Gauge invariance and spinon-dopon confinement in the t - J model: implications for Fermi surface reconstruction in the cuprates

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Abstract. We discuss the application of the two-band spin-dopon representation of the t - J model to address the issue of the Fermi surface reconstruction observed in the cuprates. We show that the electron no double occupancy (NDO) constraint plays a key role in this formulation. In particular, the auxiliary lattice spin and itinerant dopon degrees of freedom of the spin-dopon formulation of the t - J model are shown to be confined in the emergent $U(1)$ gauge theory generated by the NDO constraint. This constraint is enforced by the requirement of an infinitely large spin-dopon coupling. As a result, the t - J model is equivalent to a Kondo-Heisenberg lattice model of itinerant dopons and localized lattice spins at infinite Kondo coupling at all dopings. We show that mean-field treatment of the large vs. small Fermi surface crossing in the cuprates which leaves out the NDO constraint, leads to inconsistencies and it is automatically excluded from the t - J model framework.

1 Introduction

The observation of quantum oscillations in the lightly hole-doped cuprates [1] is an important breakthrough since it indicates that coherent electronic quasiparticles may exist even in the pseudogap (PG) regime. The PG state does not exhibit a large Fermi surface (FS) enclosing the total number of charged carriers. Instead, the FS consists of small pockets with a total area proportional to the dopant density x , rather than the $1 + x$ which is expected for conventional Fermi liquids (FL's). A possible theoretical justification for this phenomenon might be the occurrence of a simultaneous setting of a new long-range order together with the PG phase [2]. The resulting breaking of translational symmetry would cut the large FS into small pieces but the Luttinger's theorem (LT) would still hold. However, the existence or not of such translational symmetry breaking is still debatable to this date. Moreover, even in case this symmetry breaking is verified, the LT might still be violated due to the proximity to the antiferromagnetic (AF) Mott insulator transition. We thus cannot rule out the possibility that the new metallic PG state may indeed violate the traditional LT. As a result, the PG state truly qualifies as a non-Fermi liquid (NFL) state which violates the LT. If this is indeed the case, the small Fermi pockets could then be accounted for even without any symmetry breaking.

An instructive example of such a new metallic state is provided by the so-called fractionalized Fermi liquid (FL*)

which exhibits small pockets similar to what is observed in an AF metal, and at the same time keeps the translational symmetry intact. Such a state manifests itself in the context of the Kondo-Heisenberg lattice model which describes localized Heisenberg lattice spin moments coupled to a conduction band of itinerant electrons [3–6]:

$$H_{K-H} = \sum_{ij} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} + J_K \sum_i \vec{S}_i c_{i\sigma}^\dagger \vec{\tau}_{\sigma\sigma'} c_{i\sigma'} + J_H \sum_{ij} \vec{S}_i \vec{S}_j. \quad (1)$$

Here the $c_{i\sigma}$'s represent the conduction electrons and the $\vec{S}_i$'s are the spin local moments on square lattice sites, with the summation over repeated spin indices σ being implicit. A fermionic “slave-particle” representation of the local moments is:

$$\vec{S}_i = f_{i\sigma}^\dagger \vec{\tau}_{\sigma\sigma'} f_{i\sigma'}. \quad (2)$$

The $f_{i\sigma}$ describes a spinful fermion destruction operator at site i and the $\vec{\tau}$'s are Pauli matrices.

In the regime in which the Kondo coupling J_K is much greater than the Heisenberg exchange J_H , the localized spin f moments and the spin of the conduction c electrons are locked into the singlet state:

$$\frac{1}{\sqrt{2}}(|\uparrow\rangle_f |\downarrow\rangle_{c,FS} - |\downarrow\rangle_f |\uparrow\rangle_{c,FS}), \quad (3)$$

where $|\sigma\rangle_f$ represents the localized spins and $|\sigma\rangle_{c,FS}$ is a linear superposition of the conduction-electron states

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near the FS [7]. As a result of this entanglement, the local moment is readily converted into a Kondo resonance in the excitation spectrum. The conduction electrons and the Kondo resonances are then hybridized and together they produce a conventional FL state with a FS enclosing the traditional Luttinger volume, which in view of that, counts the density of both f and c electrons. If the conduction band is filled with x electrons per unit cell, this large FS encloses $1 + x$ electrons per site.

In the opposite parameter regime, $J_H \gg J_K$, a non-FL phase can show up instead, provided the Heisenberg f - f coupling is sufficiently frustrated. As a result, the localized spins are melted into a quantum spin liquid. When this phase is stable, it quenches the Kondo effect. The c electrons are effectively decoupled from the f spins and they are then solely responsible for a small FS with a volume determined only by the density of the c electrons. This violates the traditional Luttinger count and the resulting theory describes a FL* metal. Such a small FS can be associated formally with a modified LT to take into account Z_2 topological excitations (“visons”) of the fractionalized spin liquid ground state [3,4].

The idea of the Kondo-type FL*-FL transition has recently been carried over to treat the t - J model in an attempt to describe the FS reconstruction observed in the hole-doped cuprates [8,9]. After all, recent experiments have revealed striking similarities between the high- T_c cuprates and quasi two-dimensional heavy fermion materials (the CeMIn₅ family) described by the Kondo-Heisenberg model [10,11]. In fact, a variety of physical phenomena can be accounted for by that model, such as the NFL behavior, the different types of both magnetic and charge ordering as well as the unconventional superconductivity [12].

In the present paper, we show that the mean-field (MF) FL* theory of the underdoped t - J model for the underdoped cuprates is blotted out by the electron NDO constraint. In fact, the NDO constraint drives the theory to a strong-coupling regime not amenable to a MF treatment. This is a manifestation of strong electron correlations inherent in the physics of the underdoped cuprates. More specifically, the NDO constraint generates a $U(1)$ gauge theory in a confining phase: the lattice spin background and the conduction dopons are strongly coupled to the gauge field fluctuations. As a result, the weak-coupled spin-dopon MF FL* ground state is never realized in the underdoped t - J model. Instead, the lightly doped Mott regime takes place essentially at a strong spin-dopon coupling and one runs into inconsistencies if one tries to describe such a PG phase without taking proper account of the NDO.

In contrast, the overdoped regime is much simpler than that since the underlying spin background is represented by a lattice of paramagnetic spins rather than by a quantum liquid of spin singlets. Implementing the NDO constraint in this regime results in a complete magnetic screening of the background paramagnetic lattice spins, which are then dissolved into the conduction sea. This

leads to a FS with an enhanced volume which recovers the traditional Luttinger counting.

2 Spin-dopon theory

Consider the low-energy properties of the t - J model on a square lattice with

$$H_{t-J} = - \sum_{ij\sigma} t_{ij} \tilde{c}_{i\sigma}^\dagger \tilde{c}_{j\sigma} + J \sum_{ij} (\vec{Q}_i \vec{Q}_j - \frac{1}{4} \tilde{n}_i \tilde{n}_j), \quad (4)$$

where $\tilde{c}_{i\sigma} = c_{i\sigma}(1 - n_{i,-\sigma})$ is the Gutzwiller projected electron operator (to avoid the on-site double occupancy), $\vec{Q}_i = \sum_{\sigma,\sigma'} \tilde{c}_{i\sigma}^\dagger \vec{\tau}_{\sigma\sigma'} \tilde{c}_{i\sigma'}$, $\vec{\tau}^2 = 3/4$, is the electron spin operator and $\tilde{n}_i = n_{i\uparrow} + n_{i\downarrow} - 2n_{i\uparrow}n_{i\downarrow}$.

