Analysis of fractal groups of the type d -(m, r)-Cantor within the framework of Kaniadakis statistics

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ABSTRACT

We have used Kaniadakis statistics in the analysis of fractal structures of the type d -(m, r)-Cantor. The κ -entropy associated with these structures shows linearity with respect to the dimension L of the system and power-law type behavior with respect to the block size s used in the scanning of a determined sequence. The fractal dimension d_f is related to the entropic parameter κ through an inverse-type law.

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1. Introduction

In describing physical phenomena which contain many identical components, it is common to base one's description on the statistical mechanics of Boltzmann–Maxwell–Gibbs (BG) [1], if the system under consideration is classical, or those of Fermi–Dirac [2, 3] or Bose–Einstein [4,5] if we deal with a quantum system. The first of these theories, BG, offers a perfect connection between the microscopic behavior of a physical system and its macroscopic parameters through the celebrated entropy relation $S = -k_B \ln(W)$, proposed by Boltzmann in 1872. However, as with all human constructs, BG statistical mechanics is not universal; some statistical systems show small, or even large, deviations from BG predictions.¹ A simple example which we could take as evidence of this is the study of fluid turbulence. This is due, to some extent, to the fact that BG has its range of applicability reduced to a class of special phenomena found in so-called *equilibrium thermodynamics*.

On the other hand, we have the broader situation of non-equilibrium phenomena. These possess a greater descriptive complexity, but, despite this complexity, there is a greater interest as in general they deal with more realistic systems (such as climatic, biological and social systems). Since such systems cannot be

treated by traditional BG statistics, we search for a different formalism that, in some way, allows a more appropriate description of such non-equilibrium systems. Browsing the literature, we find proposals such as that of Tsallis – aimed at an area today known as *non-extensive statistics*. In 1988, Tsallis [6] presented a statistical proposal that had as one of its principal points the definition of a new form of entropy. Furthermore, based on this new theory, Tsallis leads us to reflect on traditional concepts such as extensivity and additive entropy: in this formalism, the rule for addition of the entropy for two subsystems is modified to a more general sum, such that we could have two subsystems where the relative entropies are not additive in the BG sense.² Nevertheless, q -entropy³ constitutes a generalization of BG, i.e., for some specific condition, q -statistics takes the BG form.

Tsallis' statistical proposal, which has among its bases a modified entropy, is part of a group of alternative formalisms to BG which were created to describe physical phenomena of particular interest. Within this group, we can find Druyvenstein's statistics [7,8] used to describe the mechanism for electrical discharges in gases at low pressure, and which contains the Maxwell–Boltzmann distribution as a particular case. We also have Abe's statistics [9] which, through the modification of Shannon's entropy and incorporating ideas from quantum groups, shows an entropy that is

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¹ Proof of this is obviously found in the very existence of the Bose–Einstein and Fermi–Dirac formalisms.

² When we sum the entropies of two isolated systems, the entropy of the combined system is, according to BG, the sum of the entropies of the two initially separate systems. When we use the entropy defined by Tsallis, this is not necessarily true.

³ Another way of referring to Tsallis entropy.

invariant to change of the q parameter used by Tsallis. Finally, we also have the statistics and/or entropies of Rényi [10], Sharma–Mittal [11], Landsberg–Vedral [12], the Beck–Cohen's Superstatistics [13], the (c, d) entropy [14] and the group entropies [15].

Among the above, we choose to use the alternative approach of Kaniadakis [16], which intends to answer the following questions: (i) is it possible to treat all the above-mentioned formalisms as bases for a more universal formalism; (ii) is it possible to obtain the stationary distribution of nonlinear systems in the context of this new formalism; (iii) how would the stationary system, if it existed, depend on particular form of each of the above formalisms; and finally, (iv) is there some mechanism, law, or principle behind the temporal description of this more unified formalism? Of course, the answers to these questions are treated by Kaniadakis himself, who postulates the existence of a principle that governs the dynamic interactions and imposes a form of entropy that is independent of the particular formalism used to treat the physical system, i.e., the *Kinetic Interaction Principle* (KIP).

We aim, in the present work, to use Kaniadakis' definition of an adapted entropy together with the concept of block entropy to describe the relationships between the κ parameters and the fractal dimension. In this concern, we wish to analyze the statistical relations between the elements and/or groups of elements associated with the generation of a (m, r) -Cantor. To this end, we shall apply the description and analysis used by Provata [17]. On his paper, the author demonstrated analytically that the static structures of deterministic Cantor sets with fractal dimension d_f , calculated in the framework of Tsallis' statistics, are characterized by a non-extensive q -exponent $q = 1/(d_f - d)$. Here, we shall approach the problem within the ambit of the generalized statistics of G. Kaniadakis [16,19].

