Triplet superconductivity in ferromagnets due to magnon exchange

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We consider the superconducting pairing induced by spin-wave exchange in a ferromagnet with both conduction and localized electrons, the latter being described as spins. We use the microscopic Eliashberg theory to describe the pairing of conducting electrons and the random phase approximation approach to treat the localized spins assuming an exchange coupling between the conducting electrons and spins. In the framework of a nonrelativistic Hamiltonian, we found that the spin-wave exchange results in equal spin electron pairing described by the two components of the order parameter, $\Delta^\uparrow$ (both spins up) and $\Delta^\downarrow$ (both spins down). Due to the conservation of total spin projection on the axis of the spontaneous ferromagnetic moment, the spin-wave exchange at low temperatures includes an emission of magnons and an absorption of thermal magnons by the conduction electrons. The absorption and emission processes depend differently on the temperature, with the absorption being progressively suppressed as the temperature drops. As a result, the superconducting pairing exists only if the electron–spin-wave exchange parameter g exceeds some critical value g_c . At $g > g_c$, pairing vanishes if the temperature drops below the lowest point T_{cl} or increases above the upper critical point $T_{ch} \approx T_m$ (the Curie temperature) where the spin waves cease to exist. This behavior inherent to the spin-carrying glue is in an obvious disagreement with the results of the conventional BCS approach, which assumes that the effective electron-electron attraction is simply proportional to the static magnetic susceptibility.

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I. INTRODUCTION

Eliashberg [1,2] and Scalapino, and Wilkins [3,4] developed the microscopic approach for BCS phonon-induced Cooper pairing in standard superconductors such as Nb and Pb. This electron-phonon model explicitly introduces phonons (weak ion distortions from their equilibrium positions in the crystal lattice) under the effect of electrons, resulting in effective indirect electron attraction. In the framework of this model, the involvement of phonons may be ascertained by tunneling measurements. Namely, in this model the origin of the pairing mechanism is imprinted into the superconducting order parameter via its frequency dependence on the density of states of phonons working as a glue [3,4]. Such a dependence may be revealed by the measurements of the I - V characteristics of the tunneling between the superconductor and the normal metal. Then peaks in the $I(V)$ should coincide with those observed in neutron scattering experiments. The experimental results indeed confirm the phonon mechanism and the dynamic nature of the glue in superconductors such as Nb, Al, In, and Pb. What is more, McMillan [5] has shown that tunneling data provide all the information needed to calculate the superconducting critical temperature T_c , which can then be compared with the experimental T_c to check the consistency of the model. After all of these theoretical results have been confirmed by their corresponding measurements, no doubt remains about the phonon origin of the glue in standard superconductors with singlet electron pairing.

However, there is doubt that the phonon mechanism of Cooper pairing can explain the behavior of a broad family of high-temperature superconductors, although predictions of the BCS phenomenological approach are still used to describe some of their experimental properties. It would be instructive to develop at least one more microscopic model to see whether or not the predictions of the BCS approach are universal in describing superconducting properties, at least on a phenomenological level, in other scenarios.

Recently, anomalous superconducting properties were observed in the U-based compounds UGe₂, URhGe, and UCoGe, which are quite different from the well-known singlet superconductors; see [6–8]. First, superconductivity was observed in their ferromagnetic state below the Curie temperature T_m (observed superconducting critical temperature $T_c < T_m$). The manifestation of pairing takes place deep in the ferromagnetic state, with the internal exchange field suppressing the Cooper pairing of electrons with opposite spins as an indication that, in those compounds, the pairing may be in the triplet channel rather than in the singlet one. The observation of an upper critical field well above the paramagnetic limit for singlet superconductors [7] also favors triplet pairing, although enhanced spin-orbital coupling in materials with heavy elements such as U may significantly weaken the destructive effects of the internal exchange field and of the applied magnetic field on superconducting singlet pairing [9,10]. If there is indeed triplet pairing, one can exclude the phonon mechanism because it provides stronger attraction in

the singlet channel. If phonons cannot be effective for pairing in ferromagnetic phases of U-based compounds, the only probable glue for pairing is the exchange via spin waves. This logic led McHale, Hattori [11,12], and Mineev [8] to consider, respectively, spin waves and magnetic fluctuations near T_m as the glue mechanism for the superconducting pairing at T_c just below T_m .

The replacement of phonon glue by spin-wave glue appears to be a natural move that should not lead to drastic modification of superconducting properties. However, there is an important difference between these types of glue. Namely, the spin waves (magnons) carry a projection of spin in the direction of the ferromagnetic moment, and in the nonrelativistic approach the projection of total spin of both conduction and localized electrons is preserved. Thus the exchange by magnons is restricted by this conservation law, while such a restriction does not exist for exchange by spinless phonons. One can thus anticipate that the magnon-driven superconductors may not follow the BCS scenario. In the following, we will find that this is really the case in the nonrelativistic approximation.

In this paper, we consider the nonrelativistic Hamiltonian with the Ruderman-Kittel-Kasuya-Yosida (RKKY) interaction of conducting electrons and localized spins. We use the microscopic Eliashberg theory to treat the triplet superconductivity induced by the exchange of the spin waves (magnons) in the presence of ferromagnetic ordering. To describe the superconductor at all temperatures, we use the Tyablikov [13] random phase approximation, while for the electrons and their interaction with spin waves we use the Gor'kov-Nambu formalism. We show that the spin-wave exchange mechanism results in equal spin-triplet pairing described by the two-component order parameter, while it excludes singlet pairing. We show that it is the spin conservation at electron-magnon scattering that determines the coupling of these two components of the order parameter resulting in specific low-temperature non-BCS behavior of triplet superconductors. We find the corresponding phase diagram, which is indeed drastically different from that for the standard phonon-based BCS superconductors. We also show that as in the phonon-driven superconductors, the tunneling measurements in ferromagnet superconductors reveal the origin of the dynamics of the glue, i.e., they show the peculiarities of the magnon spectrum in this case.