To establish the correspondence between the low-energy physics of the t - J model and the Kondo physics one should rewrite the degrees of freedom of the one-band t - J Hamiltonian in a two-band Kondo-Heisenberg model representation. This can be achieved within the recently proposed spin-dopon representation of the constrained electron operators [13–15]. In terms of the $su(2)$ spin and the fermionic dopon operators, the projected electron operators take the form [13,14]

$$\tilde{c}_{i\sigma}^\dagger = \frac{\text{sign}(\sigma)}{\sqrt{2}} \left[(1/2 + \text{sign}(\sigma) S_i^z) \tilde{d}_{i-\sigma} - S_i^\sigma \tilde{d}_{i\sigma} \right], \quad (5)$$

where $\text{sign}(\sigma = \uparrow\downarrow) = \pm 1$. Here $\tilde{d}_{i\sigma} = d_{i\sigma}(1 - d_{i,-\sigma}^\dagger d_{i,-\sigma})$ denotes the Gutzwiller projected dopon operator, where as $\tilde{S}_i^\sigma$ denotes the spin-raising (-lowering) operator S_i^+ (S_i^-) for $\sigma = \uparrow$ ($\downarrow$). In this framework, the holes are doped carriers in the half-filled Mott insulator which can otherwise be described exclusively in terms of spin variables.

To accommodate these new operators one obviously needs to enlarge the original onsite Hilbert space of quantum states. This enlarged space is characterized by the state vectors $|\sigma a\rangle$, with $\sigma = \uparrow, \downarrow$ labeling the spin projection of the lattice spins and with $a = 0, \uparrow, \downarrow$, labeling the dopon states (the dopon double occupancy is not allowed). In this way, the on-site enlarged Hilbert space becomes:

$$\mathcal{H}_i^{\text{enl}} = \{|\uparrow 0\rangle_i, |\downarrow 0\rangle_i, |\uparrow\uparrow\rangle_i, |\downarrow\uparrow\rangle_i, |\uparrow\downarrow\rangle_i, |\downarrow\downarrow\rangle_i\}, \quad (6)$$

while in the original Hilbert space we can either have one electron with spin $\sigma = \uparrow, \downarrow$ or a vacancy:

$$\mathcal{H}_i = \{|\uparrow\rangle_i, |\downarrow\rangle_i, |0\rangle_i\}. \quad (7)$$

The following mapping between the two spaces is then defined:

$$|\uparrow\rangle_i \leftrightarrow |\uparrow 0\rangle_i, \quad |\downarrow\rangle_i \leftrightarrow |\downarrow 0\rangle_i, \quad (8)$$

$$|0\rangle_i \leftrightarrow \frac{|\uparrow\downarrow\rangle_i - |\downarrow\uparrow\rangle_i}{\sqrt{2}}. \quad (9)$$

The remaining triplet states $(|\uparrow\downarrow\rangle_i + |\downarrow\uparrow\rangle_i)/\sqrt{2}$, $|\uparrow\uparrow\rangle_i$, $|\downarrow\downarrow\rangle_i$ are unphysical and should therefore be removed from

actual calculations. In this mapping, a vacancy corresponds to an onsite singlet state of a lattice spin and a dopon. The vacancy is a spin singlet entity which carries a unit charge e when compared to the remaining sites.

To avoid a possible confusion, the following remark is in order at this stage. Physically, one-hole doping corresponds to a removal of one electron, leaving behind an empty site, which carries a unit charge e when compared to the remaining sites. This is nothing more than a vacancy which is a dopon-spin singlet with a charge e . The total number of vacancies is then exactly equal to the total number of dopons [16]. A hole by definition is a spin-1/2 object with a charge e . A doped hole is then this vacancy which carries an extra spin 1/2 spread over the surrounding spin background. The physical hole is thus an extended nonlocal object. In the doped Mott insulator the term “hole” is often used with a different meaning. The dopons and the lattice spins are just auxiliary gauge-dependent entities, while the hole is physical and gauge-independent object.

Such a hole appears as a string-like object with much in common with the hole doped in an AF ordered lattice introduced earlier in [17,18]. This doped-hole concept was developed further to derive an effective single-hole gauge invariant AF action [19].

The original t - J Hamiltonian (4) written in terms of the constrained operators (5) vanishes when it acts on any of the unphysical states. Consequently, it automatically decouples the physical and unphysical states in the enlarged Hilbert space [13,14]. Unfortunately, the t - J Hamiltonian (4) given directly in terms of the constrained electron operators is very difficult to deal with. This is due to the fact that the algebra of the constrained electron operators is much more involved than the related algebra for conventional fermion and spin operators.

To simplify the problem, one usually relies on a MF approximation. However, some extra care needs to be exercised in this case. A MF approximation results in a MF Hamiltonian that can no longer be written solely in terms of the unique combinations given by (5). As a result, the faithful spin-dopon representation of the t - J Hamiltonian is immediately destroyed by that. Within such a MF approach, the operators $\tilde{S}_i$ and $\tilde{d}_{i\sigma}$ act in the whole enlarged Hilbert space mixing up thereby the physical and unphysical states. As a result, the unphysical states reappear in the theory in an uncontrolled way.

This is precisely the case with the MF treatment exposed in references [8,9]. In reference [8], the MF decoupling is carried out by the spin-singlet order parameter,

$$\Delta_{ff} = \langle f_{i\uparrow} f_{j\downarrow} - f_{i\downarrow} f_{j\uparrow} \rangle, \quad (10)$$

where the fermionic spinons $f_{i\sigma}$ are defined through equation (2). This parameter is used to represent a liquid of spin singlets at MF level¹. In contrast, the pairing

amplitude between conduction dopons and localized spins

$$\Delta_{df} = \langle f_{i\downarrow} d_{i\uparrow} - f_{i\uparrow} d_{i\downarrow} \rangle \quad (11)$$

describes the condensation of Kondo (or Zhang-Rice) spin singlets. Accordingly its nonvanishing value implies that the localized spins contribute to the Fermi surface volume. The underdoped FL* metallic phase is fixed by the choice $\Delta_{ff} \neq 0$, $\Delta_{df} = 0$, whereas the overdoped regime is imaged on a conventional heavy FL phase. This phase is supposed to set in under the assumption that $\Delta_{ff} = 0$ and $\Delta_{df} \neq 0$. However the precise location of the emergent small Fermi pockets in the underdoped phase has not been determined that way.

In reference [9], only the background spin-singlet order parameter (10) is used to describe the spin-liquid ground state. The fermionic amplitudes are now replaced by bosonic modes representing Schwinger bosons. Since the Z_2 bosonic spin modes are gapped in the spin-liquid phase, they can be formally integrated out. This is done perturbatively, by expanding the effective action in the bosonic MF propagator. In case the emergent effective low-energy action does indeed exist (if we assume that this series converges), the proposed theory describes the fractionalized spin liquid weakly coupled to the conduction dopons. This is essentially the FL* hypothesis discussed above in the context of the Kondo-Heisenberg model. Such an approach does not break the translational symmetry and results in a small hole-like FS for the underdoped phase around $(\pi/2, \pi/2)$ and the symmetry related points in the Brillouin zone.

However, the approach advocated in reference [9] cannot be used to describe a conventional FL state with a large FS at large doping. Within that MF scheme, the large FS can be accounted for provided the bosons are replaced by Schwinger fermions. In the cuprates, one should expect yet another reason for such a reconstruction in view of the depletion of the mobile carriers. The FS transition discussed in reference [8] is determined by the variation of the strength of the effective spin-dopon coupling rather than by a change in the doping level.

The discussed MF approaches also imply that the FL* ground state is in fact constituted of conduction dopons nearly decoupled from the lattice spins. In reference [8], this is explicitly enforced by setting $\Delta_{df} = 0$ in the underdoped phase, whereas in reference [9], this is implicit in the assumption that the perturbative expansion of the spin-dopon effective action converges. Although this appears to be a more accurate treatment of the underdoped phase than simply setting $\Delta_{df} = 0$, this convergence necessarily implies that the spinon gap is the largest energy scale in the problem and this is not the case, in the infinite Kondo coupling regime.