This paper is organized as follows. In Section 2, we give a brief description of the main considerations on the Cantor Groups and Probability. A discussion about the Kaniadakis framework is made in Section 3. In Section 4, we discuss the d -(m, r)-Cantor set in the context of κ -framework. We summarize our main conclusions in Section 5.

2. Cantor groups and probability

Since we aim to perform a statistical analysis, via the Kaniadakis formalism, of fractal structures of the type d -(m, r)-Cantor (the significance of the parameters d , m and r will become clearer later on), we shall first present a simpler structure belonging to this class of fractals, the so-called *Cantor dust* or 1-(2, 3)-Cantor in the current notation. This structure may be obtained in various ways. For the purpose of this work, we shall use the development known as a rule of iteration to construct this fractal. Thus, from the elements of the group $\{0, 1\}$, we shall build sequences of elements obeying the following rules: $0 \rightarrow 000$ and $1 \rightarrow 101$. Defining our first sequence as (1), the following sequence becomes, due to the rule of iteration, (101); then (101000101), and so on. To each of these sequences we designate a generation number N (see Fig. 1). In Fig. 2, we have a more traditional representation of the 1-(2, 3)-Cantor group. From this it is possible to give a formal definition of the parameters d , m and r in d -(m, r)-Cantor dust: d represents the topological dimension where the structure is immersed, r tells us in how many equal parts the set must be divided, and, finally m tells us that for each iteration, each subset is reduced by a fraction of its original size, thus leaving only a fraction of m/r parts. The three parameters are of course positive integers. Other examples of Cantor-like sets are the spectra of spin waves, as well as phonons, propagating in quasiperiodic superlattices [20,21].

In accordance with the work of Provata [17], within one of the generation d -(m, r)-Cantor, we obtain from the above analysis that

Fig. 1. Starting from generation zero ($N = 0$), we apply the rule of interaction $1 \rightarrow 101$ ($N = 1$). Then, we apply to each element of the first generation the rules $0 \rightarrow 000$ and $1 \rightarrow 101$, thus giving rise to a second generation. If we continue with the systematic application of the rules innumerable times such that N is very large ($N \rightarrow \infty$), we obtain a fractal called Cantor dust. The dimension d_f of this fractal is $d_f = \ln 2 / \ln 3$.

Fig. 2. Traditional representation of the 1-(2, 3)-Cantor group. Considering that the upper bar (segment) has unitary size, the rule of interaction tells us that we should divide the initial bar into three parts and eliminate the third central part. The result of this process creates the first generation ($N = 1$), with each bar now measuring $1/3$ of a unit. Thus Fig. 1 is the numerical analogous of this one. The sequence of numbers at the bottom of the figure represents the geometrical coordinates of the diverse segments.

the elements $\{0, 1\}$ are ordered such as to form blocks of a determined morphology. Fortunately, the elements $\{0, 1\}$ are arranged to form only two morphological groups. The probabilities of finding one of these groups during a block scan, taking into account the elimination of superposition, are, respectively,

$$p_{[1]} = \left[\frac{m}{r} \right]^{d \times (N - \sigma)}, \quad (1)$$

and

$$p_{[0]} = 1 - \left[\frac{m}{r} \right]^{d \times (N - \sigma)}. \quad (2)$$

The integer number σ is the parameter that incorporates the influence of the block size s (σ and s are related by the expression $s = r^\sigma$) used in the scan of a given generation N . If we take the group 1-(2, 3)-Cantor and $s = 3$, for example, we see that if $\sigma = 1$, the blocks have the form 000 and 101 for any generation N . For $d = 2$, the blocks are square matrices and for $d = 3$, the elements $\{0, 1\}$ will be figuratively arranged in space (it is sufficient to imagine graphically the cantor group 3-(2, 3)-Cantor). In Fig. 3 we illustrate another $d = 1$ arrangement. From this figure, it is noticeable that the arrangement is irrelevant for the calculation of probabilities.

