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A new compounding family of distributions: The generalized gamma power series distributions

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ABSTRACT

We propose a new four-parameter family of distributions by compounding the generalized gamma and power series distributions. The compounding procedure is based on the work by Marshall and Olkin (1997) and defines 76 sub-models. Further, it includes as special models the Weibull power series and exponential power series distributions. Some mathematical properties of the new family are studied including moments and generating function. Three special models are investigated in detail. Maximum likelihood estimation of the unknown parameters for complete sample is discussed. Two applications of the new models to real data are performed for illustrative purposes.

1. Introduction

The generalized gamma ($\mathcal{GG}$) distribution [1] is a well-known and established three-parameter distribution for modeling lifetime data and phenomenon with monotone failure rates. It is specially useful to fit bathtub hazard rate data (in addition to increasing, decreasing and unimodal shapes), thus overcoming the forms presented by the exponential, gamma and Rayleigh distributions for modeling this type of data. The $\mathcal{GG}$ distribution has been used in several research areas such as engineering, hydrology and survival analysis. However, in order to fit still more complex situations, a number of extensions have been proposed in recent years. For example, see the works by Cordeiro et al. [2], Ortega et al. [3], Cordeiro et al. [4] and the references therein.

A random variable T following the $\mathcal{GG}$ distribution with shape parameters $k > 0$, $\alpha > 0$ and scale parameter $\beta > 0$ has cumulative distribution function (cdf) given by

$$F_{\mathcal{GG}}(t) = \gamma_1\left(k, \left(\frac{t}{\beta}\right)^{\alpha}\right), \quad t > 0, \tag{1}$$

where $\gamma_1(k, z) = \gamma(k, z)/\Gamma(k)$ is the incomplete gamma function ratio and $\gamma(k, z) = \int_0^z \omega^{k-1} e^{-\omega} d\omega$ is the incomplete gamma function. The probability density function (pdf) corresponding to (1) is

$$f_{\mathcal{GG}}(t) = \frac{\alpha}{\beta\,\Gamma(k)}\left(\frac{t}{\beta}\right)^{k\alpha-1} \exp\left\{-\left(\frac{t}{\beta}\right)^{\alpha}\right\}, \quad t > 0. \tag{2}$$

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Stacy and Mihram [5] encountered some difficulties in developing maximum likelihood procedures and large sample inference for its parameters. On the other hand, Prentice [6] reparameterized it in such a way that the inference can be fairly easily handled. Lawless [7] by using Prentice's re-parametrization developed exact inference procedures concerning the quantiles and scale parameters from uncensored samples and DiCiccio [8] proposed approximate conditional inference methods for location and scale parameters. Recently, Huang and Hwang [9] presented a simple method for estimating the model parameters, using its characterization and moment estimation. An iterative estimation method for its parameters was implemented in S–PLUS by Gomes et al. [10]. Tadikamalla [11] proposed a simple rejection method for sampling directly from the $\mathcal{GG}$ distribution without generating gamma variates, but valid only for $\beta > 1$.

Our chief goal is to propose a new extension of the $\mathcal{GG}$ distribution by compounding the $\mathcal{GG}$ and power series ($\mathcal{PS}$) distributions. The generated class is called the *generalized gamma power series* ($\mathcal{GGPS}$) family. The compounding procedure follows the pioneering work of Marshall and Olkin [12]. In the same way, several classes of distributions were proposed by compounding some useful lifetime and $\mathcal{PS}$ distributions in the last few years. Chahkandi and Ganjali [13] defined the exponential power series ($\mathcal{EPS}$) class of distributions, which contains as special cases the exponential Poisson ($\mathcal{EP}$), exponential geometric ($\mathcal{EG}$) and exponential logarithmic ($\mathcal{EL}$) distributions. Morais and Barreto-Souza [14] defined the Weibull power series ($\mathcal{WPS}$) class which includes as sub-models the $\mathcal{EPS}$ distributions. The $\mathcal{WPS}$ distributions can have increasing, decreasing and upside down bathtub failure rate function. The generalized exponential power series ($\mathcal{GEPS}$) distributions were proposed by Mahmoudi and Jafari [15] following the same approach of Morais and Barreto-Souza [14]. Silva et al. [16] studied the extended Weibull power series ($\mathcal{EWPS}$) family, which includes as special models the $\mathcal{EPS}$ and $\mathcal{WPS}$ distributions. Bourguignon et al. [17] and Silva and Cordeiro [18] proposed the Birnbaum–Saunders power series ($\mathcal{BSPS}$) and Burr XII power series ($\mathcal{BXIIPS}$) classes of distributions, respectively.

The rest of the paper is organized as follows. In Section 2, we introduce and motivate the new family and present a useful representation for its density function. Section 3 gives an explicit expression for the $\mathcal{GGPS}$ moments. The moment generating function (mgf) is also derived in this section. We discuss in Section 4 three special models of the proposed family. Estimation of the parameters by maximum likelihood is addressed in Section 5. Section 6 gives two applications to real data to prove that the new family can be used quite effectively in analyzing lifetime data. Section 7 provides some conclusions.

2. The $\mathcal{GGPS}$ family of distributions

The new family of distributions is rather simple to be constructed following the same set-up carried out by Marshall and Olkin [12]. Given a discrete random variable N, let $X_1, \ldots, X_N$ be i.i.d. random variables having the $\mathcal{GG}$ distribution (1) with shape parameters $k, \alpha > 0$ and scale parameter $\beta > 0$, where N has a power series probability mass function (pmf) (truncated at zero) given by

$$p_n = P(N = n) = \frac{a_n\,\theta^n}{C(\theta)}, \quad n = 1, 2, \ldots. \tag{3}$$

The coefficients a_n's depend only on n and $C(\theta) = \sum_{n=1}^{\infty} a_n\,\theta^n$ (for $\theta > 0$) is assumed finite. It is important to remark that the probability class of distributions (3) has been considered in [19,20]. Table 1 lists some power series distributions (truncated at zero) defined by (3) such as the Poisson, logarithmic, geometric and binomial distributions. Let $X = \min\{T_i\}_{i=1}^{N}$. The conditional cumulative distribution of $X|N = n$ is given by

$$F_{X|N=n}(x) = 1 - \left[1 - \gamma_1\left(k, \left(\frac{x}{\beta}\right)^{\alpha}\right)\right]^n$$

i.e., $X|N = n$ has the exponentiated form of (1) with parameters n, k, α and β. So, we obtain

$$P(X \le x, N = n) = \frac{a_n\theta^n}{C(\theta)}\left\{1 - \left[1 - \gamma_1\left(k, \left(\frac{x}{\beta}\right)^{\alpha}\right)\right]^n\right\}, \quad x > 0,\ n = 1, 2, \ldots.$$

Then, the marginal cdf of X becomes

$$F_{\mathcal{GGPS}}(x; \theta, k, \alpha, \beta) = 1 - C(\theta)^{-1} C\left\{\theta\left[1 - \gamma_1\left(k, \left(\frac{x}{\beta}\right)^{\alpha}\right)\right]\right\}, \quad x > 0. \tag{4}$$

Eq. (4) is called the $\mathcal{GGPS}$ family of distributions.

Hereafter, a random variable X following (4) with parameters θ, k, α and β is denoted by $X \sim \mathcal{GGPS}(\theta, k, \alpha, \beta)$. Eq. (4) extends several other distributions which have been studied in the literature. The $\mathcal{EG}$ distribution [21] is obtained by taking $k = \alpha = 1$ and $C(\theta) = \theta(1-\theta)^{-1}$ with $\theta \in (0, 1)$. Further, for $k = \alpha = 1$, we obtain the $\mathcal{EP}$ [22] and $\mathcal{EL}$ [23] distributions by taking $C(\theta) = e^{\theta} - 1$, $\theta > 0$, and $C(\theta) = -\log(1-\theta)$, $\theta \in (0, 1)$, respectively. In the same way, for $k = 1$, we obtain the $\mathcal{WG}$ [24] and $\mathcal{WP}$ [25] distributions. The $\mathcal{EPS}$ distributions are obtained from (4) when $k = \alpha = 1$ for any $C(\theta)$ listed in Table 1 (see [13]). Finally, we obtain the $\mathcal{WPS}$ distributions from (4) by taking $k = 1$ for any $C(\theta)$ in Table 1 (see [14]). Some important sub-models of the $\mathcal{GG}$ distribution are listed in Table 2. This composition leads to 76 special models.

Table 1
Useful quantities for some $\mathcal{PS}$ distributions.

Distribution	a_n	$C(\theta)$	$C'(\theta)$	$C''(\theta)$	$C^{-1}(\theta)$	Θ
Poisson	$n!^{-1}$	$e^{\theta}-1$	e^{θ}	e^{θ}	$\log(\theta+1)$	$\theta\in(0,\infty)$
Logarithmic	n^{-1}	$-\log(1-\theta)$	$(1-\theta)^{-1}$	$(1-\theta)^{-2}$	$1-e^{-\theta}$	$\theta\in(0,1)$
Geometric	1	$\theta(1-\theta)^{-1}$	$(1-\theta)^{-2}$	$2(1-\theta)^{-3}$	$\theta(\theta+1)^{-1}$	$\theta\in(0,1)$
Binomial	$\binom{m}{n}$	$(\theta+1)^m-1$	$m(\theta+1)^{m-1}$	$\frac{m(m-1)}{(\theta+1)^{2-m}}$	$(\theta-1)^{1/m}-1$	$\theta\in(0,1)$

Table 2
The Stacy family of distributions (n positive integer).