A given ground-state MF theory is only reliable if it is stable against quantum fluctuations that manifest themselves beyond such zeroth MF order. In the standard slave-particle theories of strongly correlated electrons, those fluctuations are due to an emergent local $U(1)$ gauge field that takes care of the redundancy of the associated slave-particle representations. If that gauge field is in a confining

¹ In reference [8], the underdoped cuprates is described as a “Luttinger-volume violating liquid” (LvFL) with a spin liquid of fermionic spinons. The LvFL state is qualitatively the same as the FL* state.

phase, the bare slave-particle excitations are strongly coupled to each other. Accordingly, in this phase, all true physical excitations must be gauge singlets.

The gauge redundancy specific for the spin-dopon representation (5) should not be ignored (see in this respect Refs. [8,9]). Such a neglect implies that the dopons and lattice spins carry no emergent $U(1)$ gauge charges. This is indeed a necessary condition to justify a FL* phase characterized by weakly coupled well-defined dopon and lattice spin excitations. We show however that the spin-dopon theory is inevitably a strongly coupled $U(1)$ gauge theory like any slave-particle representation of strongly correlated electrons should be [20,21]. The underlying slave particles – the dopons and the lattice spins – are in fact gauge dependent and they are not present in the physical spectrum in a confining spin-dopon phase that describes strongly correlated electrons.

3 Emergent $U(1)$ gauge theory

Since the emergent $U(1)$ gauge theory plays an essential role in the spin-dopon formulation of the t - J model, we provide below a brief account of its origin.

To begin with, there is an obvious redundancy in the spin-dopon decomposition of the constrained electron operator given by equation (5), since the r.h.s. of this equation exhibits two extra degrees of freedom. This redundancy is taken care of by the emergent local $U(1)$ gauge symmetry generated by the NDO constraint.

In terms of the projected electron operators, that constraint reads

$$\sum_{\sigma} \tilde{c}_{i\sigma}^{\dagger} \tilde{c}_{i\sigma} + \tilde{c}_{i\sigma} \tilde{c}_{i\sigma}^{\dagger} = I. \quad (12)$$

It simply states that there are no on-site doubly occupied electron states. What is important is that the l.h.s. of equation (12) commutes with the constrained electron operators and, hence, with the t - J Hamiltonian (4) as well. In the spin-dopon representation (5), this constraint takes the form of a Kondo-type interaction [22]:

$$\vec{S}_i \vec{M}_i + \frac{3}{4} (\tilde{d}_{i\uparrow}^{\dagger} \tilde{d}_{i\uparrow} + \tilde{d}_{i\downarrow}^{\dagger} \tilde{d}_{i\downarrow}) = 0, \quad (13)$$

with

$$\vec{M}_i = \sum_{\sigma, \sigma'} \tilde{d}_{i\sigma}^{\dagger} \vec{\tau}_{\sigma\sigma'} \tilde{d}_{i\sigma'}.$$

This requirement excludes the unphysical spin-dopon triplet states in a self-consistent way, since the operator

$$\mathcal{R}_i^{sd} := \vec{S}_i \vec{M}_i + \frac{3}{4} (\tilde{d}_{i\uparrow}^{\dagger} \tilde{d}_{i\uparrow} + \tilde{d}_{i\downarrow}^{\dagger} \tilde{d}_{i\downarrow}), \quad (\mathcal{R}_i^{sd})^2 = \mathcal{R}_i^{sd},$$

commutes both with $\tilde{c}_{i\sigma}$ and with the t - J Hamiltonian.

In view of this commutation, the local operator $\mathcal{R}_i^{sd}$ generates a $U(1)$ gauge symmetry:

$$\tilde{c}_{i\sigma} \rightarrow e^{i\mathcal{R}_i^{sd}\theta_i} \tilde{c}_{i\sigma} e^{-i\mathcal{R}_i^{sd}\theta_i} = \tilde{c}_{i\sigma}.$$

In contrast, the slave particles – the dopons and lattice spins – are not invariant under the action of $\mathcal{R}_i^{sd}$ since

$$[\mathcal{R}_i^{sd}, \tilde{d}_{i\sigma}] \neq 0, \quad [\mathcal{R}_i^{sd}, \vec{S}_i] \neq 0.$$

In spite of that, all the physical observables constructed out of the slave operators, e.g., the on-site electron spin operator $\vec{Q}_i$, as well as the dopon number operator $\tilde{n}_i^d$ are gauge invariant quantities [16]. The MF Hamiltonians in [8,9] are gauge dependent and they do not commute with $\mathcal{R}_i^{sd}$. In other words, the MF Hamiltonians act in the enlarged Hilbert space that includes the unphysical states as well.

The origin of the emergent spin-dopon $U(1)$ gauge symmetry and that of the traditional slave-particle representations is one and the same: they are generated by the electron NDO constraint. For example, the slave boson decomposition of the electron operator

$$\tilde{c}_{i\sigma} = b_i^{\dagger} f_{i\sigma}, \quad (14)$$

where b_i is supposed to carry charge of the electron while the fermion $f_{i\sigma}$ carries the spin, implies an electron NDO constraint in the form

$$\mathcal{R}_i^{fb} := \sum_{\sigma} f_{i\sigma}^{\dagger} f_{i\sigma} + b_i^{\dagger} b_i = 1. \quad (15)$$

Again, the representation (14) is invariant under the local $U(1)$ transformations generated by $\mathcal{R}_i^{fb}$,

$$\tilde{c}_{i\sigma} \rightarrow e^{i\mathcal{R}_i^{fb}\theta_i} \tilde{c}_{i\sigma} e^{-i\mathcal{R}_i^{fb}\theta_i} = \tilde{c}_{i\sigma},$$

which takes care of the redundancy exposed in (14). However, as opposed to that, the redundant fields are not gauge invariant:

$$\begin{aligned} b_i &\rightarrow e^{i\mathcal{R}_i^{fb}\theta_i} b_i e^{-i\mathcal{R}_i^{fb}\theta_i} = e^{i\theta_i} b_i, \\ f_{i\sigma} &\rightarrow e^{i\mathcal{R}_i^{fb}\theta_i} f_{i\sigma} e^{-i\mathcal{R}_i^{fb}\theta_i} = e^{i\theta_i} f_{i\sigma}. \end{aligned}$$

Differently from the standard slave-particle representations, the spin-dopon NDO constraint does not simply reduce to an operator identity that involves only the number operators of the redundant particles. Since the spin-dopon representation engages the local $SU(2)$ spins along with the projected fermion operators, the NDO constraint $\mathcal{R}_i^{sd}$ takes on a more intricate form. It includes both the dopon number operator and the spin-dopon Kondo interaction. As in the standard slave-particle descriptions, the emergent $U(1)$ gauge field in the spin-dopon representation has no dynamics of its own and, hence, it can be considered at infinite coupling. Consequently, as we show next, the gauge dependent bare dopons and spins are strongly coupled and are necessarily confined.

This confinement is in some sense similar to the flux-charge “entanglement” observed in the fractional quantum Hall (FQH) effect. The effective low-energy theories of FQH states are $U(1)$ Chern-Simons (CS) gauge theories. The CS gauge field as well has no independent dynamics of its own: the CS coupling is a pure constraint.

The only effect of such coupling is to attach magnetic fluxes to charged particles. Within the spirit of the Anderson resonating-valence-bond concept of incompressible quantum spin liquid, this tying of flux to charge translates into a spin-flux one as discussed in references [23–25].