II. THE HAMILTONIAN

We consider a system of conduction electrons in a broad s -band with a two-dimensional dispersion $\epsilon(\mathbf{k})$ and localized spins S describing the spin degrees of localized electrons in the narrow f - or d -band positioned below the Fermi level and occupied only partially due to the strong Coulomb electron repulsion in that band. We assume that the localized spins interact with each other by means of the direct spin-spin Heisenberg exchange and also via the RKKY interaction with the conduction electrons. In the nonrelativistic approach (ignoring spin-orbit coupling), the system is described by the model Hamiltonian:

$$\mathcal{H} = \mathcal{H}_e + \mathcal{H}_s + \mathcal{H}_{\text{int}}, \quad (1)$$

$$\mathcal{H}_e = \int d\mathbf{r} \hat{\psi}_\alpha^\dagger(\mathbf{r})[\epsilon(\mathbf{k}) - \epsilon_F]\hat{\psi}_\beta(\mathbf{r}), \quad (2)$$

$$\begin{aligned} \mathcal{H}_s = \sum_{m,n} [J_{d,n,m}(S_m^x S_n^x + S_m^y S_n^y) + J'_{d,n,m} S_m^z S_n^z] \\ + \frac{1}{2} K (S_{xm}^2 + S_{ym}^2), \end{aligned} \quad (3)$$

$$\mathcal{H}_{\text{int}} = \int d\mathbf{r} \sum_{n,i,\alpha,\beta} [h_{\text{ex}}(\mathbf{r}, \mathbf{n}) \psi_\alpha^\dagger(\mathbf{r}) \sigma_{\alpha\beta}^i S_n^i \psi_\beta(\mathbf{r}) + \text{c.c.}]. \quad (4)$$

Here $\epsilon(\mathbf{k}) = k^2/2m$ is the electron dispersion with the effective mass m , and the electron momentum is $\mathbf{k} = (k_x = k \cos \phi, k_y = k \sin \phi)$, where ϕ is the polar angle in the x, y plane and the σ^i are the Pauli matrices. The indices α and β denote spin-up and -down states of the conduction electrons. The parameters J_d and J'_d describe direct exchange between localized spins, while the parameter K describes the magnetic anisotropy energy [14]. The exchange interaction between the conduction electrons and localized spins, which is characterized by the parameter h_{ex} , results in an indirect RKKY interaction of the localized spins. We do not account here for the orbital effect of the internal magnetic field induced by localized spins. It may be accounted for by the renormalization of the superconducting critical temperature. We do not account also for the Coulomb repulsion between the conduction electrons because, as we will show next, the magnon exchange results in Cooper triplet spin pairing of electrons with aligned spins only. Due to the Pauli principle, the repulsion of such electrons is significantly weakened.

We note that the simple Hamiltonian used here does not pretend to describe the real complex situation of the f -electrons in U-compounds, with the partially occupied f -band representing both localized and itinerant electrons (see Refs. [11,12] and references therein). Nevertheless, this minimal Hamiltonian model allows us to account for low-energy electron and localized spin degrees of freedom. Another important point is that it conserves the z -projection of the total spin of the system. The interplay of electron and spin degrees of freedom in the framework of the spin conservation law results in the main peculiar property of this triplet superconductivity—a two-component structure of order parameters for triplet pairing, the spin-preserved coupling of these two components, and its effect on the phase diagram of triplet superconductors.

We assume that the localized spins are ordered ferromagnetically along the z axis, and $S_z = \langle S_{z,n} \rangle_n$ is the ferromagnet order parameter. In this case, the spin waves (magnons) are collective small deviations of the magnetization along the x and y axes, and they are described by the spin operators $S_n^\pm = S_{x,n} \pm iS_{y,n}$, acting as a creation and the annihilation operators of spin waves, $|S_{x,n}|, |S_{y,n}| \ll S_z$. In the momentum representation we use the operators $S_q^\pm = S_{x,q} \pm iS_{y,q}$, where $\mathbf{q}$ is the spin-wave momentum. The commutation relations of the spin operators $S_n^\pm$ obey conventional SU(2) algebra:

$$S^+ S^- - S^- S^+ = 2S_z. \quad (5)$$

In the spin-wave random phase approximation (RPA) [13], we replace the operator $S_{z,n}$ on the right-hand side of this equation by its thermal average $\langle S_z \rangle_T$, and we replace the spin operators by $S_n^\pm = (S_{x,n} \pm iS_{y,n})/(2\langle S_z \rangle_T)$. They obey the

boson commutation relation and play the role of creation and annihilation operators of spin waves at site $\mathbf{n}$. In the momentum representation, the operators $S_{\mathbf{q}}^{\pm} = S_{x,\mathbf{q}} \pm iS_{y,\mathbf{q}}$ are the creation and annihilation operators of the spin wave (magnon) with the momentum $\mathbf{q}$ and with the dispersion

$$\epsilon_s(\mathbf{q}) = \langle S_z \rangle (K + v_s^2 \mathbf{q}^2), \quad (6)$$

where v_s is the magnon velocity so that $v_s^2 = cT_m$, with the numerical coefficient c of order unity. Indeed, the Hamiltonian Eq. (3) results in the equations of motion for spin-wave operators in the Fourier representation with respect to coordinates and imaginary time τ :

$$\dot{S}_{\mathbf{q}}^{\pm} = \pm \langle S_z \rangle \epsilon_s(\mathbf{q}) S_{\mathbf{q}}^{\pm}. \quad (7)$$

Now magnons play the same role of bosonic excitations as phonons, and thus in the spin-wave RPA we can use Wick's theorem to develop the perturbation theory with respect to the electron-magnon interaction.

Accounting for the spin-orbit coupling results in a more general form of the interaction part of the Hamiltonian,

$$\mathcal{H}_{\text{int}} = \int d\mathbf{r} \sum_{\mathbf{n}, \gamma, \kappa, \alpha, \beta} [h_{\text{ex}}^{\gamma\kappa}(\mathbf{r}, \mathbf{n}) \psi_{\alpha}^{+}(\mathbf{r}) \sigma_{\alpha\beta}^{\gamma} S_{\mathbf{n}}^{\kappa} \psi_{\beta}(\mathbf{r}) + \text{c.c.}]. \quad (8)$$

The nondiagonal terms in the indices γ and κ are absent in the nonrelativistic approximation, and their relation to the diagonal ones may be estimated as $\delta g/g$, where g is the electron gyromagnetic ratio and δg is its deviation from 2. For metals Fe ($Z = 26$), Cr ($Z = 24$), and Mn ($Z = 25$), the ratio $\delta g/g$ is 10^{-3} , 10^{-2} , and 10^{-3} , respectively [15]. Taking into account that this ratio depends on the atomic number Z as Z^4 , we see that $\delta g/g \approx 3 \times (10^{-8} - 10^{-9}) Z^4$. Thus the nonrelativistic Hamiltonian is a good approach for d -wave Fe, Mn, Cr, and rare-earth metals. For U compounds, the relativistic part of the Hamiltonian may be important, and our results may be invalid for them.