Distribution	α	β	k	Reference
Gamma	1	β	k	Johnson et al. [26]
Erlang	1	β	n	Evans et al. [27]
Standard gamma	1	1	k	Johnson et al. [26]
Scaled chi-square	1	β	$n/2$	Lee [28]
Chi-square	1	2	$n/2$	Johnson et al. [26]
Exponential	1	β	1	Johnson et al. [26]
Standard exponential	1	1	1	Johnson et al. [26]
Wien	1	β	4	Johnson et al. [26]
Nakagami	2	β	k	Nakagami [29]
Scaled chi	2	β	$n/2$	Johnson et al. [26]
Chi	2	$\sqrt{2}$	$n/2$	Johnson et al. [26]
Half-normal	2	β	1/2	Johnson et al. [26]
Folded normal	2	$\sqrt{2}$	1/2	Leone et al. [30]
Rayleigh	2	β	1	Rayleigh [31]
Maxwell–Boltzmann	2	β	3/2	Maxwell [32]
Wilson–Hilferty	3	β	k	Wilson and Hilferty [33]
Weibull	α	β	1	Weibull [34]
Stretched exponential	α	β	1	Laherrère and Sornette [35]
Log-normal	α	β	∞	Johnson et al. [26]

The survival function of X is given by

$$S_{\mathcal{GGPS}}(x;\theta,k,\alpha,\beta)=C(\theta)^{-1}C\left\{\theta\left[1-\gamma_1\left(k,\left(\frac{x}{\beta}\right)^{\alpha}\right)\right]\right\},\quad x>0.$$

The pdf of X reduces to

$$f_{\mathcal{GGPS}}(x;\theta,k,\alpha,\beta)=\frac{\theta}{C(\theta)}f_{\mathcal{GG}}(x)\,C'\left\{\theta\left[1-\gamma_1\left(k,\left(\frac{x}{\beta}\right)^{\alpha}\right)\right]\right\},\quad x>0, \tag{5}$$

where $f_{\mathcal{GG}}(\cdot)$ is given in Eq. (2). It can be shown that

$$\lim_{x\to 0}f_{\mathcal{GGPS}}(x;\theta,k,\alpha,\beta)=\begin{cases}\infty, & k\alpha<1,\\ 0, & k\alpha>1,\\ \theta\,C'(\theta)/\beta\,C(\theta), & k\alpha=1,\end{cases}\quad\text{and}\quad \lim_{x\to\infty}f_{\mathcal{GGPS}}(x;\theta,k,\alpha,\beta)=0.$$

A positive point of the current generalization is that the $\mathcal{GG}$ distribution is a basic exemplar of the proposed family. In addition, the new class is well-motivated for industrial applications and biological studies: (i) consider the time to relapse of cancer under the first-activation scheme. Suppose that the number, say N, of carcinogenic cells for an individual left active after the initial treatment follows a $\mathcal{PS}$ distribution and let T_i be the time spent for the ith carcinogenic cell to produce a detectable cancer mass, for $i\geq 1$. If $\{T_i\}_{i\geq 1}$ is a sequence of i.i.d. $\mathcal{GG}$ random variables independent of N, then the time to relapse of cancer of a susceptible individual can be modeled by a $\mathcal{GGPS}$ distribution; (ii) suppose that the failure of a device occurs due to the presence of an unknown number, say N, of initial defects of the same kind, which can be identifiable only after causing failure and are repaired perfectly. Let T_i be the time to the failure of the device due to the ith defect (for $i\geq 1$). If we assume that the T_i's are i.i.d. $\mathcal{GG}$ random variables independent of N, which follows a $\mathcal{PS}$ distribution, then the time to the first failure is appropriately modeled by the $\mathcal{GGPS}$ class; (iii) for reliability studies, from $X=\min\{T_i\}_{i=1}^{N}$ and $Y=\max\{T_i\}_{i=1}^{N}$, the proposed models can be used in serial and parallel systems with identical components, which appear in many industrial applications and biological organisms; (iv) consider that the number N of latent factors that must all be activated by failure follows a $\mathcal{PS}$ distribution and assume that T_i representing the time of resistance to a disease manifestation due to the ith latent factor has the $\mathcal{GG}$ distribution. In the last-activation scheme, the failure occurs after all N factors have been activated. So, the proposed class of distributions is able for modeling the time to the failure under last-activation scheme.

Table 3
Closed-form expressions for $\sum_{n=s}^{\infty} p_n \binom{n}{s}$.

Distribution	$\sum_{n=s}^{\infty} p_n \binom{n}{s}$
Poisson	$\frac{\theta^s}{(1-e^{-\theta})s!}$
Logarithmic	$-\frac{1}{\log(1-\theta)^s}\left(\frac{\theta}{1-\theta}\right)^s$
Geometric	$\frac{1}{1-\theta}\left(\frac{\theta}{1-\theta}\right)^{s-1}$
Binomial	$\frac{\theta^s(1+\theta)^{m-s}}{(1+\theta)^m-1}\binom{m}{s}$

The main motivation for this new family is based on four issues:

- We define the $\mathcal{GGPS}$ distributions because of the wide usage of (2) and the fact that the current generalization provides means of its continuous extension to still more complex situations. From a practical point of view, this generalization is based on the search for distributions that are more flexible in such a way that they fit better to the lifetime data, among others.
- Some distributions commonly used for parametric models in survival analysis are special cases of the $\mathcal{GGPS}$ family, which defines 76 (19×4) sub-models as special cases.
- As we shall see later, the $\mathcal{GGPS}$ hazard rate function (hrf) can be constant, decreasing, increasing, upside-down bathtub or bathtub-shaped.
- It can be applied in some interesting situations as follows: biological and reliability studies [2], failure times of fatiguing materials, medical studies (see Section 6), among others.

Proposition 1. *The classical $\mathcal{GG}$ distribution with parameters k, α and β is a limiting special case of the proposed family when $\theta \to 0$.*

Proposition 2. *The $\mathcal{GGPS}$ density function can be expressed as a mixture of densities of minimum order statistics of T.*

Proof. We know that $C'(\theta) = \sum_{n=1}^{\infty} n a_n \theta^{n-1}$. Then,

$$f_{\mathcal{GGPS}}(x; \theta, k, \alpha, \beta) = \sum_{n=1}^{\infty} p_n f_{T_{(1)}}(x; k, \alpha, \beta), \tag{6}$$

where $f_{T_{(1)}}(\cdot)$ is the density function of $T_{(1)} = \min\{T_i\}_{i=1}^{n}$, for fixed n, given by

$$f_{T_{(1)}}(t; k, \alpha, \beta) = n f_{\mathcal{GG}}(t)\left[1 - \gamma_1\left(k, \left(\frac{t}{\beta}\right)^{\alpha}\right)\right]^{n-1}, \quad t > 0. \tag{7}$$

By using the binomial theorem, we can write (7) in terms of a mixture of the exponentiated generalized gamma ($\mathcal{EGG}$) densities

$$f_{T_{(1)}}(t; k, \alpha, \beta) = \sum_{s=1}^{n} (-1)^{s-1}\binom{n}{s} f_{\mathcal{EGG}}(t; s, k, \alpha, \beta), \quad t > 0, \tag{8}$$

where the integer s is a power parameter for which the $\mathcal{GG}$ cdf is raised to yield the $\mathcal{EGG}$ [2] model. Inserting (8) in Eq. (6) gives

$$f_{\mathcal{GGPS}}(x; \theta, k, \alpha, \beta) = \sum_{n=1}^{\infty}\sum_{s=1}^{n} (-1)^{s-1} p_n \binom{n}{s} f_{\mathcal{EGG}}(x; s, k, \alpha, \beta), \quad x > 0.$$

We can substitute $\sum_{n=1}^{\infty}\sum_{s=1}^{n}$ for $\sum_{s=1}^{\infty}\sum_{n=s}^{\infty}$ to write

$$f_{\mathcal{GGPS}}(x; \theta, k, \alpha, \beta) = \sum_{s=1}^{\infty} \nu_{s,n} f_{\mathcal{EGG}}(x; s, k, \alpha, \beta) \quad x > 0, \tag{9}$$

where $\nu_{s,n} = (-1)^{s-1}\sum_{n=s}^{\infty} p_n \binom{n}{s}$. Table 3 lists closed-form expressions of $\sum_{n=s}^{\infty} p_n \binom{n}{s}$ for the Poisson, logarithmic, geometric and binomial distributions. These expressions can be obtained using programs for symbolic computations such as MAPLE, MATLAB or MATHEMATICA. The $\mathcal{EGG}$ distribution is a well-established model, which has its main properties studied in detail by Cordeiro et al. [2]. For example, they provided explicit algebraic formulae for the moments which hold in generality for any parameter values. These results can be applied to the generating function and incomplete moments. Further, they obtained an infinite weighted sum for the moments of the order statistics. Notice that Eq. (9) is an important result since we can obtain various structural quantities of X such as moments and generating function from those of the $\mathcal{EGG}$ class.