As known, the standard slave-particle theory can be explicitly reformulated as a $U(1)$ gauge theory [26] in its confining phase [20,21]. This can be done in this way because the underlying NDO constraint has a very simple form: it just fixes the total number of the on-site auxiliary particles. In contrast, the spin-dopon NDO constraint goes beyond that and this hinders the explicit derivation of the corresponding gauge theory for the t - J model. In spite of that, the NDO constraint in the spin-dopon representation offers a different way to prove explicitly that the bare spins and dopon excitations are indeed strongly coupled to each other. To see this we demonstrate below the equivalence of the t - J Hamiltonian in the spin-dopon representation and the Kondo-Heisenberg lattice model with an *infinitely* large Kondo coupling, $J_K \rightarrow +\infty$. As we already mentioned, the necessary condition for the onset of the FL* phase is $J_H \gg J_K$. This condition is never realized in the t - J model. The infinitely strong Kondo coupling regime obviously rules out such a possibility.

t-J model vs. Kondo-Heisenberg model

To establish explicitly the correspondence between the infinitely coupled Kondo-Heisenberg lattice model and the t - J model, let us notice that the set of local constraints $\mathcal{Y}_i^{sd} = 0$, one for each lattice site, is equivalent to the global condition $\mathcal{Y}^{sd} := \sum_i \mathcal{Y}_i^{sd} = 0$. This simplification holds true because the unphysical states manifest themselves as the degenerate eigenvectors of $\mathcal{Y}_i^{sd}$ with an eigenvalue 1. Therefore, if it acts on an unphysical state, $\mathcal{Y}^{sd}$ simply produces the same state multiplied by a positive number. Contrary to that, acting on a physical state, $\mathcal{Y}^{sd}$ always gives zero. We can therefore enforce the local constraint $\mathcal{Y}_i^{sd} = 0$ by adding an extra piece to the Hamiltonian

$$\Delta H_\lambda = \lambda \sum_i \mathcal{Y}_i^{sd},$$

with the *global* parameter λ being sent to $+\infty$. In this way, all the unphysical states are separated from the physical spectrum by an energy gap $\sim \lambda$. In the limit $\lambda \rightarrow +\infty$, they are automatically excluded.

The t - J Hamiltonian in the spin-dopon representation then reads [16]

$$H_{t-J} = \sum_{ij\sigma} \left(2t_{ij} + \frac{3\lambda}{4}\delta_{ij} \right) \tilde{d}_{i\sigma}^\dagger \tilde{d}_{j\sigma} + \lambda \sum_i \vec{S}_i \vec{M}_i + J \sum_{ij} \left(\vec{S}_i \vec{S}_j - \frac{1}{4} \right) (1 - \tilde{n}_i^d) (1 - \tilde{n}_j^d), \quad (16)$$

with λ being sent to $+\infty$ to ensure the selection of the physical subspace. As we show in Appendix A, this representation is indeed equivalent to the standard t - J model, and it reproduces the well-known 1d exact result.

Close to half filling, where the density of doped holes is small $x := \langle \tilde{n}_i^d \rangle \ll 1$, one can also make the change $J \rightarrow \tilde{J} = J(1-x)^2$. One can safely ignore the “tilde” sign for the dopon operators as well, since the NDO constraint for the dopons is already taken care of by the requirement $\mathcal{Y}_i^{sd} = 0$. The spin-dopon representation of the t - J Hamiltonian for the underdoped cuprates then takes a form of the Kondo-Heisenberg lattice model, namely

$$H_{t-J} = \sum_{ij\sigma} T_{ij} d_{i\sigma}^\dagger d_{j\sigma} + \tilde{J} \sum_{ij} \left(\vec{S}_i \vec{S}_j - \frac{1}{4} \right) + \lambda \sum_i \vec{S}_i \vec{M}_i \quad (17)$$

where $T_{ij} = 2t_{ij} + (3\lambda/4 - \mu)\delta_{ij}$. The global parameter λ should be sent to $+\infty$ only after the thermodynamic limit is explicitly carried out. Although $\mathcal{Y}_i^{sd}$ no longer commutes with H_{MF} , the limit $\lambda \rightarrow +\infty$ still singles out the on-site physical subspace self-consistently at any instance of time. This is precisely the case because the operator $\mathcal{Y}_i^{sd}$ has no negative eigenvalues.

Notice that the Kondo coupling λ is present in the dopon dispersion as well. This ensures that the energy of the system remains finite even when $\lambda \rightarrow \infty$. This important renormalization of the dopon dispersion is absent in earlier attempts to establish the Kondo and the t - J model correspondence [27]. Note that it is precisely the local NDO constraint that is behind such a correspondence.

It is also important to stress that the parameter λ cannot be absorbed in the dopon chemical potential. To see that suppose we include λ into μ and take the limit, $\lambda \rightarrow \infty$. If there were no more λ -dependent terms in the Hamiltonian, this would result in the constraint $n_i^d = 0$, which means that the dopon band becomes empty in this limit. However, λ enters the Kondo term as well. This implies instead that $3/4(n_i^d) + \vec{S}_i \vec{M}_i = 0$, which immediately brings an occupied dopon band back to the stage.

The conventional slave-particle representations allow for a similar treatment in terms of the gauge independent variables. For instance, one can use the slave-boson representation (14) of the t - J Hamiltonian free of any constraints, provided an extra term

$$\lambda \sum_i \left(\mathcal{Y}_i^{fb} - 1 \right)^2, \quad \lambda \rightarrow +\infty \quad (18)$$

is added to the Hamiltonian. It explicitly singles out the physical subspace. It is also clear that this extra term results in an infinitely strong interaction between the slave particles. A similar approach that involves an infinitely large coupling to fix an appropriate physical Hilbert space was successfully used to describe the Kondo effect in metals [28], as well as the thermodynamics of the quantum Heisenberg model [29].

An explicit MF treatment of the Kondo-Heisenberg model (1) at large though finite values of the Kondo coupling can be found in reference [30]. It has been established that the competition between the Kondo coupling and the Heisenberg exchange does lead to a doping driven phase transition between states with different FS volumes. For small J_H , a nonvanishing solution $\Delta_{df} \neq 0$ exists down

to $x = 0$. Accordingly, there is no phase transition down to $x = 0$ for small enough J_H/t . If J_H increases, there is an extended range of small x where $\Delta_{df} = 0$. This implies that for large enough J_H/t , there is a crossover at some x_c . For $x < x_c$, the MF theory predicts a spin liquid ($\Delta_{ff} \neq 0$) with a small FS around (π, π) . This disagrees with experiment because the pockets are observed at $(\pi/2, \pi/2)$ and other symmetry related points. This deficiency of the MF treatment is attributed to the neglect of correlations between the localized spins and the conduction holes, which are clearly present for large J_K/t . One may therefore expect that a strong coupling of the conduction hole pocket to the AF spin fluctuations will eventually create hole pockets centered at $(\pm\pi/2, \pm\pi/2)$.

4 Strong coupling limit

The physical regime of the parameters to discuss the t - J model within the representation (17) is $\lambda \gg t \gg J$. A description of both large and low doping phases in the strong-coupling picture is required, which may be expected to hold best in the limit $\lambda/t \gg 1$. In the present section, we show that the overdoped phase does admit a reliable description in this limit, although for the underdoped phase the appropriate strong-coupling theory is not yet complete.

4.1 Overdoped regime

Let us consider first the overdoped cuprates which is expected to be described by a standard FL. In the limit $\lambda \rightarrow \infty$, we can employ a framework which was originally put forward to treat the full Kondo screening regime in reference [27] (see also Refs. [7,30]). Namely, in the limit $\lambda \rightarrow \infty$, the bare vacuum state reads

$$|\Psi_0\rangle^{\text{overdoped}} = \prod_i |0\rangle_i = 2^{-N/2} \prod_i (|\uparrow\downarrow\rangle_i - |\downarrow\uparrow\rangle_i). \quad (19)$$

This is a product of local Kondo (Zhang-Rice) singlets and it is the ground state of the model for $t/\lambda = J/\lambda = 0$ at $x = 1$. It then follows that the on-site vacancy state is destroyed by the operators $d_{i\sigma}^\dagger$:

$$d_{i\sigma}^\dagger |0\rangle_i = 0. \quad (20)$$

In the truncated Hilbert space, the only possible excitation above the ground state takes the form [27]

$$|\sigma, 0\rangle_i = \sqrt{2} \text{sign}(\sigma) d_{i,-\sigma} |0\rangle_i, \quad \text{sign}(\sigma = \uparrow, \downarrow) = \pm 1. \quad (21)$$

A free local spin moment thus behaves as an anti-particle excitation of the d -field above the ground state (19) with a kinetic energy of order $D \sim t$, where D is the conduction dopon bandwidth. Notice that for an infinite λ , the paramagnetic susceptibility is not given by $1/T_K$, which is zero here, but by $1/D$ ². The excitations $S_i^\pm |0\rangle_i$ do not

appear in the theory because the states $|\uparrow\uparrow\rangle_i$ and $|\downarrow\downarrow\rangle_i$ have been already excluded by the NDO constraint.