Though the description of the phonon glue and the magnon glue look very similar, an important difference exists between their interactions with electrons. Phonons do not carry spins, while magnons carry ± 1 spin projections for $S_{\mathbf{q}}^{\pm}$, respectively. As a result, the effective electron-electron interaction becomes spin-dependent with on-site vertexes proportional to $\sum_i \sigma_i S_i = (\sigma^{+} S^{-} + \sigma^{-} S^{+}) \langle S_z \rangle$. Due to spin conservation for the nonrelativistic electron-spin interaction Hamiltonian, the complete cycle of magnon exchange should include both the emission and the absorption of magnons. In view of that, the effective pairing electron-electron interaction $\mathcal{E}_{ee}$ is proportional to the product of those two matrix elements. In other words, $\mathcal{E}_{ee}$ is not proportional to the static magnetic susceptibility, as was assumed in Refs. [8, 12, 16] in the BCS approach for ferromagnetic superconductors. Consequently, $\mathcal{E}_{ee}$ itself becomes strongly temperature-dependent because the absorption of magnons is strongly suppressed at low temperatures. Such a limitation is absent in the case of phonon glue.

At low temperatures $T \rightarrow 0$ the RPA replacement gives exact results because the amplitudes of the spin waves are negligible. At nonzero temperature in the RPA approach, one needs to find $\langle S_z \rangle_T$ by solving appropriate self-consistent equations. For spin $S = 1$, which will be considered later on,

it has the form

$$\frac{1}{\langle S_z \rangle} = \frac{1}{\pi} \int_0^{\pi} d\mathbf{q} \coth \frac{\epsilon_s(\mathbf{q})}{2T}. \quad (9)$$

The magnon correlation function in the Fourier representation with respect to coordinate and imaginary time τ is, in turn,

$$\langle S_{\mathbf{n}}^{\mp}(\tau) S_{\mathbf{m}}^{\pm}(\tau') \rangle_{v_n, \mathbf{q}} = \langle S \rangle_T [i v_n \mp \epsilon_s(\mathbf{q})]^{-1}, \quad v_n = 2\pi n. \quad (10)$$

Note that in the RPA approach, the frequencies of all magnetic excitations drop proportionally to $\langle S_z \rangle_T$ as T approaches T_m , and they vanish at $T > T_m$. Thus this approach does not account for strong magnetic fluctuation near the Curie temperature.

In our calculations, we will consider the spin dynamics unaffected by superconducting pairing. This approach is valid because the free energy of the magnetic system is much larger than the superconducting condensation energy at all temperatures, except at the very narrow region near T_m . Indeed, the free energy of localized spins is of the order of T_m per spin, and it drops as $(T_m - T)^2/T_m$ as T approaches T_m . In contrast, the energy of the superconducting pairing is of the order of $(T_c - T)^2 N(0)$ and $T_c \leq T_m \ll \epsilon_F$. Here $N(0)$ is the electron density of states per spin and $N(0) \approx 1/\epsilon_F$, with ϵ_F being the electron Fermi energy. Thus only at temperatures at which $(T_m - T)/(T_m - T_c) \approx (T_m/\epsilon_F)^{1/2}$ do the magnetism and superconducting pairing strongly affect each other.

The Curie temperature T_m of ferromagnetic ordering is determined by the sum of the RKKY interaction and of the direct spin-coupling contribution $T_d = \sum_m J'_{d, \mathbf{n}, \mathbf{m}}$, i.e., $T_m = c_1 J_{\text{ex}} + c_2 J_d$, where $J_{\text{ex}} \approx h_{\text{ex}}^2 N(0)$, and c_1, c_2 are numerical coefficients.

III. NONUNITARY TRIPLET ORDER PARAMETER

The description of the triplet superconducting state is quite different from that of a singlet state because, in the first place, the order parameter should be characterized by a vector rather than a scalar. For spin-triplet pairing, the superconducting order parameter $\hat{\Delta}$ is characterized by the vector $\mathbf{d}$ as

$$\hat{\Delta}(\mathbf{r}, \mathbf{k}) = i d_i(\mathbf{r}, \mathbf{k}) \sigma_i \sigma_y = \sum_i D_i(\mathbf{r}, \mathbf{k}) \sigma_i, \quad (11)$$

$$D_0 = i d_y, \quad D_x = d_z, \quad D_y = 0, \quad D_z = -d_x, \quad (12)$$

where summation over repeated indices $i = 0, x, y, z$ is assumed. Here and in the following, we denote $\sigma_0 = \hat{1}$. The order parameter depends on the center of mass $\mathbf{r}$ and on the momentum $\mathbf{k}$ of the Cooper pair. It is odd in both coordinates and in momenta. In the momentum space, the vector $\mathbf{d}$ is related to the amplitudes of the spin-up, spin-down, and zero-spin projections of the superconducting order parameter $\Delta^{\uparrow}, \Delta^{\downarrow}, \Delta^0$ as follows [8]:

$$d_x = (1/2)(-\Delta^{\uparrow} + \Delta^{\downarrow}), \quad (13)$$

$$d_y = -(i/2)(\Delta^{\uparrow} + \Delta^{\downarrow}), \quad (14)$$

$$d_z = \Delta_0. \quad (15)$$

Here $\Delta^\uparrow$ is the amplitude of the state $|\uparrow\uparrow\rangle$, $\Delta^\downarrow$ is the amplitude of the state $|\downarrow\downarrow\rangle$, while Δ_0 is the amplitude of the state $|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle$. We note that while the superconducting state characterized by the parameter d_z corresponds to an electron pair with spin oriented in the direction perpendicular to the z -axis, states with parameters (d_x, d_y) correspond to an electron system of at least two up-spins or two down-spins along the z axis. Exchange by spin waves does not lead to the interaction of electrons with all spins aligned because the spin wave carries unity spin and thus the spins of two interacting electrons should be opposite to each other. Hence, pairing in the system of two or more electrons with aligned spins does not exist. In contrast, in systems of four or more electrons with different spins, the ordering characterized by the parameters (d_x, d_y) is possible due to the presence of attraction between electrons with opposite spins.