Remark. Let $Y = \max\{T_i\}_{i=1}^{N}$, where $N \sim \mathcal{PS}(\theta)$. Then, the cdf and pdf of Y are, respectively, given by

$$F_Y(y) = C(\theta)^{-1} C\left[\theta\gamma_1\left(k, \left(\frac{x}{\beta}\right)^{\alpha}\right)\right], \quad y > 0$$

and

$$f_Y(y) = \frac{\theta}{C(\theta)} f_{\mathcal{GG}}(y)\, C'\left[\theta\gamma_1\left(k, \left(\frac{x}{\beta}\right)^{\alpha}\right)\right], \quad y > 0. \tag{10}$$

Proposition 3. *The density function of Y defined by Eq. (10) can be expressed as a mixture of densities of maximum order statistics of T.*

Proof. We can use a similar argument to prove Proposition 2.

The class with cdf (10) is called the *complementary generalized gamma power series* ($\mathcal{CGGPS}$) family of distributions. This family is suitable in a complementary risk problem based in the presence of latent risks which arise in several areas such as public health, actuarial science, biomedical studies, demography and industrial reliability [36]. However, in this work, we do not focus on this alternative family.

The $\mathcal{GGPS}$ hrf is given by

$$h_{\mathcal{GGPS}}(x; \theta, k, \alpha, \beta) = \theta f_{\mathcal{GG}}(x) \frac{C'\left\{\theta\left[1 - \gamma_1\left(k, (x/\beta)^{\alpha}\right)\right]\right\}}{C\left\{\theta\left[1 - \gamma_1\left(k, (x/\beta)^{\alpha}\right)\right]\right\}}, \quad x > 0,$$

and the corresponding reverse hrf becomes

$$r_{\mathcal{GGPS}}(x; \theta, k, \alpha, \beta) = \frac{\theta f_{\mathcal{GG}}(x)\, C'\left\{\theta\left[1 - \gamma_1\left(k, (x/\beta)^{\alpha}\right)\right]\right\}}{C(\theta) - C\left\{\theta\left[1 - \gamma_1\left(k, (x/\beta)^{\alpha}\right)\right]\right\}}, \quad x > 0.$$

Theorem 1. *Let $X \sim \mathcal{GGPS}(\theta, k, \alpha, \beta)$. Then:*

(i) *For any constant $c > 0$, it follows that $cX \sim \mathcal{GGPS}(\theta, k, \alpha, c\beta)$, i.e., the $\mathcal{GGPS}$ distribution is closed under scale transformations;*
(ii) *If $X \sim \mathcal{GGPS}(\theta, k, \alpha, \beta)$, then the random variable $Z = X^m$, $m > 0$, has the generalized gamma power series distribution with parameters θ, α/m, β^m and k, i.e., the $\mathcal{GGPS}$ distribution is closed under power transformation;*
(iii) *If $X \sim \mathcal{GGPS}(\theta, k, \alpha, \beta)$, then the random variable $Z = X^{-1}$ has the inverse $\mathcal{GGPS}$ distribution, i.e., the $\mathcal{GGPS}$ distribution is closed under inverse transformation;*
(iv) *$Z = \frac{k}{\beta}X \sim \mathcal{GGPS}(\theta, k, \alpha, k)$, i.e., Z follows a three-parameter $\mathcal{GGPS}$ distribution.*

Proof. The proof is given in the Appendix.

3. Quantiles, moments and generating function

The inverse of the incomplete gamma function ratio $\gamma_1^{-1}(\cdot, \cdot)$ yields a simple quantile function (qf) given by

$$Q(u) = \beta\,\gamma_1^{-1}\left\{k, 1 - \theta^{-1}C^{-1}\left[(1-u)C(\theta)\right]\right\}^{1/\alpha}, \quad u \in (0, 1), \tag{11}$$

where $C^{-1}(\cdot)$ is the inverse function of $C(\cdot)$.

We generate $\mathcal{GGPS}$ random variables following the procedure:

- *Step* 1: Generate $U \sim U(0, 1)$;
- *Step* 2: Set values for θ, k, α and β of $X \sim \mathcal{GGPS}(\theta, k, \alpha, \beta)$;
- *Step* 3: Specify the function $C^{-1}(\cdot)$ such as anyone in Table 1 and use (11);
- *Step* 4: Obtain an outcome of X by $X = Q(U)$;
- *Step* 5: Repeat Steps 1–4 until the required amount of random numbers be completed.

Many of the important characteristics and features of a distribution are obtained through the ordinary moments and generating function. The rth ordinary moment of X is given by $\mu'_r = \mathbb{E}(X^r) = \int_0^{\infty} x^r f_{\mathcal{GGPS}}(x)dx$. From Proposition 2 (and the monotone convergence theorem) and Eq. (9), we obtain

$$\mu'_r = \sum_{s=1}^{\infty} \nu_{s,n}\, \mathbb{E}(Z^r), \tag{12}$$

where from now on $Z \sim \mathcal{EGG}(s, k, \alpha, \beta)$. The rth ordinary moment of Z can be found in equations (7) and (8) by Cordeiro et al. [2]. Tables 4–6 compare the first five moments of X for three special $\mathcal{GGPS}$ models determined numerically from

Table 4
Moments of $X \sim \mathcal{GGG}(\theta, k, \alpha, \beta)$ for some parameter values: $k = 0.5, \alpha = 2$ and $\beta = 3$.

μ'_r	$\theta = 0.3$		$\theta = 0.7$	
	Expression (12)	Expression (19)	Expression (12)	Expression (19)
μ'_1	1.451767	1.451768	0.962619	0.962531
μ'_2	3.561044	3.561044	1.934723	1.934703
μ'_3	11.511850	11.511850	5.651680	5.651674
μ'_4	44.656620	44.656620	20.757380	20.757380
μ'_5	198.129200	198.129200	89.276610	89.276510

Table 5
Moments of $X \sim \mathcal{GGP}(\theta, k, \alpha, \beta)$ for some parameter values: $k = 0.5, \alpha = 2$ and $\beta = 1$.

μ'_r	$\theta = 0.3$		$\theta = 0.7$	
	Expression (12)	Expression (21)	Expression (12)	Expression (21)
μ'_1	0.607780	0.607779	0.569781	0.569790
μ'_2	0.484590	0.484590	0.435174	0.435170
μ'_3	0.453412	0.453415	0.395260	0.395256
μ'_4	0.474276	0.474271	0.404853	0.404853
μ'_5	0.540001	0.540001	0.454062	0.454062

Table 6
Moments of $X \sim \mathcal{GGL}(\theta, k, \alpha, \beta)$ for some parameter values: $k = 1, \alpha = 2$ and $\beta = 3$.

μ'_r	$\theta = 0.3$		$\theta = 0.7$	
	Expression (12)	Expression (23)	Expression (12)	Expression (23)
μ'_1	1.570591	1.570591	1.307648	1.307641
μ'_2	4.016867	4.016867	3.077573	3.077572
μ'_3	13.299440	13.299440	9.766135	9.766134
μ'_4	52.334060	52.334060	37.539000	37.539000
μ'_5	234.250800	234.250800	165.739500	165.739500

Eq. (12) and using numerical integration from their own density functions. We consider the first 50 terms in (12). The computations are performed using the R program.

The central moments (μ_r) and cumulants (κ_r) of X can be calculated from (12) as

$$\mu_r = \sum_{k=0}^{p} \binom{r}{k} (-1)^k \mu_1'^r \mu'_{r-k} \quad \text{and} \quad \kappa_r = \mu'_r - \sum_{k=1}^{r-1} \binom{r-1}{k-1} \kappa_k \mu'_{r-k},$$

respectively, where $\kappa_1 = \mu'_1$. The skewness and kurtosis of X are given by $\zeta_1 = \kappa_3/\kappa_2^{3/2}$ and $\zeta_2 = \kappa_4/\kappa_2^2$, respectively.

The incomplete moments and generating function of X follow from (9) using the monotone convergence theorem:

$$\mathbb{I}_X(y) = \int_0^y x^r f_{\mathcal{GGPS}}(x)\mathrm{d}x = \sum_{s=1}^{\infty} \nu_{s,n}\, \mathbb{I}_Z(y)$$

and

$$\mathbb{M}_X(t) = \sum_{s=1}^{\infty} \nu_{s,n}\, \mathbb{E}(\mathrm{e}^{tZ}),$$

where $Z \sim \mathcal{EGG}(s, k, \alpha, \beta)$ and $\mathbb{I}_Z(y) = \int_0^y z^r f_Z(z)\mathrm{d}z$. We can calculate $\mathbb{E}(\mathrm{e}^{tZ})$ numerically for any special case of the $\mathcal{EGG}$ model.