Since the dopons represent holes, the local spin moment now behaves as the conduction “electron” with the quantum numbers, spin 1/2 and charge $-e$ (when compared to the vacuum state). This is a direct consequence of the infinitely strong Kondo screening, or equivalently, of the exact resolution of the NDO constraint. Under the assumption that the LT holds in this case, we can then conclude that the FS encloses $1 - x$ electron-like particles or, equivalently, $(1 + x)$ holes per unit cell (there are two possible states per unit cell). This is a large hole-like FS. This phase only sets in provided the dopon hopping effectively destroys all the spin singlets when the local AF order disappears. This marks the termination of the PG phase. In this way, the necessary energy to break the spin singlet is roughly J . Since the dopon kinetic energy is of order $2tx$, for a hole doped Mott insulator, this only happens when $x \geq x_c = J/2t \approx 0.15$ (for $J = t/3$).

4.2 Underdoped phase

Next we switch to the lightly doped regime, $x \ll 1$. This is a more involved case, since the physics behind this phase is still unclear. A common belief is that it is essentially determined by strong electron correlations encoded in the NDO constraint. To illustrate a generic difficulty that hinders a non-MF treatment in this case, we briefly discuss a recently proposed approach [30] to deal with the underdoped phase seemingly beyond a MF approximation. In that paper, the strong-coupling theory ($\lambda \gg t \gg J$) of the Kondo-Heisenberg model is considered at small doping. The corresponding λ -stabilizing term is not included in that approach, however. A similar approach has been employed to treat the Hubbard model in the limit $U/t \gg 1$ and slightly away from half-filling [31].

The following remark is in order at this stage. The authors of references [30,31] claim that the single-band Hubbard model as well as the t - J model can be derived from the Kondo-Heisenberg lattice model in the limit of large Kondo coupling. However, they provide no explicit derivation of that. As will be argued in Appendix A, the t - J model is indeed identical to the strongly coupled Kondo-Heisenberg lattice model. However, this correspondence implies both the infinitely large Kondo coupling regime and a *simultaneous* renormalization of the hopping amplitude, $t_{ij} \rightarrow T_{ij}(\lambda)$ as given by our representation (17).

In the region $x \ll 1$, the Kondo-Heisenberg model is assumed to display short-range AF spin fluctuations. The bare vacuum at $x = 0$ is then taken to be a spin-liquid state $|\Psi_0\rangle^{\text{underdoped}}$ [30,31]. In contrast to the overdoped regime where the state $|\Psi_0\rangle^{\text{overdoped}}$ is the exact eigenstate of the strongly coupled Kondo-Heisenberg Hamiltonian, at $x = 1$, the proposed bare vacuum state is not an eigenstate of the Hamiltonian at $x = 0$. A precise form of this state is therefore not specified. What is important is that this state has exactly one spin per site, has momentum zero, and is a spin singlet. In particular, one can write it in the

² For infinite λ , the Kondo temperature, which is essentially the singlet-triplet separation, is also infinite. The exponential variation $T_K = De^{-1/\rho\lambda}$ is only valid for small λ [27].

form of the resonating valence bond (RVB) spin singlet:

$$|\Psi_0\rangle^{\text{underdoped}} \equiv |\Psi_0\rangle = |RVB\rangle \otimes |vac\rangle, \quad (22)$$

where

$$|RVB\rangle \sim \sum_{c_{\{ij\}}} \prod_{ij} c_{\{ij\}} (|\uparrow_i \downarrow_j\rangle - |\downarrow_i \uparrow_j\rangle),$$

and $|vac\rangle$ stands for a canonical fermionic vacuum state. The coefficients $c_{\{ij\}}$'s are such that the resulting spin-spin correlation length is finite. The only physical quantity which is claimed to be relevant for the calculation of the quasiparticle spectrum is the static spin-spin correlation function [30,31]:

$$\chi_{ij} = \langle \Psi_0 | \vec{S}_i \vec{S}_j | \Psi_0 \rangle. \quad (23)$$

This is considered as an input parameter. Upon fixing in this way the spin sector of the Hilbert space, the authors proceed to a description of the charge excitations on top of it. It is clear that such an approach displays no dynamical correlations between the spin and charge degrees of freedom.

In the limit of the large Kondo coupling, the charge sector comprises the on-site spin-dopon singlet and triplet states. The triplet state corresponds to a higher energy and it is separated from the lowest singlet state by a gap $\sim \lambda$. By an appropriate redefinition, the energy of the spin-singlet state can be taken to be finite in the limit $\lambda \rightarrow \infty$. This limit then pushes the triplet states out of the physical spectrum. Since it is precisely this case that has a direct relevance for the t - J and Hubbard models, we adjust the results exposed in references [30,31] to that situation exclusively.

The charged quasiparticle excitations above the spin ground state can then be taken in the form

$$|\vec{k}, \sigma\rangle = \sum_i \tilde{a}_{i\sigma}^\dagger e^{i\vec{k}\vec{R}_i} |\Psi_0\rangle. \quad (24)$$

Here

$$\tilde{a}_{i\sigma}^\dagger = \tilde{c}_{i,-\sigma}, \quad (25)$$

where the constrained electron (Hubbard) operator $\tilde{c}_{i\sigma}$ is given by our equation (5). The action of the fermionic operators $\tilde{a}_{i,-\sigma}^\dagger$ on a localized spin state $|\sigma, 0\rangle_i$ produces a vacancy state, e.g.

$$\tilde{a}_{i\downarrow}^\dagger |\uparrow, 0\rangle_i = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle_i - |\downarrow\uparrow\rangle_i). \quad (26)$$

This process effectively describes the effect of hole doping in AF spin background. It appears as a vacancy surrounded by a locally disturbed spin-liquid background.

The single-hole spin-dopon wave function (24) describes a “dressed” hole in analogy with the many-body wave function which was used to describe, in the context of a spin-wave approximation, the AF string or the spin polaron associated with the t - J model in the presence of AF ordering [32,33]. The important distinction, in our case,

is the fact that the vacancies are now inserted in a spin-liquid background rather than in the Néel state which was used to characterize the AF lattice.

The fermion operator $\tilde{a}_{i\sigma}^\dagger$, which transforms itself in the fundamental $SU(2)$ representation [34], creates a quasiparticle with spin σ and charge e . The dopon operator $d_{i\sigma}^\dagger$ produces the same effect when acting on the on-site canonical vacuum state $|0\rangle_i$. At $\lambda = 0$, the Kondo-Heisenberg model reduces to a gas of noninteracting dopons decoupled from the spin background. Let now the spin-dopon interaction λ be adiabatically turned on towards large values. It is then assumed that the resulting final state is a gas of the quasiparticles (24) weakly coupled to the same spin background. In other words, those quasiparticles are assumed to be low-energy excitations in the physical spectrum. This is the key assumption in references [30,31]. In this case, the low-energy excitations in the quasiparticle sector take the form

$$E_{n\vec{k}} = \sum_{\vec{k}\sigma} E_{\vec{k}} d_{\vec{k}\sigma}^\dagger d_{\vec{k}\sigma} + \dots, \quad (27)$$

for some $E_{\vec{k}}$. The omitted terms in (27) describe weak interactions between the quasiparticles. The crucial point is the replacement of the constrained fermion operator $\tilde{a}_{\vec{k}\sigma}$ by the conventional unconstrained dopon operator, $d_{\vec{k}\sigma}$. One thus arrives at a FL state described in terms of conventional quasiparticles. As shown in reference [30], the FS in such a scheme encloses a volume proportional to the density of the doped holes. This does not occur as a consequence of the backfolding of the Brillouin zone due to any kind of broken symmetry. If one further assumes that the spin-liquid background forms a Z_2 spin liquid, one can easily prove that the modified LT holds true in this case as well. The derived FS is then directly associated with both dopon and Z_2 gauge excitations with a FL* state set up as advocated earlier in other MF treatments [8,9].