For triplet pairing, the dependence of the amplitudes d_x, d_y on momentum $\mathbf{k}$ of Cooper pair electrons should be odd. The function d_z should also be odd in $\mathbf{k}$, but we will see that $d_z = 0$ in the magnon exchange model. Thus the two functions $d_{x,y}$ fully characterize the triplet order parameter in the magnon exchange model. Singlet pairing is characterized by the scalar $d_0(\mathbf{k})$ and $\hat{\Delta} = id_0\sigma_y$, with $D_y = id_0$.

IV. SELF-CONSISTENT EQUATIONS FOR SUPERCONDUCTING ORDER PARAMETERS

We neglect the vertex corrections for the electron-magnon coupling, as was done for the electron-phonon case, because we assume $\epsilon_s(\mathbf{q}) \ll \epsilon_F$. We consider a two-dimensional electron system with momenta $\mathbf{k} = k_F(\sin\phi, \cos\phi)$, while a change of electron momentum from $\mathbf{k}$ to $\mathbf{k}'$ at the electron-magnon scattering requires conservation of the momentum,

$$\mathbf{k} - \mathbf{k}' = \mathbf{q}. \quad (16)$$

In the superconducting state, we write down the electron Green's function in the Nambu representation as a 2×2 matrix $\hat{G}(\omega, \mathbf{k})$, whose diagonal components G_{11} and G_{22} are the conventional Green's functions, while the off-diagonal elements G_{12} and G_{21} are the Gor'kov's functions F and F^+ describing the pairing condensation. They are matrices in the spin coordinates, and in a general case they may be expanded as a sum in the σ_i matrices:

$$\hat{F}^+(\omega_n, \mathbf{k}) = \sum_i a_i(\mathbf{k}, i\omega_n)\sigma_i, \quad i = 0, x, y, z. \quad (17)$$

The Eliashberg equations for the Green's functions in the electron-phonon model in the Matsubara representation have the form (see Refs. [1,2])

$$\begin{aligned} [i\omega - \xi(\mathbf{k})]G(\omega, \mathbf{k}) &= 1 + g^2 \int d\mathbf{q} G(\omega - \nu, \mathbf{k} - \mathbf{q})D(\nu, \mathbf{q})G(\omega, \mathbf{k}) \\ &+ g^2 \int d\mathbf{q} F(\omega - \nu, \omega, \mathbf{k} - \mathbf{q})D(\nu, \mathbf{q})F^+(\omega, \mathbf{q}), \end{aligned} \quad (18)$$

$$\begin{aligned} [-i\omega - \xi(\mathbf{k})]F^+(\omega, \mathbf{k}) &= 1 + g^2 \int d\mathbf{q} G(\omega - \nu, \mathbf{k} - \mathbf{q})D(\nu, \mathbf{q})F^+(\omega, \mathbf{k}) \\ &+ g^2 \int d\mathbf{q} F^+(\omega - \nu, \mathbf{k} - \mathbf{q})D(\nu, \mathbf{q})G(\omega, \mathbf{k}), \end{aligned} \quad (19)$$

where ω denotes $\omega_n = 2\pi(n + 1/2)$, while $\xi(\mathbf{k})$ is the electron energy accounted from the Fermi energy. Further, $D(\nu, \mathbf{k}) = -(T[\varphi(x_1)\varphi(x_2)])$ is the phonon Green function, $\nu = 2\pi n$, and g is the electron-phonon coupling parameter.

For the electron-magnon model, we need to replace the phonon Green function by the spin correlation functions Eq. (10) and the coupling parameter g by $J(k, k')$ and account for spin conservation at the vertex, i.e., we replace

$$\begin{aligned} g^2 \sum_\nu \int d\mathbf{q} D(\nu, \mathbf{q})F^+(\omega - \nu, \mathbf{k} - \mathbf{q}) &\Rightarrow \sum_\nu \int d\mathbf{q} |J(\mathbf{k}, \mathbf{k} - \mathbf{q})|^2 \\ &\times [S^{+-}(\nu, \mathbf{q})\sigma^- F^+(\omega - \nu, \mathbf{k} - \mathbf{q})\sigma^+ \\ &+ S^{-+}(\nu, \mathbf{q})\sigma^+ F^+(\omega - \nu, \mathbf{k} - \mathbf{q})\sigma^-]. \end{aligned} \quad (20)$$

Neglecting the renormalization of the electron Green function due to electron-magnon coupling, the electron diagonal Green's function in the absence of an external magnetic field is

$$G_0(\omega, \mathbf{k})^{-1} = i\omega_n - \xi - h_z\sigma_z. \quad (21)$$

The exchange field h_z splits the electron energy levels into two electron bands with spins up and down. At strong h_z one of these bands becomes empty, while the other is filled with spins of the corresponding kind. As was mentioned above, superconducting pairing is absent in this case because electrons of the same kind do not interact with each other via spin-wave exchange. At low h_z , electrons with both spin up and down are present close to the Fermi level and contribute to pairing. This case is discussed in the following.

From Eq. (19), in the second-order perturbation theory with respect to the electron-spin exchange interaction term in Eq. (4), we obtain the self-consistent equation for the Gor'kov's Green function $F^+(\omega, \mathbf{k})$ at T_c :

$$\begin{aligned} F^+(\omega_n, \mathbf{k}) &= \sum_i a_i\sigma_i \\ &= -2T \sum_{\mathbf{k}', \omega_{n'}} |J(\mathbf{k}, \mathbf{k}')|^2 G_0(-\omega_{n'}, \mathbf{k}')G_0(\omega_n, \mathbf{k}') \\ &\times [S^{+-}(\nu, \mathbf{q})\sigma^- F^+(\omega_{n'} - \nu, \mathbf{k} - \mathbf{q})\sigma^+ \\ &+ S^{-+}(\nu, \mathbf{q})\sigma^+ F^+(\omega_{n'} - \nu, \mathbf{k} - \mathbf{q})\sigma^-] \end{aligned} \quad (22)$$

with the spin correlation functions given by Eq. (10) and

$$J_{\mathbf{k}\mathbf{k}'} = \int d\mathbf{r} h_{\text{ex}}(\mathbf{r})e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}. \quad (23)$$

Here on the right side we introduced the magnon Green's functions $S_{ij}(i\nu_n, \mathbf{q})$,

$$S_{+-}(\tau - \tau', \mathbf{n} - \mathbf{m}) = \langle S_{\mathbf{n}}^+(\tau)S_{\mathbf{m}}^-(\tau') \rangle, \quad (24)$$

$$S_{-+}(iv_m, \mathbf{q}) = \int d\nu (iv_m - \nu)^{-1} \delta[\nu - \epsilon_s(\mathbf{q})], \quad (25)$$

$$S_{+-}(iv_m, \mathbf{q}) = \int d\nu (iv_m + \nu)^{-1} \delta[\nu - \epsilon_s(\mathbf{q})]. \quad (26)$$