4. Maximum likelihood estimation

Estimation of the unknown parameters of the $\mathcal{GGPS}$ family by the method of maximum likelihood is addressed in this section. Let $x_1, \ldots, x_n$ be a sample of size n from the $\mathcal{GGPS}(\theta, k, \alpha, \beta)$ distribution. Let $\boldsymbol{\eta} = (\theta, k, \alpha, \beta)^\top$ be the parameter vector of interest. The log-likelihood function for $\boldsymbol{\eta}$ based on this sample becomes

$$\ell_n = \ell_n(\boldsymbol{\eta}) = n \log\left[\frac{\theta\alpha}{\beta\,\Gamma(k)}\right] + (k\alpha - 1)\sum_{i=1}^{n} \log\left(\frac{x_i}{\beta}\right) - \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha} - n\log[C(\theta)]$$
$$+ \sum_{i=1}^{n} \log C'\left\{\theta\left[1 - \gamma_1\left(k, \left(\frac{x_i}{\beta}\right)^{\alpha}\right)\right]\right\}. \tag{13}$$

The log-likelihood can be maximized either directly by using the SAS (PROC NLMIXED) or the Ox program (subroutine MaxBFGS) or by solving the nonlinear likelihood equations obtained by differentiating (13). The score components corresponding to the parameters in $\boldsymbol{\eta}$ are given by

$$\frac{\partial \ell_n}{\partial \theta} = \frac{n}{\theta} - n\frac{C'(\theta)}{C(\theta)} + \sum_{i=1}^{n} [1-\gamma_1(k,u_i)] \frac{C''\{\theta[1-\gamma_1(k,u_i)]\}}{C'\{\theta[1-\gamma_1(k,u_i)]\}},$$

$$\frac{\partial \ell_n}{\partial k} = -n\psi(k) + \sum_{i=1}^{n}\log(u_i) - \theta\sum_{i=1}^{n}\frac{C''\{\theta[1-\gamma_1(k,u_i)]\}}{C'\{\theta[1-\gamma_1(k,u_i)]\}}\left[\dot{\gamma}_1(k,u_i) - \psi(k)\gamma_1(k,u_i)\right],$$

$$\frac{\partial \ell_n}{\partial \alpha} = \frac{n}{\alpha} + \frac{k}{\alpha}\sum_{i=1}^{n}\log(u_i) - \frac{1}{\alpha\beta^{\alpha}}\sum_{i=1}^{n}x_i^{\alpha}\log(u_i) - \frac{\theta}{\alpha\Gamma(k)}\sum_{i=1}^{n}\frac{C''\{\theta[1-\gamma_1(k,u_i)]\}}{C'\{\theta[1-\gamma_1(k,u_i)]\}}\frac{u_i^k\exp(-u_i)\log(u_i)}{\gamma_i(k,u_i)}$$

and

$$\frac{\partial \ell_n}{\partial \beta} = -\frac{n}{\beta} - n\frac{(k\alpha-1)}{\beta} + \frac{\alpha}{\beta^{\alpha+1}}\sum_{i=1}^{n}x_i^{\alpha} - \frac{\theta}{\alpha}\sum_{i=1}^{n}\frac{C''\{\theta[1-\gamma_1(k,u_i)]\}}{C'\{\theta[1-\gamma_1(k,u_i)]\}}\frac{u_i^k\exp(-u_i)}{\gamma_i(k,u_i)},$$

where $u_i = (x_i/\beta)^{\alpha}$, $\psi(\cdot)$ is the digamma function and $\dot{\gamma}_1(k,u_i) = \Gamma(k)^{-1}\int_0^{u_i}\omega^{k-1}\mathrm{e}^{-\omega}\log(\omega)\mathrm{d}\omega$. The maximum likelihood estimator (MLE) $\widehat{\boldsymbol{\eta}} = (\widehat{\theta}, \widehat{k}, \widehat{\alpha}, \widehat{\beta})$ can be obtained by solving simultaneously the nonlinear equations

$$\frac{\partial \ell_n}{\partial \theta} = \frac{\partial \ell_n}{\partial k} = \frac{\partial \ell_n}{\partial \alpha} = \frac{\partial \ell_n}{\partial \beta} = 0.$$

We can provide interval estimation and hypothesis tests for the model parameters based on the normal approximation for $\widehat{\boldsymbol{\eta}}$. Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\widehat{\boldsymbol{\eta}} - \boldsymbol{\eta})$ is multivariate normal $N_4(0, I^{-1}(\boldsymbol{\eta}))$, where $I(\boldsymbol{\eta}) = \lim_{n\to\infty} n^{-1}J_n(\boldsymbol{\eta})$ and $J_n(\boldsymbol{\eta})$ is the total observed information matrix, whose elements are listed in the Appendix. We obtain $100(1-\gamma)\%$ $(0 < \gamma < 1/2)$ asymptotic confidence interval for the ith parameter ϑ_i in $\boldsymbol{\eta}$ by

$$ACI_i = \left(\widehat{\vartheta}_i - z_{1-\gamma/2}\sqrt{\widehat{J}^{\vartheta_i,\vartheta_i}},\ \hat{\vartheta}_i + z_{1-\gamma/2}\sqrt{\widehat{J}^{\vartheta_i,\vartheta_i}}\right),$$

where $\widehat{J}^{\vartheta_i,\vartheta_i}$ stands for the ith diagonal element of the inverse of the observed information matrix estimated at $\widehat{\boldsymbol{\eta}}$, i.e., $J_n(\widehat{\boldsymbol{\eta}})^{-1}$, for $i = 1, \ldots, 4$, and $z_{1-\gamma/2}$ is the $1-\gamma/2$ standard normal quantile.

Expectation–Maximization (EM) [37] is another technique for estimating $\boldsymbol{\eta}$. It is a recurrent method such that each step consists of an estimate of the expected value of a hypothetical random variable and then maximizes the log-likelihood for the complete data. Consider the complete-data $X_1, \ldots, X_n$ with observed values $x_1, \ldots, x_n$ and the hypothetical random variables $Z_1, \ldots, Z_n$. The joint density function is such that the marginal density of $X_1, \ldots, X_n$ is the likelihood of interest. Then, we define a hypothetical complete-data distribution for each pair $(X_i, Z_i)^{\top}$ $(i = 1, \ldots, n)$ having the joint density function

$$g(x,z;\boldsymbol{\eta}) = \frac{\alpha\, z\, a_z\, \theta^z}{C(\theta)}\left(\frac{x}{\beta}\right)^{k\alpha-1}\exp\left\{-\left(\frac{x}{\beta}\right)^{\alpha}\right\}\left\{1-\gamma_1\left[k,\left(\frac{x}{\beta}\right)^{\alpha}\right]\right\}^{z-1},$$

where $\theta \in (0,1)$, $k, \alpha, \beta > 0$, $x > 0$ and $z \in \mathbb{N}$. Under this formulation, the E-step of an EM cycle requires the expectation of $Z|X; \boldsymbol{\eta}^{(r)}$, where $\boldsymbol{\eta}^{(r)} = (\theta^{(r)}, k^{(r)}, \alpha^{(r)}, \beta^{(r)})^{\top}$ is the current estimate (in the rth iteration) of $\boldsymbol{\eta}$. The pdf of Z given X, say $g(z|x)$, is given by

$$g(z|x;\boldsymbol{\eta}) = z\, a_z\, \theta^{z-1}\left\{1-\gamma_1\left[k,\left(\frac{x}{\beta}\right)^{\alpha}\right]\right\}^{z-1} C'\left\{\theta\left[1-\gamma_1\left(k,\left(\frac{x}{\beta}\right)^{\alpha}\right)\right]\right\}^{-1}$$

and its expected value is

$$\mathbb{E}(Z|X) = 1 + \theta\left[1-\gamma_1\left(k,\left(\frac{x}{\beta}\right)^{\alpha}\right)\right]\frac{C''\{\theta[1-\gamma_1(k,(x/\beta)^{\alpha})]\}}{C'\{\theta[1-\gamma_1(k,(x/\beta)^{\alpha})]\}}.$$

The EM cycle is completed with the M-step by using the maximum likelihood estimation over $\boldsymbol{\eta}$, with the missing Z's replaced by their conditional expectations given before. The log-likelihood for the complete data is given by

$$\ell_n^* = \ell_n^*(x_1,\ldots,x_n;z_1,\ldots,z_n;\boldsymbol{\eta}) \propto n\log\alpha - n\log\beta - \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha} + (k\alpha-1)\sum_{i=1}^{n}\log\left(\frac{x_i}{\beta}\right)$$
$$+ \log(\theta)\sum_{i=1}^{n}z_i + \sum_{i=1}^{n}(z_i-1)\log\left[1-\gamma_1\left(k,\left(\frac{x_i}{\beta}\right)^{\alpha}\right)\right] - n\log C(\theta).$$