Let us see now in what way the NDO constraint modifies the theory discussed in references [30,31]. If we assume that at low doping the NDO constraint is not so important and it is relaxed, the bilinear form

$$\sum_{ij\sigma} t_{ij} \tilde{a}_{i\sigma}^\dagger \tilde{a}_{j\sigma} \quad (28)$$

constructed out of the constrained electron operators is replaced by the new kinetic term

$$\sum_{ij\sigma} t_{ij}^{\text{eff}} d_{i\sigma}^\dagger d_{j\sigma}, \quad (29)$$

where the $d_{i\sigma}$'s represent the canonical unrestricted dopon operators and t_{ij}^{eff} is a certain effective hopping amplitude (see Appendix C).

However, the unphysical states in this formalism are not just doubly occupied dopon states. The triplet spin-dopon states that involve only single occupied dopon states are unphysical as well. It is the NDO constraint that eliminates all of the unphysical states. In spite of the fact that the operators $\tilde{a}_{i\sigma}$ bear the same quantum numbers

as the dopon operators, the algebra they are closed into is much more complicated than the canonical fermionic algebra. In fact, the constrained fermion operators obey in the configuration space the $su(2|1)$ superalgebra commutation/anticommutation relations [35] that mix up the bosonic and fermionic degrees of freedom. In general, the bilinear form (28) cannot be diagonalized neither in the configuration nor in the momentum spaces [36]. The only exceptions are the 1d case discussed in Appendix A and in the case of an exactly one hole doped into an AF spin background, the so-called Nagaoka phase [37].

Although there is indeed a low probability for two holes to hop on the same site in the low doping regime, relaxing the on-site NDO constraint drastically affects the physics at any doping level, not just at high dopings as usually claimed elsewhere. The operators $\tilde{a}_{i\sigma}$ act in the physical Hilbert space. However, the substitution

$$\tilde{a}_{i\sigma} \rightarrow d_{i\sigma} \quad (30)$$

brings the unphysical triplet states back to the theory at any doping, in spite of the fact that the strong-coupling Kondo regime is at work.

The MF relaxation of the NDO constraint modifies the underlying Hilbert space. Such a modification results in dramatic consequences for the low-energy properties of the electron systems and this is totally ignored by the substitution (30). For instance, the approach advocated in [30,31] is expected to work well for $U \gg t$ in the lightly doped Hubbard model. The limit $U \rightarrow +\infty$ directly eliminates doubly occupied states, so that the resulting Hamiltonian describes a system of strongly correlated electrons (see Appendix B). However, the resulting effective quasiparticle Hamiltonian given by equation (10) in reference [31] reads:

$$H_{U=\infty} = \sum_{ij\sigma} \tilde{t}_{ij} h_{i\sigma}^\dagger h_{j\sigma}, \quad (31)$$

where $h_{i\sigma}$ is a canonical hole-like fermion operator and

$$\tilde{t}_{ij} = t_{ij} \left(\frac{1}{2} + 2\chi_{ij} \right). \quad (32)$$

This is a trivial problem that admits an exact solution in any dimensions. It is well known, however, that the $U = \infty$ Hubbard model (given by Eq. (A.3) in Appendix A) captures an extreme limit of the physics of strong electron correlations. That model is certainly far from trivial and it admits an exact solution only in 1d. The substitution (30) which is the key assumption behind such approximation obviously leaves out the essence of the physics of the underdoped t - J model, i.e., the strong electron correlations. This approach therefore reduces to a kind of uncontrolled MF treatment. It starts with the MF ansatz (23) to fix a spin-liquid structure for the lattice spin background and proceeds by considering the dopons to be nearly decoupled from the static spins.

Within this theory, the positions of the hole pockets of the Hubbard model is centered at $(\pm\pi/2, \pm\pi/2)$. The

pockets move to the inner side of the magnetic Brillouin zone, as the strength of the AF correlator χ_{ij} is increased. Basically the same conclusion was reached in the MF FL* theory of the underdoped t - J model. This finding agrees with experimental data. However, this conclusion is solely based on a choice of the input parameter (32). If one sets $\chi_{ij} = 0$, the hole pocket moves back to (π, π) . This is obviously an artifact of the MF approach rather than a true physical property of the model.

To justify the FL* theory, the final state needs to be adiabatically connected to the state of weakly interacting spinons and dopons. Moreover, the stability of the MF FL* theory implies that the resulting physical quasiparticles are just renormalized spinon and dopon excitations. However, this is not the case for the underdoped t - J model. Whatever small but non-zero the doping concentration x may be, the dopons couple infinitely strongly to the lattice spins. The true final state cannot therefore be adiabatically connected to a state of weakly interacting dopons and spinons (in the 1d case, this is explicitly demonstrated in our Appendix A). The spin-dopon entanglement due to the NDO constraint is in fact the key ingredient to discuss the underdoped phase. To work out the relevant true low-energy spectrum one needs to resolve the NDO constraint explicitly prior to any MF treatment.

5 Conclusion

The spin-dopon decomposition of the constrained electron operator is shown to be invariant under the local $U(1)$ gauge transformations generated by the local NDO constraint. This symmetry emerges from the redundancy inherent in the spin-dopon representation. It has been missed in earlier developments on the FS reconstruction addressed in the framework of the spin-dopon representation. Since the emergent gauge field is at an infinitely strong coupling, it is necessarily a confining gauge field: the lattice spin background and the conduction dopons are always strongly coupled to each other through the confining $U(1)$ gauge field. In the $U(1)$ confining phase, the unphysical states are naturally excluded from the spectrum.

On the other hand, the stability of the MF FL* ground state necessarily implies that the $U(1)$ gauge symmetry is spontaneously broken. This contradicts a well-known assertion that a local gauge theory can never be broken [38]. Thanks to the NDO constraint there is never a deconfining phase in which the spinons and dopons are weakly coupled to each other. At the moment, we cannot formulate explicitly the resulting strongly coupled $U(1)$ gauge theory of the t - J model within the spin-dopon representation. However we show explicitly that the t - J model is in fact equivalent to a Kondo-Heisenberg lattice model of dopons and lattice spins at infinite Kondo coupling, for all dopings. This observation leads to the conclusion that the dopons and spinons are always confined.

MF Hamiltonians that ignore NDO are gauge dependent: they do not commute with the local operator that enforces the constraint. They act in the enlarged Hilbert space that includes both physical and unphysical states.

The NDO constraint plays a key role in describing the FS crossover as a function of doping in the t - J model. In both the overdoped as well as the underdoped phases, there is a strong entanglement of the spin-dopon degrees of freedom due to the NDO constraint. The weak-coupling MF treatment of the large vs. small Fermi surface crossing in the cuprates is blotted out by the NDO constraint that drives this model to a strong-coupling regime.