3. Kaniadakis framework

Recent studies on the kinetic foundations of the so-called κ -statistics led to the power-law distribution function and the κ -entropy which emerge naturally in the framework of the kinetic interaction principle (see, e.g., Refs. [16,18]). Formally, the

Fig. 3. Irrelevance of arrangement of a block for the probability. For a given 1-(3,5)-Cantor set, the rule of iteration can assume any of the following forms $1 \rightarrow 10101$, $1 \rightarrow 11010$, $1 \rightarrow 10110$.

κ -framework is based on the κ -exponential and the κ -logarithm functions defined as [16,18]

$$\exp_{\kappa}(x) = (\sqrt{1 + \kappa^2 x^2} + \kappa x)^{1/\kappa} \quad (3)$$

$$\ln_{\kappa}(x) = \frac{x^{\kappa} - x^{-\kappa}}{2\kappa} \quad (4)$$

with

$$\ln_{\kappa}(\exp_{\kappa}(x)) = \exp_{\kappa}(\ln_{\kappa}(x)) = x. \quad (5)$$

The κ -parameter belongs to the mathematical interval $|\kappa| < 1$ and in the case $\kappa = 0$ these expressions reduce to the usual exponential and logarithmic functions. The κ -entropy associated with the κ -framework is given by

$$S_{\kappa} = - \int d^3p f \ln_{\kappa} f \quad (6)$$

which fully recovers standard Boltzmann–Gibbs entropy, $S_{\kappa \rightarrow 0}(f) = - \int f \ln f d^3p$. Here, p and f represent the momenta and its distribution function, respectively. Several physical features of the κ -distribution have also been theoretically investigated as, for instance, the self-consistent relativistic statistical theory [18,22], nonlinear kinetics [23], and the H-theorem from a generalization of the chaos molecular hypothesis [24,25]. As a matter of fact, the Kaniadakis entropy also can be a particular case of two-parameter entropies [26–29] or of three-parameter entropies [29].

4. Block entropy and Kaniadakis framework

In order to analyze the d -(m, r)-Cantor structures statistically, we shall use a recent definition attributed to entropy, established in 2001 and further developed in 2002, in the context of special relativity [18,22]. In our formalism, the entropy which describes any sequence of elements, is constructed by scanning the structure with a block of size s , and may be determined from the expression [18,22]

$$S_{\kappa} = - \sum_i p_i \ln_{\kappa} p_i = - \sum_i \frac{[p_i(s)]^{1+\kappa} - [p_i(s)]^{1-\kappa}}{2\kappa}. \quad (7)$$

A special case of Eq. (7) is the Shannon–Gibbs–Boltzmann entropy: one need only take the limit of κ tending to zero.⁴ In our case, the index i goes from 1 to 2, such that $p_1(s) = p_{[0]}(s)$ and $p_2(s) = p_{[1]}(s)$. In terms of block probabilities, $p_1(s) = (m/r)^{N-\sigma}$ and $p_2(s) = 1 - p_1(s)$, so $p_1(s)$ is the probability of finding a given arrangement of 1's and $p_2(s)$ is the probability of occurrence of 0's [17]. Inserting Eqs. (1) and (2) into (7), we find, for sequences of the type d -(m, r)-Cantor, that the κ block entropy assumes the form

$$S_{\kappa} = - \frac{[(\frac{m}{r})^{N-\sigma}]^{1+\kappa} - [(\frac{m}{r})^{N-\sigma}]^{1-\kappa}}{2\kappa} - \frac{[1 - (\frac{m}{r})^{N-\sigma}]^{1+\kappa} - [1 - (\frac{m}{r})^{N-\sigma}]^{1-\kappa}}{2\kappa} \quad (8)$$

⁴ Formally, $\lim_{\kappa \rightarrow 0} S_{\kappa}$.

Fig. 4. S_{κ} behavior with increasing L for the configuration $m = 2$, $r = 3$ and $s = 9$. The value of κ that makes S_{κ} linear with L is $\kappa = \kappa_L = 3.895 \pm 0.059$. For values of κ greater than κ_L , the entropy grows rapidly and hence diverges. When κ is smaller than the above value, the entropy increases slowly with L .