So, the components of the score function are

$$\frac{\partial \ell_n^*}{\partial \theta} = \frac{1}{\theta}\sum_{i=1}^n z_i - n\frac{C'(\theta)}{C(\theta)}, \qquad \frac{\partial \ell_n^*}{\partial k} = \sum_{i=1}^n \log u_i - \sum_{i=1}^n (z_i - 1)\frac{[\dot{\gamma}_1(k,u_i) - \psi(k)\gamma_1(k,u_i)]}{1-\gamma_1(k,u_i)},$$

$$\frac{\partial \ell_n^*}{\partial \alpha} = \frac{n}{\alpha} - \frac{1}{\alpha}\sum_{i=1}^n u_i \log u_i + \frac{k}{\alpha}\sum_{i=1}^n \log u_i - \frac{1}{\alpha}\sum_{i=1}^n \frac{u_i^{1/\alpha}\log(u_i)\exp(-u_i)}{1-\gamma_1(k,u_i)}$$

and

$$\frac{\partial \ell_n^*}{\partial \beta} = -\frac{n\,k\,\alpha}{\beta} + \frac{\alpha}{\beta}\sum_{i=1}^n u_i + \frac{\alpha}{\beta}\sum_{i=1}^n (z_i - 1)\frac{u_i^k \exp(-u_i)}{1-\gamma_1(k,u_i)},$$

where $\dot{\gamma}_1(\cdot,\cdot)$ is defined as before. From a nonlinear system of equations $\partial\ell_n^*/\partial\theta = \partial\ell_n^*/\partial k = \partial\ell_n^*/\partial\alpha = \partial\ell_n^*/\partial\beta = 0$, it follows the iterative procedure of the EM algorithm

$$\hat{\theta}^{(t+1)} = \frac{C(\hat{\theta}^{(t+1)})}{nC'(\hat{\theta}^{(t+1)})}\sum_{i=1}^n z_i^{(t)}, \qquad \sum_{i=1}^n \log u_i - \sum_{i=1}^n \left(z_i^{(t)} - 1\right)\frac{\left[\dot{\gamma}_1(\hat{k}^{(t+1)},u_i) - \psi(\hat{k}^{(t+1)})\gamma_1(\hat{k}^{(t+1)},u_i)\right]}{1-\gamma_1(\hat{k}^{(t+1)},u_i)} = 0,$$

$$\frac{n}{\hat{\alpha}^{(t+1)}} - \frac{1}{\hat{\alpha}^{(t+1)}}\sum_{i=1}^n u_i\log u_i + \frac{\hat{k}^{(t)}}{\hat{\alpha}^{(t+1)}}\sum_{i=1}^n \log u_i - \frac{1}{\hat{\alpha}^{(t+1)}}\sum_{i=1}^n \frac{u_i^{1/\hat{\alpha}^{(t+1)}}\log(u_i)\exp(-u_i)}{1-\gamma_1(\hat{k}^{(t)},u_i)} = 0$$

and

$$-\frac{n\,\hat{k}^{(t)}\,\hat{\alpha}^{(t)}}{\hat{\beta}^{(t+1)}} + \frac{\hat{\alpha}^{(t)}}{\hat{\beta}^{(t+1)}}\sum_{i=1}^n u_i + \frac{\hat{\alpha}^{(t)}}{\hat{\beta}^{(t+1)}}\sum_{i=1}^n \left(z_i^{(t)} - 1\right)\frac{u_i^{\hat{k}^{(t)}}\exp(-u_i)}{1-\gamma_1(\hat{k}^{(t)},u_i)} = 0,$$

where $\hat{\theta}^{(t+1)}$, $\hat{k}^{(t+1)}$, $\hat{\alpha}^{(t+1)}$ and $\hat{\beta}^{(t+1)}$ are determined numerically. Here, for $i = 1,\ldots,n$, we have

$$z_i^{(t)} = 1 + \theta^{(t)}\left[1 - \gamma_1\left(k^{(t)}, \left(\frac{x_i}{\beta^{(t)}}\right)^{\alpha^{(t)}}\right)\right]\frac{C''\left\{\theta^{(t)}\left[1-\gamma_1\left(k^{(t)},\left(x_i/\beta^{(t)}\right)^{\alpha^{(t)}}\right)\right]\right\}}{C'\left\{\theta^{(t)}\left[1-\gamma_1\left(k^{(t)},\left(x_i/\beta^{(t)}\right)^{\alpha^{(t)}}\right)\right]\right\}}.$$

For each step, θ, k, α and β are estimated independently.

5. Special models

In this section, we study, in detail, three special models of the $\mathcal{GGPS}$ family. We provide plots of the density and hazard rate functions for selected parameter values to illustrate the flexibility of these distributions. We offer explicit expressions for the moments.

5.1. The generalized gamma geometric ($\mathcal{GGG}$) distribution

The $\mathcal{GGG}$ pdf is defined by the cdf (4) with $C(\theta) = \theta(1-\theta)^{-1}$ leading to

$$f_{\mathcal{GGG}}(x;\theta,k,\alpha,\beta) = (1-\theta)f_{\mathcal{GG}}(x)\left\{1-\theta\left[1-\gamma_1\left(k,\left(\frac{x}{\beta}\right)^{\alpha}\right)\right]\right\}^{-2}, \quad x>0. \tag{14}$$

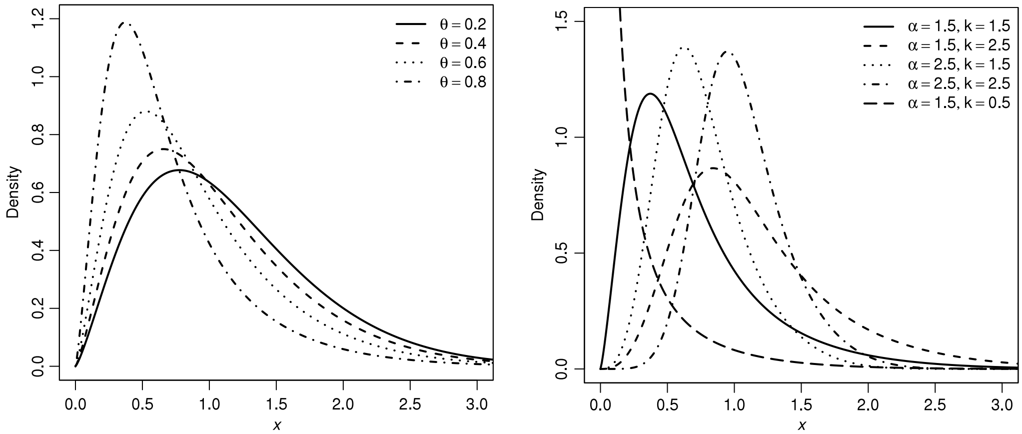
Even when $\theta \le 0$, Eq. (14) is a density function. We can then define the $\mathcal{GGG}$ distribution by (14) for any $\theta < 1$. The study of the new distribution seems important since it extends some distributions previously considered in the literature. In fact, the $\mathcal{GGG}$ distribution is obtained by taking $\theta = 0$ [1]. For $\theta \to 1^-$, the $\mathcal{GGG}$ distribution tends to a distribution degenerate in zero. Hence, the parameter θ can be interpreted as a concentration parameter. Eq. (14) extends several distributions which have been studied in the literature. The $\mathcal{WG}$ distribution is obtained by taking $k = 1$. Further, for $\beta = k = 1$, we obtain the $\mathcal{EG}$ distribution. Fig. 1 illustrates its pdf shapes for some parameter values.

Proposition 4. *The distribution of the form (14) is geometric extreme stable.*

Proof. See [12, p. 648].

The corresponding cumulative distribution and hazard rate functions (for $x > 0$) are

$$F_{\mathcal{GGG}}(x;\theta,k,\alpha,\beta) = \frac{\gamma_1\left(k,(x/\beta)^{\alpha}\right)}{1-\theta\left[1-\gamma_1\left(k,(x/\beta)^{\alpha}\right)\right]}, \tag{15}$$



(a) $\alpha = k = 1.5$ and $\beta = 1.0$.

(b) $\theta = 0.8$ and $\beta = 1.0$.

Fig. 1. Plots of the $\mathcal{GGG}$ density function for some parameter values.

and

$$h_{\mathcal{GGG}}(x;\theta,k,\alpha,\beta) = \frac{h_{\mathcal{GG}}(x)}{1-\theta\,[1-\gamma_1\,(k,(x/\beta)^{\alpha})]}, \tag{16}$$

respectively, where $h_{\mathcal{GG}}(x)$ is the hrf of the $\mathcal{GG}$ distribution.

From (16), note that $h_{\mathcal{GGP}}(x)/h_{\mathcal{GG}}(x)$ is increasing in x for $\theta \le 0$ and decreasing in x for $0 \le \theta < 1$. Additionally, we have (for $\theta \le 0$)

$$\frac{h_{\mathcal{GG}}(x)}{1-\theta} \le h_{\mathcal{GGG}}(x) \le h_{\mathcal{GG}}(x), \qquad S_{\mathcal{GG}}(x) \le S_{\mathcal{GGG}}(x) \le S_{\mathcal{GG}}(x)^{(1-\theta)^{-1}}$$

and

$$h_{\mathcal{GG}}(x) \le h_{\mathcal{GGG}}(x) \le \frac{h_{\mathcal{GG}}(x)}{1-\theta}, \qquad S_{\mathcal{GG}}(x)^{(1-\theta)^{-1}} \le S_{\mathcal{GGG}}(x) \le S_{\mathcal{GG}}(x),$$

for $0 \le \theta < 1$, where $S_{\mathcal{GG}}(x)$ and $S_{\mathcal{GGG}}(x)$ are the survival functions of the $\mathcal{GG}$ and $\mathcal{GGG}$ models, respectively.