First, we prove that the self-consistency equation results in the relation $D_x = 0$, i.e., $d_z = 0$. For that we write the electron Green's function $\hat{G}$ in the form Eq. (17). Then on the right-hand side of Eq. (22) we have that

$$\begin{aligned} \sigma_+ \sigma_{x,y} \sigma_- &= \sigma_- \sigma_{x,y} \sigma_+ = 0, \\ \sigma_{\pm} \sigma_0 \sigma_{\mp} &= 2(\sigma_0 \pm \sigma_z), \\ \sigma_{\pm} \sigma_z \sigma_{\mp} &= \mp 2(\sigma_0 \pm \sigma_z). \end{aligned} \quad (27)$$

Due to the first relation above, the matrices σ_x and σ_y are absent on the right-hand side of Eq. (22) and, consequently, on the left-hand side they are absent, too. As a result, only the components d_x, d_y are nonzero. This means that the magnon exchange results in equal-spin triplet states $\hat{\Delta} = (\Delta^\uparrow |\uparrow\uparrow\rangle, \Delta^\downarrow |\downarrow\downarrow\rangle)$. The opposite statement is also true, namely if we have a superconductor with such $\hat{\Delta}$, the glue should carry spin $S_z = \pm 1$ because otherwise the coupling of its component does not obey the spin conservation law. It is important to note that relations (27) *exclude also singlet pairing induced by the magnon exchange mechanism*.

To find the superconducting critical temperature of the transition from normal state to pairing state, T_c , and the structure of the superconducting order parameter at T_c , it is sufficient to know F^+ at the lowest order in the components of the order parameter, d_x and d_y . Thus we write Eq. (22) in the form

$$F^+(\omega_n, \mathbf{k}) = -2T \sum_{\mathbf{k}', \omega'} |J_{\mathbf{k}\mathbf{k}'}|^2 \{ S_{-+}(i\omega - i\omega', \phi - \phi') \Lambda \sigma_- (D_0 \sigma_0 + D_z \sigma_z) \sigma_+ + S_{+-}(i\omega - i\omega', \phi - \phi') \Lambda \sigma_+ (D_0 \sigma_0 + D_z \sigma_z) \sigma_- \}, \quad (28)$$

$$\Lambda = G_0(-\omega, \mathbf{k}') G_0(\omega, \mathbf{k}') = \frac{\xi'^2 + \omega^2 + h_z^2 - 2\xi' h \sigma_z}{[(\xi' + h_z)^2 + \omega^2][(\xi' - h_z)^2 + \omega^2]}. \quad (29)$$

Next we integrate over ξ' and take into account that at $h_z \ll \epsilon_F$ we may put $h_z = 0$. Summing over Matsubara frequencies ω of Green's function $F^+(\omega, \mathbf{k})$, we replace by the integral over real frequencies ω' , as was done in (2.12) of Ref. [3] for the superconductor with electron-phonon coupling. We obtain the equation

$$F^+(\omega_n, \mathbf{k}) = - \sum_{\mathbf{k}'} |J_{\mathbf{k}\mathbf{k}'}|^2 \int_{-\infty}^{+\infty} \frac{d\omega'}{\omega'} \left\{ \frac{\sigma_+ (D_0 \sigma_0 + D_z \sigma_z) \sigma_-}{\omega' - (i\omega_n + \epsilon_s(\mathbf{k} - \mathbf{k}'))} [f(\omega') + n(\epsilon_s)] - \frac{\sigma_- (D_0 \sigma_0 + D_z \sigma_z) \sigma_+}{[\omega' - i\omega_n + \epsilon_s(\mathbf{k} - \mathbf{k}')] } [f(-\omega') + n(\epsilon_s)] \right\}, \quad (30)$$

$$f(\omega) = [e^{\beta\omega} + 1]^{-1}, \quad n(\epsilon_s) = [e^{\beta\epsilon_s} - 1]^{-1}, \quad (31)$$

where $f(\omega)$ and $n(\omega)$ are the fermion and the boson distribution functions, and $\beta = 1/T$. Now the expression (28) for $F^+(\omega_n)$ can be analytically continued with respect to $i\omega_n$ to the real axis from the upper half-plane by replacing $i\omega_n$ by $\omega + i\delta$:

$$\begin{aligned} F^+(\omega, \phi) = D_0 \sigma_0 + D_z \sigma_z &= - \int_0^{2\pi} d\phi' |J(\phi, \phi')|^2 \int_{-\infty}^{+\infty} \frac{d\omega'}{\omega'} \left\{ \frac{[\text{Im}(D_0 - D_z)](\sigma_0 + \sigma_z)}{\omega' - i\delta - \omega - \epsilon_s(\phi - \phi')} \left[\frac{1}{1 + e^{\beta\omega'}} + \frac{1}{e^{\beta\epsilon_s} - 1} \right] \right. \\ &\quad \left. - \frac{[\text{Im}(D_0 + D_z)](\sigma_0 - \sigma_z)}{\omega' - i\delta - \omega + \epsilon_s(\phi - \phi')} \left[\frac{1}{1 + e^{-\beta\omega'}} + \frac{1}{e^{\beta\epsilon_s} - 1} \right] \right\}. \end{aligned} \quad (32)$$

We introduce the notations

$$\begin{aligned} \mathcal{D}(\omega, \phi) &= D_0(\omega, \phi) - D_z(\omega, \phi) = \Delta^\downarrow(\omega, \phi), \\ \mathcal{R}(\omega, \phi) &= D_0(\omega, \phi) + D_z(\omega, \phi) = \Delta^\uparrow(\omega, \phi). \end{aligned}$$

