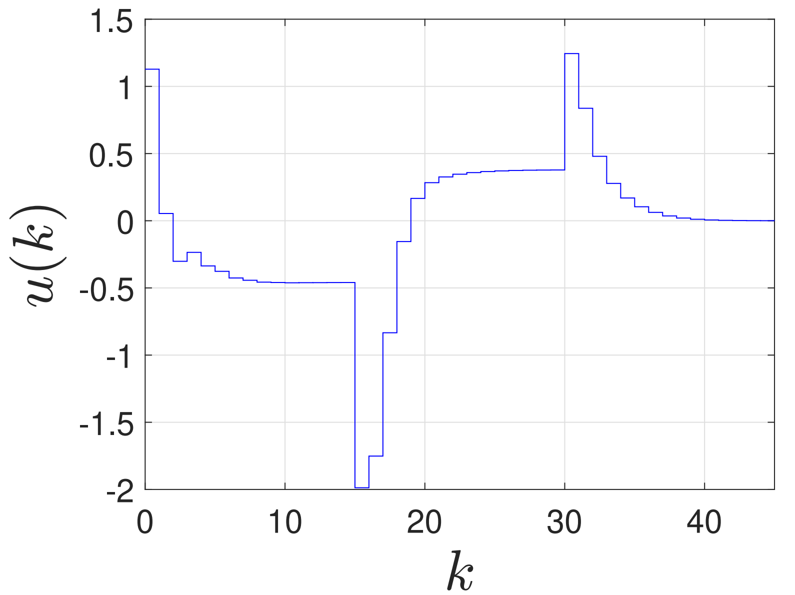


Figure 4.12: Control trajectory - Case **B**.

4.7 Conclusions

The proposed approach explores the concept of robust positive invariance to design a local stabilizing I-P tracking controller for fuzzy T-S systems. Analogous to the conditions for robust invariance of polyhedral sets, sufficient conditions are established for the invariance of polyhedral sets in presence of a piecewise constant reference $r(k)$, which can be interpreted as a bounded disturbance.

Based on these conditions, a bilinear programming problem is formulated to obtain a solution that yields a PI polyhedron, and the controller gains. Moreover, these gains guarantee the local asymptotic convergence of the output to the reference since any stabilizer controller with the presented structure ensures offset-free asymptotic set-point tracking for a piecewise constant reference.

To enhance the tracking performance of the system, two optimization strategies were employed. The first strategy maximizes the set of admissible reference signals, enabling effective tracking of a wide range of set-points and enhancing the controller's adaptability to various operating conditions. The second strategy focuses on minimizing the bounds of the integral of the tracking error, aiming for a faster transient response and more efficient control action. Overall, these strategies contribute to improve the tracking performance, aligning with the specified design objectives.

Chapter 5

Conclusion

In this work, initially a theoretical foundation of positively invariant sets and their application in control problems subject to constraints was presented. In particular the property of positive invariance concerning polyhedral domains in state-space of discrete-time fuzzy T-S systems was addressed. It was seen that in order to keep the state trajectory completely confined to the positive invariant polyhedron, standard techniques assume that the state is entirely measurable, which is not always possible in practice. An static output feedback control approach was presented to deal with the impossibility of measuring the complete state vector.

This approach is limited to the situation where the membership functions of the T-S model exhibit only depends on the system output. However, it is crucial to acknowledge that, in general, the membership functions may rely on unmeasured states. A natural way to deal with this situation is to design a state observer. In this regard, we proposed a dynamic feedback approach based on the fuzzy T-S observer for both scenarios. The estimation process is essential for the general case, since the membership functions depend on non-accessible variables. In the simplest case, although the membership functions depend only on the output, the estimated state feedback results, in general, in controllers with better performance and with larger sets of admissible states associated with them than the static output feedback control.

In this work, the designs of the controller and the observer are based on a methodology that incorporates the estimation error and state variables into a unified augmented vector (see, e.g., [Wang, H.O., & Tanaka, K. 2004, Tanaka, K. et al. 1998, Tanaka, K., & Wang, H.O. 1997]). The concept of positive invariance of polyhedral domains in augmented state-space (state + estimation error) established in Chapter 3 is a cornerstone of the proposed approach. In this regard, PI polyhedral sets are used to ensure that state, estimation error and input control constraints are satisfied at all times. Furthermore, given that the PI set is also contractive, the augmented state trajectory converges to the origin, thus guaranteeing the solution to the constrained regulator problem.

The problem of tracking a constant reference signal was also considered, for which the concept of robust positive invariance is used along a stabilizing Integral-Proportional (I-P) controller. The positive invariance of polyhedra sets in augmented state-space (state + estimation error + tracking error integral state) presented in Chapter 4 is the key element of the proposed technic.

Comparison with other approaches available in the literature revealed that our proposed approach enables the following main contributions:

- Compute both the PI set, and the associated controller, which distinguishes it from the approaches presented in [Ariño, C. et al. 2013, Ariño, C. et al. 2014];
- Obtain larger sets of admissible states when compared to static output feedback control techniques;
- Outperform the methods discussed in [Ding, B., & Pan, H. 2016, Ping, X. et al. 2021], in terms of computational cost and ease of implementation;
- Address the scenario where the membership functions depend on non-accessible variables, setting it apart from the approaches presented in [Dórea, C.E. et al. 2020] and [Lopes, A.N. et al. 2020].

In future works, we aim to extend our proposal to address systems that are subject to bounded disturbances. Specifically, we plan to explore this extension in both the constrained regulator problem and the constrained tracking problem. The possibility of enforcing state trajectories into a given polyhedral set for persistently disturbed fuzzy T-S systems can be characterized through the concept of robust invariance, as in [Dórea, C.E. et al. 2020]. In this regard, PI polyhedra in augmented state-space are used to ensure that the design constraints are satisfied, upon the presence of a bounded disturbance $d(k) \in \mathcal{D}$, where $\mathcal{D}$ is a compact polyhedral set containing the origin. Consequently, a nonlinear optimization problem can be formulated based on the robust invariance conditions, in accordance with the methodology proposed in this thesis. As a result, we obtain the control gains and the admissible state set, which is characterized by a Robust Positively Invariant (RPI) polyhedron. Offset-free constant reference tracking in this context of PI sets under disturbances can be treated as in, e.g., [Almeida, T.A. 2020, Almeida, T.A. et al. 2021], by introducing a disturbance observer.

The following articles were published based on the concepts discussed in this Thesis:

- Isidório, I.D., Dórea, C.E., & Castelan, E.B. (2022), Controle por realimentação de saída baseado em observador de estado para sistemas fuzzy T-S sujeitos a restrições, In: 'Preprints of the XXIV Congresso Brasileiro de Automática (in Portuguese)';
- Isidório, I.D., Dórea, C.E., & Castelan, E.B. (2023), 'Observer-based output feedback control using invariant polyhedral sets for fuzzy t–s models under constraints', *Journal of Control, Automation and Electrical Systems*, **34**, 752–765. `https://doi.org/10.1007/s40313-023-01011-7`.

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