Fig. 2 displays plots of the $\mathcal{GGG}$ hrf for some parameter values. This hrf can have an upside-down bathtub, bathtub-shaped, constant ($\alpha = k = 1$), decreasing or increasing depending on the parameter values.

Proposition 5. *Let $X \sim \mathcal{GGG}(\theta, k, \alpha, \beta)$. Then:*

(i) *The cdf of the nth order statistic corresponding to the pdf (14) is given by*

$$F_n(x) = F_{\mathcal{GGG}}(x;\theta,k,\alpha,\beta)^n = \left\{\frac{\gamma(k,(t/\beta)^{\alpha})/\Gamma(k)}{1-\theta\,[1-\gamma(k,(t/\beta)^{\alpha})/\Gamma(k)]}\right\}^n. \tag{17}$$

(ii) *The density function of the nth order statistic is given by*

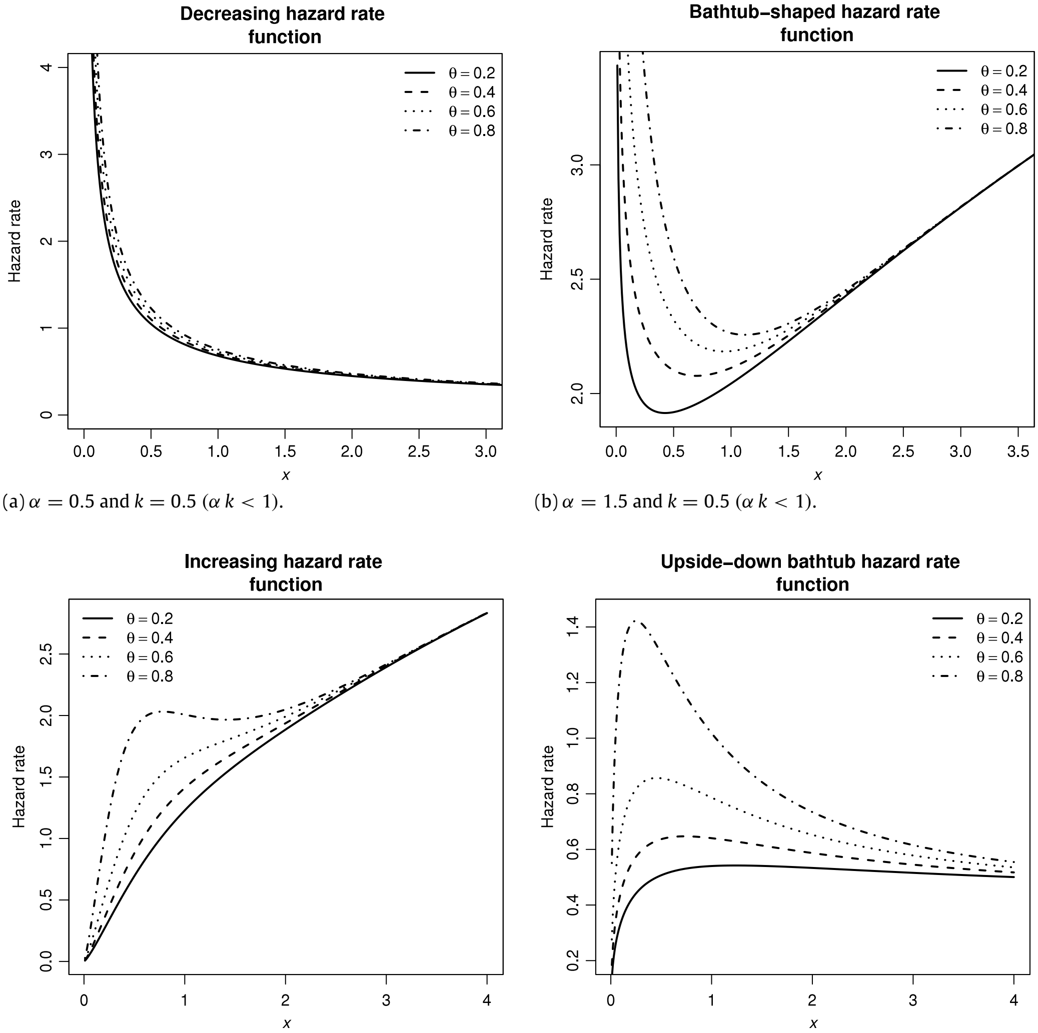
$$f_n(x) = \frac{n(1-\theta)f_{\mathcal{GG}}(x;k,\alpha,\beta)[F_{\mathcal{GG}}(x;k,\alpha,\beta)]^{n-1}}{\{1-\theta[1-F_{\mathcal{GG}}(x;k,\alpha,\beta)]\}^{n+1}}. \tag{18}$$

Proof. The proof of Proposition 5 is omitted here since it is a straightforward consequence of a result obtained by Bidram and Alavi [38].

According to Bidram and Alavi [38], Proposition 5 provides two important motivations. First, the cdf of the nth order statistic can be easily obtained using the cdf (15). Second, Eq. (17) reveals that there exists a relationship between the cdf of the nth order statistic and the $\mathcal{GGG}$ distribution.

Suppose that $F_{\mathcal{GG}}(x)$ is a heavy-tailed distribution [39]. Then, $S_{\mathcal{GG}}(x) = 1 - F_{\mathcal{GG}}(x)$ verifies that $S_{\mathcal{GG}}(x) > 0$ (for $x \ge 0$) and, also for any $y \ge 0$, we obtain

$$\begin{aligned}
\lim_{x\to\infty}\frac{S_{\mathcal{GGG}}(x+y)}{S_{\mathcal{GGG}}(x)} &= \lim_{x\to\infty}\frac{S_{\mathcal{GG}}(x+y)}{S_{\mathcal{GG}}(x)}\cdot\frac{1-\theta\,S_{\mathcal{GG}}(x)}{1-\theta\,S_{\mathcal{GG}}(x+y)} \\
&= \lim_{x\to\infty}\frac{1-\gamma_1\,(k,((x+y)/\beta)^{\alpha})}{1-\gamma_1\,(k,(x/\beta)^{\alpha})}\cdot\frac{1-\theta\,[1-\gamma_1\,(k,(x/\beta)^{\alpha})]}{1-\theta\left[1-\gamma_1\,(k,((x+y)/\beta)^{\alpha})\right]} = 1,
\end{aligned}$$



(a) $\alpha = 0.5$ and $k = 0.5$ ($\alpha k < 1$).
(b) $\alpha = 1.5$ and $k = 0.5$ ($\alpha k < 1$).
(c) $\alpha = 1.5$ and $k = 1.5$ ($\alpha k > 1$).
(d) $\alpha = 0.8$ and $k = 1.8$ ($\alpha k > 1$).

Fig. 2. The $\mathcal{GGG}$ hrf for some parameter values; $\beta = 1$.

where we use the fact that $\lim_{u\to\infty} 1 - \gamma_1(k, (u/\beta)^\alpha) = 0$ and $F_{\mathcal{GG}}(x)$ is heavy-tailed, i.e., $\lim_{x\to\infty} \frac{1-\gamma_1\left(k,((x+y)/\beta)^\alpha\right)}{1-\gamma_1(k,(x/\beta)^\alpha)} = 1$. Then, $F_{\mathcal{GGG}}(x)$ is also heavy-tailed.

The rth raw moment of the random variable X having the $\mathcal{GGG}$ distribution is given by

$$\mathbb{E}(X^r) = \frac{\alpha\,(1-\theta)}{\beta\,\Gamma(k)} \int_0^\infty x^r \left(\frac{x}{\beta}\right)^{k\alpha-1} \exp\left\{-\left(\frac{x}{\beta}\right)^\alpha\right\} \left\{1-\theta\left[1-\gamma_1\left(k,\left(\frac{x}{\beta}\right)^\alpha\right)\right]\right\}^{-2} \mathrm{d}x, \tag{19}$$

where the integral requires to be computed numerically. Table 4 lists the first five moments of X obtained from Eqs. (12) and (19) for some parameter values.