Since $L = r^N$ and $s = r^{\sigma}$, it is more convenient to write a shorter version of the above equation. By considering the property $y^{\ln(x)} = x^{\ln(y)}$, we rewrite Eq. (8) as

$$S_{\kappa}(s, L) = - \frac{(\frac{s}{L})^{(d-d_f)(1+\kappa)} - (\frac{s}{L})^{(d-d_f)(1-\kappa)}}{2\kappa} - \frac{[1 - (\frac{s}{L})^{(d-d_f)(1+\kappa)}] - [1 - (\frac{s}{L})^{(d-d_f)(1-\kappa)}]}{2\kappa}. \quad (9)$$

The constant d_f is a fractal dimension corresponding to the N th generation d -(m, r)-Cantor, with $N \rightarrow \infty$. There now arises a second question: which of the various values of the deformation parameter κ will represent the entropy of a given system of the type d -(m, r)-Cantor in the simplest form? The choice will be made in accordance with the following observation: for certain values of κ , S_{κ} may grow rapidly and consequently diverge; in other cases, the entropy grows slowly with L ; however, there exists a unique value for κ where S_{κ} exhibits a linear behavior with L for a fixed value of s . It is this value that we shall choose to represent the entropy S_{κ} of a given d -(m, r)-Cantor group (Fig. 4).

We should also consider the question of the behavior of S_{κ} with the block size s used to scan a given d -(m, r)-Cantor generation. In general, it is observed that, for fixed L values, S_{κ} shows a power law type behavior in relation to s . Through the calculation of the correlation coefficient, we can show that the dependence of S_{κ} on s , taking each on a logarithmic scale, shows very close to a linear behavior with negative gradient, i.e., when s increases, S_{κ} decreases (Fig. 5). This behavior is the one that would be expected since, each time we decrease the size s of the block, the system acquires more components (in the language of statistical mechanics, we would say that the system augments the number of accessible microstates) thus further increasing the entropy; if the opposite occurs, the system tends to possess less elements thus reducing the entropy.

For the values of κ which make the entropy S_{κ} linear with L for a d -(m, r)-Cantor group, we find that the dependence on the fractal dimension obeys, in the unidimensional case, the relation

$$\kappa = 1 + \frac{1}{d - d_f} \quad (10)$$

where d is the topological dimension. Eq. (10) is obtained analytically by Taylor's expansion of Eq. (9), and by demanding that the entropy must be linear with the sample size. It also can be obtained numerically, from Fig. 4. In both cases the results agree. In Fig. 6, we show the relationship between κ and d_f for different dimensions d .

Fig. 5. Log-log graph of S_κ as a function of block size s for a sequence containing $L = 10^{10}$ elements. Here, we use $r = 12$ and $m = 2$.

Fig. 6. Dependence of κ on the fractal dimension for two and three dimensions. On this graph, d is the topological dimension corresponding to the fractal object under study. As the errors in the determination of κ are of order $\sim 10^{-2}$, we have multiplied each error by 10 for better visualization.

5. Conclusions

An analysis of random Cantor-type fractal structures immersed in a space of dimension d was performed using the Kaniadakis formalism adapted to the block concept. Such systems are non-extensive from the point of views of the Boltzmann–Gibbs–Shannon formalism. However, in the ambit of κ -entropy, we can always find a suitable κ that makes the system linear with L and, consequently, extensive in this direction. We also find that, for such structures, the κ -entropy shows linear dependence with s (on a logarithmic scale) for any value of κ , demonstrating that S_κ shows a power law type comportment in relation to s . This behavior occurs simply because, when we increase s , the degree of disorder decreases (larger blocks imply a greater knowledge of information and consequently a reduction of entropy). Also, we find that the values of κ that make S_κ linear with L are contained in a region where κ -entropy shows no maximum since they are outside the limit $|\kappa| \leq 1$. Exploring the dependence of κ on frac-

tal dimension, we show numerically that κ behaves analytically as an inverse shifted function. We have also explored regions where $d > 1$. The results are analogous to the analysis in one dimension, i.e., linearity of S_κ with L for a given κ and power law type behavior of s with S_κ for any value of κ . The extension of the analysis performed in this work to other fractal groups is viable, in principle, depending on the structural nature of the fractal under consideration; for Fibonacci sequences, this could be realized only for unidimensional blocks ($s = 1$), limiting the use of blocks greater than unity; for Thue–Morse sequences, the *a priori* equality of the probabilities prevails and, in this case, we would be within the limits where S_κ is allowed a maximum value. For more complicated sequences we would have to improve on the concepts used with regard to scanning these sequences in search of a morphological tendency, as was found in the case of a d -(m, r)-Cantor set. Although the present work relies on a purely theoretical analysis, we are using the techniques and skills learned from it to analyze (the results are indeed ready for submission) the κ -block behavior of DNA sequences on the view of Kaniadakis' framework.

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