Appendix A

Here we prove that equation (16) is indeed equivalent to the original representation (4). To see this, we employ the effective Hamiltonian approach worked out in reference [39] to treat the strong-coupling regime of the Kondo-lattice model. We start by rewriting the local lattice spin operators in the fermion-oscillator representation (2): $\vec{S}_i = \sum_{\sigma,\sigma'} f_{i\sigma}^\dagger \vec{\tau}_{\sigma\sigma'} f_{i\sigma'}$, where $f_{i\sigma}$ denotes the fermion operator subject to the on-site constraint, $\sum_{\sigma} f_{i\sigma}^\dagger f_{i\sigma} = 1$.

The new creation (annihilation) operators can then be introduced [39]:

$$\tilde{c}_{i\sigma}^\dagger = (1 - n_i^d) f_{i\sigma}^\dagger, \quad \tilde{c}_{i\sigma} = (1 - n_i^d) f_{i\sigma}$$

with $n_i^d = \sum_{\sigma} d_{i\sigma}^\dagger d_{i\sigma}$. These operators are restricted to $\tilde{n}_{i\sigma}^c \tilde{n}_{i-\sigma}^c = 0$ for all sites, i.e., no double occupancy of $\tilde{c}$ states is allowed. It is also clear that:

$$\tilde{n}_i^c = \sum_{\sigma} \tilde{c}_{i\sigma}^\dagger \tilde{c}_{i\sigma} = (1 - \tilde{n}_i^d) = (1 - n_i^d).$$

In reference [39], it is shown that the Kondo-lattice Hamiltonian

$$H = \sum_{ij\sigma} 2t_{ij} d_{i\sigma}^\dagger d_{j\sigma} + \lambda \sum_i \vec{S}_i \vec{M}_i$$

in the limit $\lambda \rightarrow \infty$ takes on the form

$$H = - \sum_{ij\sigma} t_{ij} \tilde{c}_{i\sigma}^\dagger \tilde{c}_{j\sigma} - \frac{3\lambda}{4} \sum_i \tilde{n}_i^d + \mathcal{O}(t^2/\lambda). \quad (\text{A.1})$$

Notice now that the term $\propto 3\lambda/4$ in our representation (16) exactly cancels out the second term in equation (A.1). As for the spin exchange contribution $\propto J$ in (16), it takes the form

$$J \sum_{ij} \left(\vec{Q}_i^c \vec{Q}_j^c - \frac{1}{4} \tilde{n}_i^c \tilde{n}_j^c \right). \quad (\text{A.2})$$

Collecting all this together, we get that, in the limit $\lambda \rightarrow \infty$, equation (16) is equivalent to the original representation (4).

Let us now demonstrate this equivalence rederiving the ground-state energy of the 1d Hubbard model at $U = \infty$ in terms of the Kondo-type representation of the t - J model as given by equation (17) at $J = 0$. If this is the case our representation (17) is indeed in agreement with the well-known exact result.

The exact ground-state energy of the $U = \infty$ Hubbard Hamiltonian

$$H_{U=\infty} = - \sum_{ij\sigma} t_{ij} \tilde{c}_{i\sigma}^\dagger \tilde{c}_{j\sigma}, \quad \tilde{c}_{i\sigma} = c_{i\sigma} (1 - n_{i,-\sigma}) \quad (\text{A.3})$$

takes in 1d the form [40]

$$E_{gr}^{U=\infty}/N_{site} = -\frac{2t}{\pi} \sin(\pi x), \quad (\text{A.4})$$

where $x = 1 - \sum_{\sigma} \langle \tilde{c}_{i\sigma}^\dagger \tilde{c}_{i\sigma} \rangle$ is the density of vacancies. On the other hand, for $J \propto t^2/U = 0$ equation (17) reads:

$$H_{U=\infty} = \sum_{ij\sigma} \left(2t_{ij} + \frac{3\lambda}{4} \delta_{ij} \right) d_{i\sigma}^\dagger d_{j\sigma} + \lambda \sum_i \vec{S}_i \vec{M}_i, \quad (\text{A.5})$$

where $\lambda \rightarrow \infty$. This is the exact representation of the $U = \infty$ Hubbard model Hamiltonian.

If equation (A.5) is correct it must reproduce exactly equation (A.4). To show this, consider the 1D strong-coupling Kondo Hamiltonian

$$H^{Kondo} = - \sum_{ij\sigma} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} + \lambda \sum_i \vec{S}_i \vec{s}_i, \quad \lambda \rightarrow +\infty, \quad (\text{A.6})$$

where now $c_{i\sigma}$ stands for a conduction electron operator, and $\vec{s}_i$ denotes the conduction electron spin operator. The ground-state energy is found to be [39]

$$E_{gr}^{Kondo}/N_{site} = \frac{t}{\pi} \sin(\pi x) - \frac{3}{4} \lambda x + \mathcal{O}(1/\lambda). \quad (\text{A.7})$$

Comparing now equations (A.5) and (A.6) immediately gives for the ground-state energy of the Hamiltonian (A.5)

$$E_{gr}/N_{site} = -\frac{2t}{\pi} \sin(\pi x) = E_{gr}^{U=\infty}/N_{site}, \quad (\text{A.8})$$

as desired. Note once more that the $3\lambda/4$ term in equation (A.5) plays an essential role in stabilizing the ground state energy in the limit $\lambda \rightarrow +\infty$.

The ground state of the Hamiltonian (A.5) is represented by the noninteracting *spinless* fermions [40]:

$$H_{gr} = - \sum_{ij} t_{ij} c_i^\dagger c_j, \quad \{c_i^\dagger, c_j\} = \delta_{ij}.$$

This state cannot be adiabatically connected to a state of weakly interacting dopons and lattice spins.

Appendix B

There is a formal analogy between the present formulation and that of the $U = \infty$ Hubbard model. Consider the Hamiltonian

$$H_{Hubb} = \sum_{ij\sigma} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow}. \quad (\text{B.1})$$

In the case $U \rightarrow \infty$, the system is subject to the constraint $n_{i\uparrow} + n_{i\downarrow} \leq 1$. This constraint is equivalent to

$\mathcal{Y}_i^G = n_{i\uparrow}n_{i\downarrow} = 0$. In this way, when $\mathcal{Y}_i^G$ acts on the unphysical (doubly occupied) states we have $\hat{\mathcal{Y}}_i^G|unphys\rangle_i = |unphys\rangle_i$. Therefore, $P_i^G = 1 - n_{i\uparrow}n_{i\downarrow}$ is a projection operator that eliminates the unphysical state at site i . The gauge transformation generated by this constraint:

$$c_{i\downarrow} \rightarrow c_{i\downarrow} e^{i\theta_i n_{i\uparrow}}, \quad c_{i\uparrow} \rightarrow c_{i\uparrow} e^{i\theta_i n_{i\downarrow}},$$

leaves the projected electron operators $\tilde{c}_{i\sigma} = P_i^G c_{i\sigma} P_i^G = c_{i\sigma}(1 - n_{i-\sigma})$ intact. The global projection operator is the well-known Gutzwiller projector $P^G = \prod_i P_i^G$. We can then impose the constraint writing $H_{Hubb} = P^G \sum_{ij\sigma} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} P^G$, which is equivalent to:

$$H_{Hubb} = \sum_{ij\sigma} t_{ij} \tilde{c}_{i\sigma}^\dagger \tilde{c}_{j\sigma}. \quad (\text{B.2})$$

This representations is equivalent to equation (B.1) at $U \rightarrow +\infty$. From this point of view, the Kondo coupling parameter λ in the spin-dopon representation of the t - J model plays the role of the Coulomb repulsion parameter U in the Hubbard model at infinitely large U .