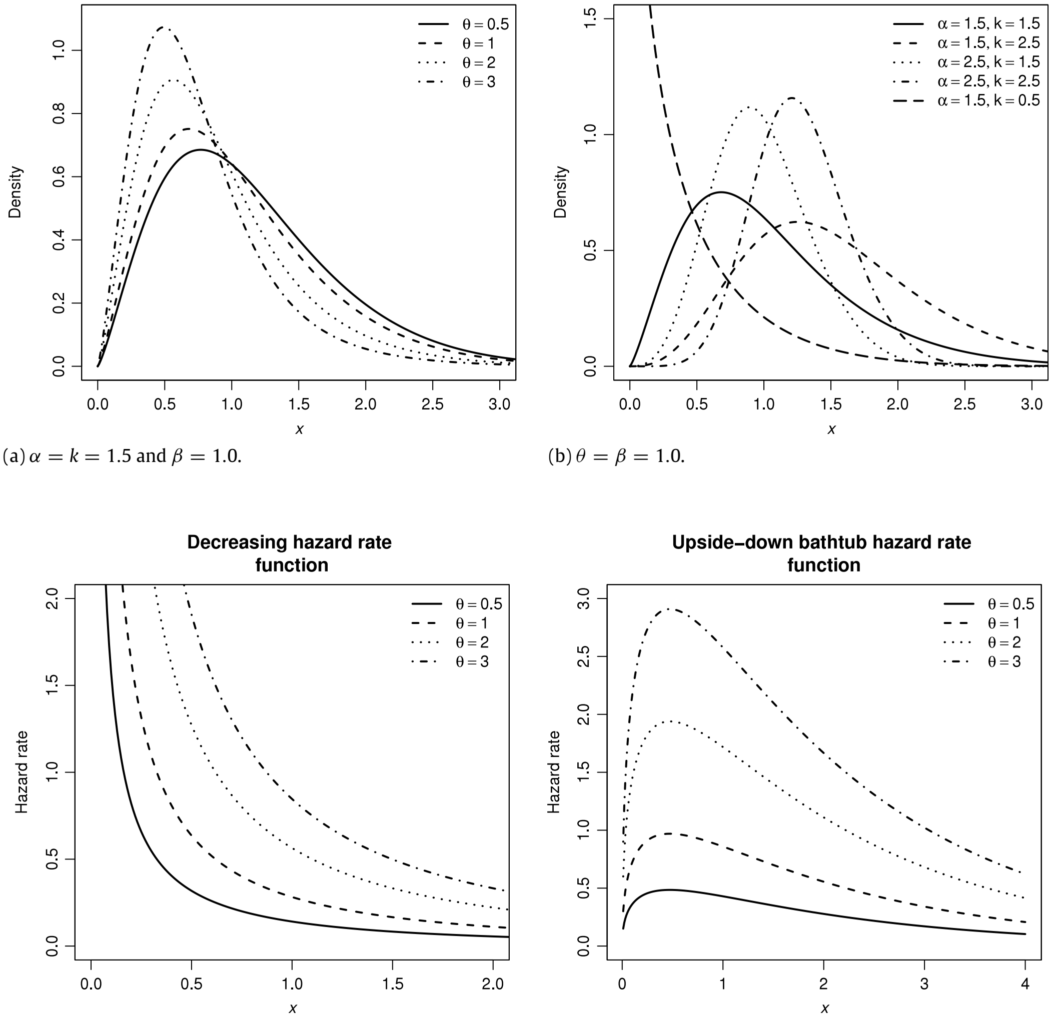
5.2. The generalized gamma Poisson ($\mathcal{GGP}$) distribution

The cdf of the $\mathcal{GGP}$ distribution is defined by (4) with $C(\theta) = \mathrm{e}^\theta - 1$ ($\theta > 0$) corresponding to the Poisson distribution. We obtain

$$F_{\mathcal{GGP}}(x;\theta,k,\alpha,\beta) = 1 - \frac{1}{\mathrm{e}^\theta - 1}\left\{\exp\left[\theta\left(1-\gamma_1\left(k,\left(\frac{x}{\beta}\right)^\alpha\right)\right)\right] - 1\right\}, \quad x > 0.$$

The associated density and hazard rate functions (for $x > 0$) are

$$f_{\mathcal{GGP}}(x;\theta,k,\alpha,\beta) = \frac{\theta}{\mathrm{e}^\theta - 1} f_{\mathcal{GG}}(x) \exp\left\{\theta\left[1-\gamma_1\left(k,\left(\frac{x}{\beta}\right)^\alpha\right)\right]\right\} \tag{20}$$



(a) $\alpha = k = 1.5$ and $\beta = 1.0$.

(b) $\theta = \beta = 1.0$.

(c) $\beta = 1$, $\alpha = 0.5$ and $k = 0.5$ ($\alpha\, k < 1$).

(d) $\beta = 1$, $\alpha = 0.8$ and $k = 1.8$ ($\alpha\, k > 1$).

Fig. 3. Plots of the $\mathcal{GGP}$ density and hrf for some parameter values.

and

$$h_{\mathcal{GGP}}(x;\theta,k,\alpha,\beta) = \frac{\theta\, f_{\mathcal{GG}}(x)\exp\left\{\theta\left[1-\gamma_1\left(k,\left(\frac{x}{\beta}\right)^{\alpha}\right)\right]\right\}}{\exp\left[\theta\left(1-\gamma_1\left(k,\left(\frac{x}{\beta}\right)^{\alpha}\right)\right)\right]-1},$$

respectively.

The $\mathcal{WP}$ [25] distribution is obtained when $\alpha = k = 1$ in (20). In the same way, for $\alpha = k = 1$, we have the $\mathcal{EP}$ distribution [22]. In Fig. 3, we plot the density and hazard rate functions of the $\mathcal{GGP}$ distribution for some parameter values.

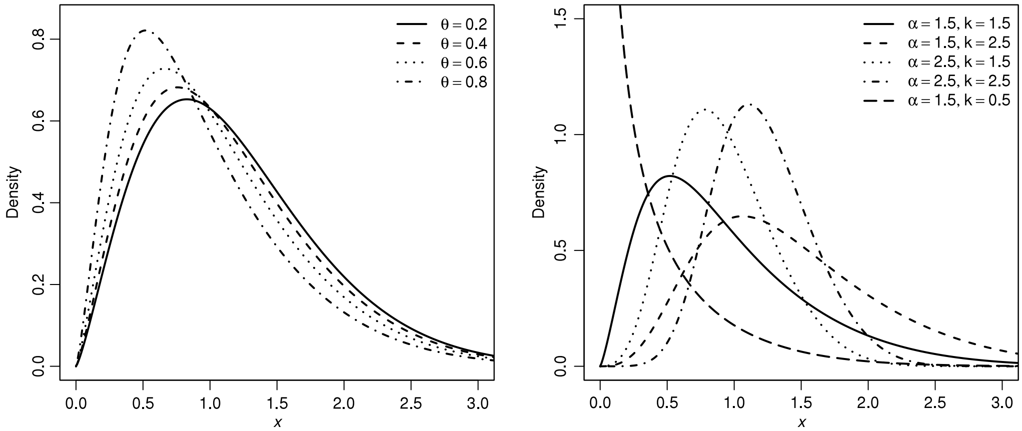
Suppose that $F_{\mathcal{GG}}(x)$ is a heavy-tailed distribution [39]. Then, $S_{\mathcal{GG}}(x)$ verifies that $S_{\mathcal{GG}}(x) > 0$, for $x \geq 0$, and also for any $y \geq 0$, we obtain

$$\lim_{x\to\infty}\frac{S_{\mathcal{GGP}}(x+y)}{S_{\mathcal{GGP}}(x)} = \lim_{x\to\infty}\frac{e^{S_{\mathcal{GG}}(x+y)}-1}{e^{S_{\mathcal{GG}}(x)}-1}.$$

Then, by using L'Hôpital's rule, we have

$$\lim_{x\to\infty}\frac{S_{\mathcal{GGP}}(x+y)}{S_{\mathcal{GGP}}(x)} = \lim_{x\to\infty}\frac{e^{S_{\mathcal{GG}}(x+y)}}{e^{S_{\mathcal{GG}}(x)}}\cdot\frac{f_{\mathcal{GG}}(x+y)}{f_{\mathcal{GG}}(x)} = 1,$$

based on the fact that $F_{\mathcal{GG}}(x)$ is heavy-tailed. Then, $F_{\mathcal{GGP}}(x)$ is also heavy-tailed.



(a) $\alpha = k = 1.5$ and $\beta = 1.0$. (b) $\theta = 0.8$ and $\beta = 1.0$.

Fig. 4. Plots of the $\mathcal{GGL}$ density function for some parameter values.

The $\mathcal{GGP}$ distribution can be obtained through the exponentiated-G (exp-G) class [40] or truncated-exponential skew-symmetric distribution [41], and it can be generalized by extending the parameter space with respect to θ, i.e., by taking $\theta \in \mathbb{R}$. In particular, if $\theta < 0$, N is a discrete random variable with zero-truncated Poisson distribution with parameter $-\theta$ and the sequence $\{X_i\}_{i=1}^N$ is defined as before. Then, the random variable $X = \max(X_1, \ldots, X_N)$ has pdf (20). By considering the $\mathcal{GGP}$ distribution with density (20) ($\theta \in \mathbb{R}$), its rth raw moment can be expressed as

$$\mu_r' = \frac{\alpha\,\theta\,\beta^{r-1}}{\Gamma(k)(e^{\theta}-1)} \int_0^{\infty} \left(\frac{x}{\beta}\right)^{k\alpha+r-1} \exp\left\{\theta\left[1-\gamma_1\left(k,\left(\frac{x}{\beta}\right)^{\alpha}\right)\right] - \left(\frac{x}{\beta}\right)^{\alpha}\right\} dx, \tag{21}$$

where the integral should be computed numerically. Table 5 lists its first five moments obtained from Eqs. (12) and (21) for some parameter values. The figures in this table indicate a good agreement between the two formulae.