Equating the coefficients in front of operators σ_0 and σ_z on both sides of Eq. (32), we obtain two coupled equations connecting the order parameters $\Delta^\downarrow$ and $\Delta^\uparrow$ at the temperature of the second-order phase transition T_c . This is the temperature of the vanishing superconducting order parameter at cooling, and we will show in the following that so obtained critical temperature decreases with g . Here we introduce the dimensionless parameter of electron-magnon coupling $g = 2|J_{\mathbf{k}\mathbf{k}'} S|^2 N(0)/T_m$. The first equation is

$$\text{Im} \Delta^\downarrow(\omega, \phi) = 4g \int_0^{2\pi} d\phi' \int_{-\infty}^{+\infty} \frac{d\omega'}{\omega'} [\text{Im} \Delta^\uparrow(\omega', \phi')] \left[\frac{1}{1 + e^{-\beta\omega'}} + \frac{1}{e^{\beta\epsilon_s(\phi - \phi')} - 1} \right] \delta[\omega' - \omega + \epsilon_s(\phi - \phi')]. \quad (33)$$

This equation describes the creation of a magnon with energy ϵ_s by an electron that changes its spin by -1 , while its energy goes from ω' to $\omega' - \epsilon_s$. The second equation,

$$[\text{Im}\Delta^\uparrow(\omega, \phi)] = -4g \int_0^{2\pi} d\phi' \int_{-\infty}^{+\infty} \frac{d\omega'}{\omega'} [\text{Im}\Delta^\downarrow(\omega', \phi')] \left[\frac{1}{1 + e^{\beta\omega'}} + \frac{1}{e^{\beta\epsilon_s(\phi - \phi')} - 1} \right] \delta[\omega' - \omega - \epsilon_s(\phi - \phi')], \quad (34)$$

describes the absorption of a magnon with energy ϵ_s by the electron that changes its spin by $+1$ and suffers the energy change from ω' to $\omega + \epsilon_s$. Here $\epsilon_s(\phi - \phi') = \kappa + \gamma^2[1 - \cos(\phi - \phi')]$ with $\gamma = 2ak_F$, and a is the spacing between localized spins while $\kappa = K/T_m$. Writing the spin-wave energy in this form, we assume that $ak_F \lesssim 1$. In the following, we take $\gamma = 1$. We also express all quantities with the dimension of energy in units of T_m .

Now we should account for the space dependence of the order parameters inherent to the triplet pairing. It should be odd in space variables because it is even in spin variables. Namely, the triplet superconducting order parameters $d_x(x, y)$ and $d_y(x, y)$ should change sign at the replacement $x \rightarrow -x$ and $y \rightarrow -y$. Thus both order parameters should change sign under the transformation $\phi \rightarrow \phi + \pi$. Note that for dispersionless magnons, the triplet pair order parameter vanishes because it must be odd in coordinate dependence (as opposed to the singlet ordering). To account for the change of sign at the transformation $\phi \rightarrow \phi + \pi$, we expand $\Delta^{\downarrow, \uparrow}(\omega, \phi)$ in a Fourier series with respect to the angle ϕ :

$$[\text{Im}\Delta^\downarrow(\omega, \phi)] = \sum_{n=1} \Delta_n^\downarrow(\omega) \sin[(2n-1)\phi], \quad (35)$$

$$[\text{Im}\Delta^\uparrow(\omega, \phi)] = \sum_{n=1} \Delta_n^\uparrow(\omega) \sin[(2n-1)\phi]. \quad (36)$$

We perform the integration over angles on the right-hand sides of Eqs. (33) and (34) by changing the variables in the integral. We get

$$\sin \phi \sin \phi' = (\cos \alpha - \cos \beta)/2, \quad (37)$$

$$\int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' f(\phi - \phi', \phi + \phi') = \frac{1}{2} \int_{-2\pi}^{2\pi} d\alpha \int_{\alpha}^{4\pi-\alpha} d\beta f(\alpha, \beta), \quad (38)$$

where $\alpha = \phi - \phi'$, while $\beta = \phi + \phi'$. Integral $\int_{\alpha}^{4\pi-\alpha} d\beta \cos \beta$ gives $2 \sin \alpha$. Subsequent integration over α in the interval $(-2\pi, 2\pi)$ with a function even in α gives 0. Then we integrate over angles and note that the states with different n do not mix in the linear approximation. Thus the minimum temperature T at a given g , which we call $T_{cl}(g)$, comes from the coupled equations for the least nonuniform state $n = 1$. Hence, we obtain finally the pair of integral equations

$$[\text{Im}\Delta^\downarrow(\omega)] = 8g \int_{-\infty}^{+\infty} \frac{d\omega'}{\omega'} \Delta^\uparrow(\omega') \frac{\omega' - \omega + \tilde{\kappa}}{[1 - (\omega' - \omega + \tilde{\kappa})^2]^{1/2}} \left[\frac{1}{e^{-\beta\omega'} + 1} + \frac{1}{e^{\beta(\omega - \omega')} - 1} \right], \quad (39)$$

$$[\text{Im}\Delta^\uparrow(\omega)] = 8g \int_{-\infty}^{+\infty} \frac{d\omega'}{\omega'} \Delta^\downarrow(\omega') \frac{\omega' - \omega - \tilde{\kappa}}{[1 - (\omega' - \omega - \tilde{\kappa})^2]^{1/2}} \left[\frac{1}{e^{\beta\omega'} + 1} + \frac{1}{e^{-\beta(\omega - \omega')} - 1} \right]. \quad (40)$$

Here and in the following, the integration over ω' is limited additionally by the condition that the expression under the square root should be positive-definite. Eliminating $\Delta^\uparrow(\omega')$ in Eq. (33), we obtain a single equation for $\Delta^\downarrow(\omega)$:

$$\Delta^\downarrow(\omega) = -64g^2 \int_{-\infty}^{+\infty} \frac{d\omega''}{\omega''} \Delta^\downarrow(\omega'') K(\omega, \omega''), \quad (41)$$

$$K(\omega, \omega'') = \int_{-\infty}^{+\infty} \frac{d\omega'}{\omega'} \frac{\omega' - \omega + \tilde{\kappa}}{[1 - (\omega' - \omega + \tilde{\kappa})^2]^{1/2}} \frac{\omega' - \omega'' + \tilde{\kappa}}{[1 - (\omega' - \omega'' + \tilde{\kappa})^2]^{1/2}} \left[\frac{1}{e^{-\beta\omega'} + 1} + \frac{1}{e^{\beta(\omega - \omega')} - 1} \right] \left[\frac{1}{e^{\beta\omega''} + 1} + \frac{1}{e^{\beta(\omega'' - \omega')} - 1} \right]. \quad (42)$$

Note that two absorptions and two emissions of spin waves are needed to accomplish the cycle $\Delta^\uparrow \Rightarrow \Delta^\downarrow \Rightarrow \Delta^\uparrow$ for the Cooper pair. Both transitions are necessary because neither state $\Delta^\uparrow$ nor $\Delta^\downarrow$ can exist without each other since the spin wave carries spin 1 and thus the attraction of electrons with the same spins due to spin-wave exchange is completely absent.