Appendix C

The explicit form of the effective hopping amplitude $t_{ij}^{\text{eff}}(t, \chi)$ in equation (29) is determined in references [30,31] by equating matrix elements of a physical operator in the reduced Hilbert space spanned by the basis vectors (the triplet states are discarded)

$$\sim \prod_i \tilde{a}_{i\sigma_i}^\dagger |\Psi_0\rangle$$

to matrix elements of a certain bilinear form of the canonical fermion operators $d_{i\sigma}$ in the Hilbert space with the canonical basis

$$\prod_i d_{i\sigma_i}^\dagger |0\rangle.$$

Since simply equating two operators acting in different Hilbert spaces (not isomorphic to each other) is in fact a meaningless procedure, we provide below a more accurate treatment to explicitly bring out the actual meaning of the conjecture made in references [30,31].

To this end, we need the following representations that can be found in [16]:

$$\tilde{a}_{i\downarrow} = \sqrt{2}P_i \tilde{d}_{i\downarrow} P_i, \quad \tilde{a}_{i\uparrow} = -\sqrt{2}P_i \tilde{d}_{i\uparrow} P_i. \quad (\text{C.1})$$

Here $\tilde{d}_{i\sigma} = d_{i\sigma}(1 - n_{i\bar{\sigma}}^d)$ is the on-site Gutzwiller projected dopon operator and the projection operator $P_i = 1 - \mathcal{Y}_i^{sd}$ singles out the subspace spanned by the local spin-1/2 states and the spin-dopon singlet state.

Let us now consider the matrix element [30]

$$\langle \Psi_0 | \tilde{a}_{j\uparrow} H_t \tilde{a}_{i\uparrow}^\dagger | \Psi_0 \rangle, \quad (\text{C.2})$$

where

$$H_t = \sum_{ij\sigma} t_{ij} d_{i\sigma}^\dagger d_{j\sigma}.$$

In view of equations (C.1), this can be rewritten as ($P \equiv \prod_i P_i$)

$$\sim \langle \Psi_0 | P \tilde{d}_{j\uparrow} P H_t P \tilde{d}_{i\uparrow}^\dagger P | \Psi_0 \rangle. \quad (\text{C.3})$$

At low doping concentration $x \ll 1$, one can drop the “tilde” sign over the dopon operators, which brings the matrix element to the form ($P|\Psi_0\rangle = |\Psi_0\rangle$)

$$\sim \langle \Psi_0 | d_{j\uparrow} \tilde{H}_t d_{i\uparrow}^\dagger | \Psi_0 \rangle, \quad (\text{C.4})$$

where

$$\tilde{H}_t \equiv P H_t P = \sum_{ij\sigma} t_{ij} \tilde{a}_{i\sigma}^\dagger \tilde{a}_{j\sigma}.$$

The key approximation made in references [30,31] amounts then to discarding the P projection accompanied by a simultaneous renormalization of the hopping amplitude:

$$\tilde{a}_{i\sigma} = \text{sign}(\sigma) \sqrt{2} P_i d_{i\sigma} P_i \rightarrow \text{sign}(\sigma) \sqrt{2} d_{i\sigma}, \quad t_{ij} \rightarrow t_{ij}^{\text{eff}},$$

which yields for the matrix element

$$\sim \langle \Psi_0 | d_{j\uparrow} H_{t_{\text{eff}}} (d^\dagger, d) d_{i\uparrow}^\dagger | \Psi_0 \rangle. \quad (\text{C.5})$$

To explicitly fix t_{ij}^{eff} , the matrix elements (C.2) and (C.5) are then equated to each other. It should be stressed that while the Gutzwiller projection for the dopon operators can indeed be safely discarded at low doping, this is obviously not the case for the NDO projection operator P .

References

1. N. Doiron-Leyraud et al., *Nature* **447**, 565 (2007)
2. A.V. Chubukov, D.K. Morr, *Phys. Rep.* **288**, 355 (1997)
3. T. Senthil, S. Sachdev, M. Vojta, *Phys. Rev. Lett.* **90**, 216403 (2003)
4. T. Senthil, M. Vojta, S. Sachdev, *Phys. Rev. B* **69**, 035111 (2004)
5. I. Paul, C. Pepin, M.R. Norman, *Phys. Rev. Lett.* **98**, 026402 (2007)
6. P. Coleman, N. Andrei, *Phys. Rev. Lett.* **62**, 595 (1989)
7. Q. Si, in *Understanding Quantum Phase Transitions*, edited by L.D. Carr (CRC Press/Taylor Francis, Boca Raton, 2010)
8. J.-W. Mei, S. Kawasaki, G.-Q. Zheng, Z.-Y. Weng, X.-G. Wen, *Phys. Rev. B* **85**, 134519 (2012)
9. M. Punk, S. Sachdev, *Phys. Rev. B* **85**, 195123 (2012)
10. V.A. Sidorov et al., *Phys. Rev. Lett.* **89**, 157004 (2002)
11. R. Bel et al., *Phys. Rev. Lett.* **92**, 217002 (2004)
12. P. Coleman, in *Lectures on the Physics of Highly Correlated Electron Systems VI*, edited by F. Manichini (American Institute of Physics, New York, 2002), p. 79

13. T.C. Ribeiro, X.-G. Wen, Phys. Rev. Lett. **95**, 057001 (2005)
14. T.C. Ribeiro, X.-G. Wen, Phys. Rev. B **74**, 155113 (2006)
15. A. Ferraz, E. Kochetov, M. Mierzejewski, Phys. Rev. B **73**, 064516 (2006)
16. R.C. Pepino, A. Ferraz, E. Kochetov, Phys. Rev. B **77**, 035130 (2008)
17. L.N. Bulaevskii, E.L. Nagaev, D.I. Khomskii, Sov. Phys. J. Exp. Theor. Phys. **27**, 836 (1968)
18. W.F. Brinkman, T.M. Rice, Phys. Rev. B **2**, 1324 (1970)
19. B.I. Shraiman, E.D. Siggia, Phys. Rev. B **42**, 2485 (1990)
20. C. Nayak, Phys. Rev. Lett. **85**, 178 (2000)
21. C. Nayak, Phys. Rev. Lett. **86**, 943 (2001)
22. A. Ferraz, E. Kochetov, B. Uchoa, Phys. Rev. Lett. **98**, 069701 (2007)
23. V. Kalmeyer, R.B. Laughlin, Phys. Rev. Lett. **59**, 2095 (1987)
24. R.B. Laughlin, Phys. Rev. Lett. **60**, 2677 (1988)
25. A.L. Fetter, C.B. Hanna, R.B. Laughlin, Phys. Rev. B **39**, 9679R (1989)
26. G. Baskaran, P.W. Anderson, Phys. Rev. B **37**, 580 (1988)
27. C. Lacroix, Solid State Commun. **54**, 991 (1985)
28. A.A. Abrikosov, Physics **2**, 5 (1965)
29. V.G. Vaks, A.I. Larkin, S.A. Pikin, J. Exp. Theor. Phys. Lett. **26**, 188 (1968)
30. R. Eder, P. Wróbel, Phys. Rev. B **84**, 035118 (2011)
31. R. Eder, P. Wróbel, Y. Ohta, Phys. Rev. B **82**, 155109 (2010)
32. V.I. Belinicher, A.L. Chernyshev, V.A. Shubin, Phys. Rev. B **56**, 3381 (1997)
33. V.I. Belinicher, A.L. Chernyshev, A.V. Dotsenko, O.P. Sushkov, Phys. Rev. B **51**, 6076 (1995)
34. M. Scheunert, W. Nahm, V. Rittenberg, J. Math. Phys. **18**, 155 (1977)
35. P.B. Wiegmann, Phys. Rev. Lett. **60**, 821 (1988)
36. A. Ferraz, E.A. Kochetov, Nucl. Phys. B **853**, 710 (2011)
37. Y. Nagaoka, Phys. Rev. **147**, 392 (1966)
38. S. Elitzur, Phys. Rev. D **12**, 3978 (1975)
39. M. Sigrist, H. Tsunetsugu, K. Ueda, T.M. Rice, Phys. Rev. B **46**, 13838 (1992)
40. H. Shiba, M. Ogata, Int. J. Mod. Phys. B **5**, 31 (1991)