The temperature affects differently the rates of these two transformations $\Delta^\uparrow \Rightarrow \Delta^\downarrow \Rightarrow \Delta^\uparrow$. According to Eqs. (41) and (42), the probability of the emission and the subsequent absorption process are described by the factor

$$[f(\omega) + n(\epsilon_s)][f(-\omega) + n(\epsilon_s)] = \frac{1}{4} \left[\cosh^{-2} \frac{\beta\omega}{2} + \sinh^{-2} \frac{\beta\epsilon_s}{2} \right]. \quad (43)$$

This factor, describing the temperature dependence of the triplet order parameter, drops exponentially as the temperature drops well below the anisotropy gap in the spin-wave energy (if the gap is absent, the factor drops as a power law with T). This means that T_{cl} drops as the coupling parameter g increases. Such a tendency is opposite to the dependence T_c in singlet phonon-mediated superconducting pairing described by a single order parameter in the BCS approach. In the latter case, the temperature dependence of the order parameter is described by the factor $f(-\omega/T) - f(\omega/T) = \tanh(\omega/2T)$ at any T because absorption of the phonon is not needed to exchange by phonons.

V. NUMERICAL CALCULATION

As we have integrated over angles, we are no longer concerned with the angular dependence of the order parameter. Thus it suffices to solve the homogeneous integral equations (41) and (42) for the functions of a single variable ω . Toward that end, we convert the integral equation into an algebraic eigenvalue problem of the linearized Eliashberg equations. The first step is to convert the integrations over ω'' into a summation by using some quadrature scheme. Here we use the Gauss-Legendre quadrature $\int_a^b f(t)dt = \sum_{i=1}^N w_i f(x_i)$, which is one of the many Gaussian-type quadrature formulas, where the x_i 's, also known as the abscissa, are the zeros of the Legendre polynomial—the integrand is evaluated at these points, and w_i are the weights. More details of the quadrature method can be found in [17].

Thus, we construct the one-dimensional matrix $\mathcal{A}_{mn''}$ corresponding to $K(\omega_n, \omega_{n''})$ to solve the equation $\mathbf{1} - \lambda \hat{\mathcal{A}} = 0$ by following a standard diagonalization procedure for the matrix $\mathcal{A}_{mn''}$.

After solving the equation $\mathbf{1} - \lambda \hat{\mathcal{A}} = 0$, we find the eigenvalues of the corresponding matrix at a given temperature T . The maximum eigenvalue g defines temperature T_{cl} as well as the corresponding eigenvectors $\Delta^\downarrow(\omega)$ and $\Delta^\uparrow(\omega)$. We will summarize our results in the next section.

VI. BOUNDARIES OF TRIPLET SUPERCONDUCTING PHASE IN A FERROMAGNET

The temperature at which $\Delta^\downarrow(\omega)$ becomes nonzero determines the lower superconducting critical temperature T_{cl} for the second-order phase transition from the normal ferromagnetic state to the triplet superconducting state. The dependencies of $g(T_{cl})$ for $\kappa = 0.1$ and 0.2 are shown in Fig. 1.

In accordance with the previous discussion and in accordance with Eq. (43), T_{cl} is a descending function of g , which is opposite to the behavior of $T_c(g)$ in phonon-based superconductors. Note, however, that in our case T_{cl} is the transition temperature from a normal ferromagnetic phase NF to a superconducting ferromagnetic phase TS+F as a result of the increase in the thermal magnon number and the enhancement of coupling between the components $\Delta^\uparrow$ and $\Delta^\downarrow$ of the order parameter. In contrast, in the phonon-driven pairing, the transition from the superconducting to the normal phase upon warming is due to smoothening of the step in the electron population at the Fermi surface by thermal motion. Such drastically different behavior of spin-wave-induced pairing in

FIG. 1. The phase diagram of the ferromagnet in the plane of temperature T_{cl} and the strength g of the exchange coupling of localized spins and conducting electrons. As the temperature increases from $T = 0$, the superconducting triplet pairing establishes itself due to the exchange of electrons with spin waves at the second-order transition line $T_{cl}(g, \kappa)$. The descending line $g(T_{cl})$ is a direct consequence of the spin conservation at the magnon exchange, which is effective only in the presence of thermal magnons. Note that this drop in g vs T_{cl} becomes more pronounced as the anisotropy parameter and thus the gap in the spin-wave spectrum increase. Warming up the superconductor results in the destruction of spin waves and the Cooper pairs as the temperature approaches the Curie temperature T_m . This change from triplet superconducting phase to normal phase is signaled by a first-order phase transition; its exact position may be found if magnetic fluctuations are accounted for.

a ferromagnet is the direct consequence of spin conservation at magnon exchange.

As the anisotropy of the ferromagnet increases, the suppression of the coupling of components $\Delta^\uparrow, \Delta^\downarrow$ becomes stronger because of the increase in the magnon energy ϵ_s and the subsequent drop in the number of thermal magnons. Consequently, a bigger g is needed to obtain the same T_{cl} ; see Fig. 1.

Upon heating up the system toward T_m , the decrease in the component coupling is widespread, while the effective electron-electron interaction falters because the decrease in the spin-wave amplitudes, proportional to the RPA factor $\langle S_z \rangle$, is compensated for by the decrease in magnon frequencies. Approaching T_m , we enter into the region of strong magnetic fluctuations, and our RPA to find spin correlation functions becomes invalid. To find the position of the first-order phase transition, $T_{ch}(g)$, there must be a more accurate treatment of the spin correlation functions. We think that T_{ch} probably lies below T_m . Note that the second-order phase transition between N and TS phase is excluded because there is only one solution for T_c with a vanishing order parameter, i.e., the line $T_{cl}(g)$ found previously.

Mineev [8] discussed the effect of magnetic fluctuations near T_m on the coupling parameter of triplet pairing in the framework of the BCS (static) approach to find T_c . However, in the framework of a static BCS approach for spin fluctuation exchange, the cutoff frequency, which essentially characterizes the dynamics of the system and determines T_c , cannot be well-defined. What is more, we see that for the spin-dependent glue, the dependence of the superconducting critical temperature on the coupling parameter is not of the BCS type.

Remarkably, the superconducting phase is no longer present for $g < g_c(\kappa)$. We determined the critical value of the

FIG. 2. The frequency dependence of the order parameter $\Delta^\downarrow$ in the ferromagnet with magnetic anisotropy $\kappa = 0.1$. The dependence at $\kappa < \omega < 2 + \kappa$ reflects the spectrum of magnons for $\epsilon(\theta) = \kappa + 1 - \cos \theta$, where θ is the polar angle in the x, y plane.

parameter $g_c \approx 0.192$. Note that in our model, the parameter $g = 2|J_{\mathbf{k}\mathbf{k}'}S|^2N(0)/T_m$ essentially represents the fraction of the exchange RKKY contribution to the Curie temperature.

The order parameters $\Delta^\uparrow$ and $\Delta^\downarrow$ depend on the angle ϕ in the x, y plane perpendicular to the direction of the magnetization. At T_{cl} this dependence reduces to $\sin \phi$. Such a dependence may be observed in tunneling measurements between the triplet superconductor and the normal metal.

Let us consider now the dependence of the order parameter on the frequency. For $\kappa = 0.1$ this is displayed in Figs. 2 and 3 for dimensionless temperatures $T/T_m = 0.02, 0.08, 0.15, 1$. The component $\Delta^\downarrow$ transforms into $\Delta^\uparrow$ by means of a magnon emission (positive ω). Those excitations have a gap κ at $0 < \omega < \kappa$ and at $-2 - \kappa < \omega < 2$. The dependence $\Delta^\downarrow(\omega)$ perfectly reflects the magnon spectrum, as was discussed previously in the case of phonon-induced superconductors in Ref. [3]. The dependence $\Delta^\uparrow(\omega)$ corresponds to the magnon absorption (negative ω) with singularities near the edges of the spectrum at $\omega = \kappa$ and $\omega = 2 + \kappa$. Anomalies near the edges of the spectrum become sharper at higher temperatures due

FIG. 3. The frequency dependence of the order parameter $\Delta^\uparrow$ in the ferromagnet with magnetic anisotropy $\kappa = 0.1$. The dependence at $-2 - \kappa < \omega < -\kappa$ reflects the spectrum of magnons $\epsilon(\mathbf{q}) = \kappa + 1 - \cos \theta$.

to induced emission or absorption by thermal magnons. Note that $\Delta^\downarrow(\omega)$ and $\Delta^\uparrow(\omega)$ satisfy the symmetry relation

$$\Delta^\downarrow(\omega + \kappa) = \Delta^\uparrow(-\omega - \kappa). \quad (44)$$

The frequency dependence of the order parameter results in similar anomalies in the I - V characteristics of tunneling between the TS and the normal metal, as was observed in the phonon-induced superconductors (see Ref. [3]).

Let us compare our results with the magnetic exciton mediated superconductivity attributed to UPd₂Al₃ [11]. Such an exciton arises from crystal-field-split U4+ levels. It is coupled to delocalized f -electrons by electron spin operator σ_z . Thus the electron part of the electron-exciton Hamiltonian is the same as Eqs. (1) and (2), but Eqs. (3) and (4) now have the form

$$\mathcal{H} = \sum_{\mathbf{q}} \omega_{\mathbf{q}} \left[\alpha_{\mathbf{q}}^+ \alpha_{\mathbf{q}} - I \int d\mathbf{r} \Psi_{\alpha}^+(\mathbf{r}) \sigma_{\alpha\beta}^z \Psi_{\beta}(\mathbf{r}) \right] \Phi(\mathbf{r}), \quad (45)$$

$$\Phi(\mathbf{r}) = \sum_{\mathbf{q}} \tilde{\lambda}_{\mathbf{q}} (\alpha_{\mathbf{q}} + \alpha_{-\mathbf{q}}^+) e^{i\mathbf{q}\cdot\mathbf{r}}, \quad (46)$$

where $\omega_{\mathbf{q}}$ is the dispersion of the magnetic exciton, while $\alpha_{\mathbf{q}}^+$ and $\alpha_{\mathbf{q}}$ are the creation and annihilation of the magnetic exciton. For such a coupling, the magnetic exciton does not carry spin and acts similarly to phonons. Thus the behavior of these triplet superconductors with respect to the temperature is standard BCS-like behavior. Only triplet superconductors with a spin-carrying mediator (spin wave) differ drastically from superconductors with spin-neutral mediators.

VII. CONCLUSIONS

We used an appropriate spin-electron model Hamiltonian to discuss the spin-wave-mediated triplet pairing in ferromagnets. We found a very particular phase diagram in the plane (T_{cl}, g) , shown in Fig. 1, for the triplet superconducting state described by the two-component order parameter. The pairing exists only at $g > g_c$ and only in the temperature interval $T_{cl}(g) < T < T_{ch} < T_m$. The low-temperature boundary of the TS phase is determined by the conservation of the z -component of the spin in the electron-spin-wave interaction. It is spin-carrying glue and the spin conservation law that make this two-component superconducting phase diagram so different from that of pairing mediated by spinless phonon glue resulting in a single-component order parameter.

The question now is whether this phase diagram is specific for our model Hamiltonian or more generally applicable for any triplet spin-wave mediated pairing. The more general nonrelativistic Hamiltonian to treat the triplet pairing for electrons in s -, d -, or f -bands must take into account the strong Coulomb repulsion in the latter narrow bands. In the ferromagnetic phase with the spontaneous magnetization along the z -axis, the collective spin-wave mode always exists and it is described by the two-electron Green function. The coupling of conduction electrons with this mode results in a triplet pairing as in our model system. This coupling is still governed by the conservation of the spin z -component. Thus the low-temperature boundary of the triplet phase should always exist, as in our model. What is different is that we need to derive the strength of the electron-spin-wave coupling

g as well as the new T_{cl} and T_{ch} boundaries of the triplet phase. Thus, we think that the qualitative behavior of the triplet superconductor phase diagram in ferromagnets may be qualitatively the same as that shown in Fig. 1, but the quantitative behavior may be different in each case.

Experimental data for U-based superconductors show a BCS-like phase diagram, i.e., the pairing phase exists at all temperatures below T_c . Thus we are inclined to think that their superconducting phase is not a triplet state of a spin exchange mechanism of pairing. It may still be phonon-based pairing in the presence of strong spin-orbit interaction, which suppresses

the destructive effect of an exchange field for singlet pairing, or it may be some other hypothetical mechanism resulting in static effective attraction of electrons.

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