5.3. *The generalized gamma logarithmic ($\mathcal{GGL}$) distribution*

The $\mathcal{GGL}$ distribution is defined by the pdf (5) with $C(\theta) = -\log(1-\theta)$ leading to

$$f_{\mathcal{GGL}}(x; \theta, k, \alpha, \beta) = -\frac{\theta}{\log(1-\theta)} f_{\mathcal{GG}}(x) \left\{1-\theta\left[1-\gamma_1\left(k,\left(\frac{x}{\beta}\right)^{\alpha}\right)\right]\right\}^{-1}, \tag{22}$$

where $\theta \in (0, 1)$. The Weibull logarithmic [14] is obtained by setting $k = 1$ in (22). For $\alpha = k = 1$, we have the exponential logarithmic [23] distribution. Fig. 4 displays $\mathcal{GGL}$ shapes for some parameter values.

The associated cdf and hrf (for $x > 0$) are given by

$$F_{\mathcal{GGL}}(x; \theta, k, \alpha, \beta) = 1 - \frac{\log\{1-\theta\,[1-\gamma_1(k, (x/\beta)^{\alpha})]\}}{\log(1-\theta)}$$

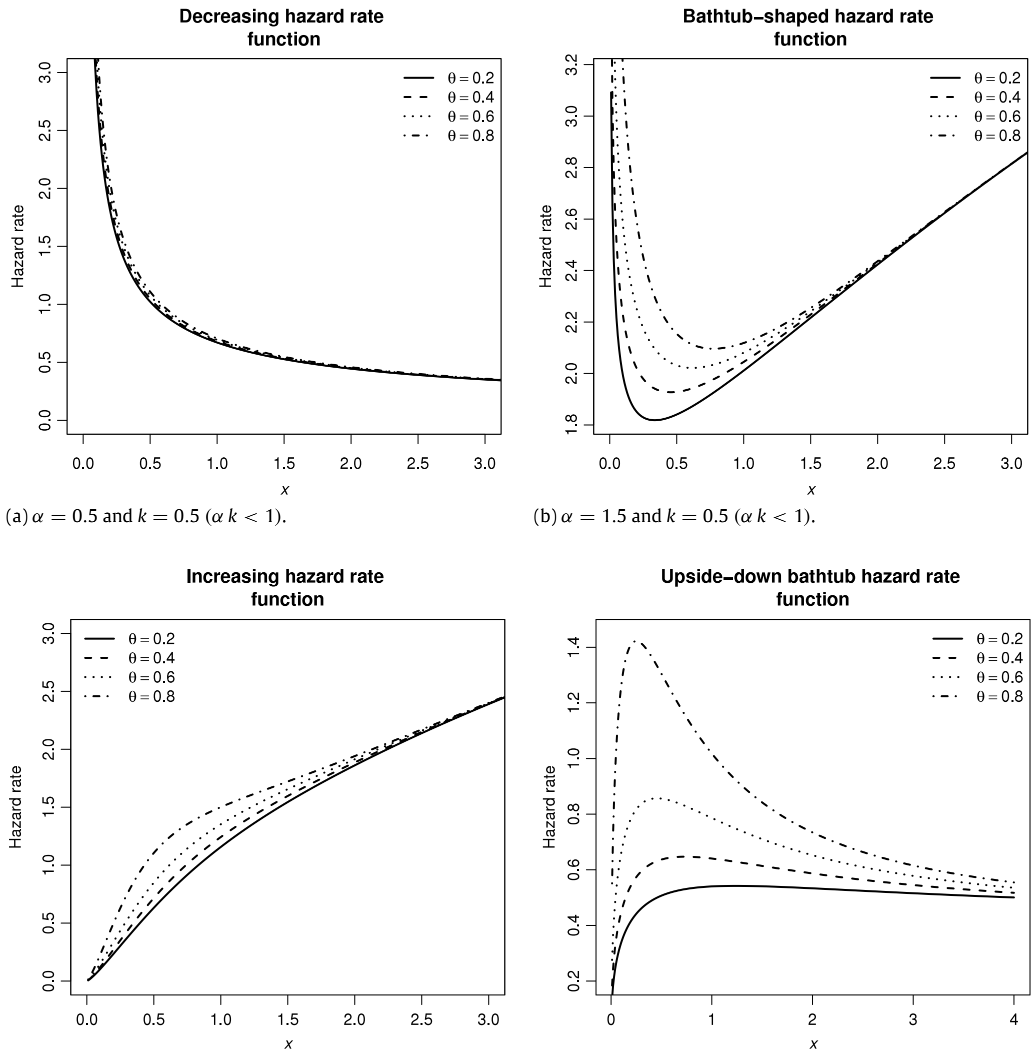
and

$$h_{\mathcal{GGL}}(x; \theta, k, \alpha, \beta) = -\frac{\theta f_{\mathcal{GG}}(x)}{\log\{1-\theta\,[1-\gamma_1(k, (x/\beta)^{\alpha})]\}\,\{1-\theta\,[1-\gamma_1(k, (x/\beta)^{\alpha})]\}},$$

respectively. Fig. 5 displays plots of the $\mathcal{GGL}$ hrf for some parameter values. We can verify that this distribution has an upside-down bathtub, bathtub-shaped, constant ($\alpha = k = 1$), decreasing or increasing hrf depending on the parameter values.

Suppose that $F_{\mathcal{GG}}(x)$ is a heavy-tailed distribution [39]. Then, $S_{\mathcal{GG}}(x)$ verifies that $S_{\mathcal{GG}}(x) > 0$, for $x \geq 0$, and also for any $y \geq 0$, we obtain

$$\lim_{x\to\infty} \frac{S_{\mathcal{GGL}}(x+y)}{S_{\mathcal{GGL}}(x)} = \lim_{x\to\infty} \frac{\log\{1-\theta S_{\mathcal{GG}}(x+y)\}}{\log\{1-\theta S_{\mathcal{GG}}(x)\}}.$$



(a) $\alpha = 0.5$ and $k = 0.5$ ($\alpha\, k < 1$).

(b) $\alpha = 1.5$ and $k = 0.5$ ($\alpha\, k < 1$).

(c) $\alpha = 1.5$ and $k = 1.5$ ($\alpha\, k > 1$).

(d) $\alpha = 0.8$ and $k = 1.8$ ($\alpha\, k > 1$).

Fig. 5. The $\mathcal{GGL}$ hrf for some parameter values; $\beta = 1$.

Then, by using L'Hôpital's rule, we have

$$\lim_{x\to\infty} \frac{S_{\mathcal{GGL}}(x+y)}{S_{\mathcal{GGL}}(x)} = \lim_{x\to\infty} \frac{1-\theta S_{\mathcal{GG}}(x)}{1-\theta S_{\mathcal{GG}}(x+y)} \cdot \frac{f_{\mathcal{GG}}(x+y)}{f_{\mathcal{GG}}(x)} = 1,$$

based on the fact that $F_{\mathcal{GG}}(x)$ is heavy-tailed. Then, $F_{\mathcal{GGL}}(x)$ is also heavy-tailed.

The rth raw moment of the $\mathcal{GGL}$ distribution is given by

$$\mu_r' = -\frac{\alpha\,\theta\,\beta^{r-1}}{\log(1-\theta)\Gamma(k)} \int_0^\infty \left(\frac{x}{\beta}\right)^{k\alpha+r-1} \exp\left\{-\left(\frac{x}{\beta}\right)^{\alpha}\right\} \left\{1-\theta\left[1-\gamma_1\left(k, \left(\frac{x}{\beta}\right)^{\alpha}\right)\right]\right\}^{-1} \mathrm{d}x. \tag{23}$$

This integral should be computed numerically. Table 6 lists the first five moments of X obtained from Eqs. (12) and (23) for some parameter values.

6. Empirical illustrations

Here, we present applications of the proposed family to two real data sets for illustrative purposes. The applications demonstrate the potentiality of the new family in modeling positive data. All the computations are performed using the `SAS (PROC NLMIXED)` programming language.

Table 7
Times to infection for AIDS.

0.25	0.75	0.75	0.75	1.00	1.00	1.00	1.00	1.00	1.25	1.25	1.25	1.25	1.50	1.50	1.50
1.50	1.50	1.50	1.75	1.75	1.75	1.75	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
2.00	2.00	2.25	2.25	2.25	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50	2.50
2.50	2.75	2.75	2.75	2.75	2.75	2.75	2.75	2.75	2.75	2.75	2.75	2.75	2.75	2.75	2.75
2.75	2.75	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00
3.25	3.25	3.25	3.25	3.25	3.25	3.25	3.25	3.25	3.25	3.25	3.50	3.50	3.50	3.50	3.50
3.50	3.50	3.50	3.50	3.50	3.50	3.50	3.50	3.50	3.50	3.50	3.75	3.75	3.75	3.75	3.75
3.75	3.75	3.75	3.75	3.75	3.75	3.75	3.75	3.75	3.75	3.75	3.75	4.00	4.00	4.00	4.00
4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.25	4.25	4.25	4.25	4.25	4.25	4.25	4.25	4.25
4.25	4.25	4.25	4.25	4.25	4.50	4.50	4.50	4.50	4.50	4.50	4.50	4.50	4.50	4.50	4.50
4.50	4.50	4.50	4.50	4.50	4.50	4.50	4.50	4.50	4.50	4.75	4.75	4.75	4.75	4.75	4.75
4.75	4.75	4.75	4.75	4.75	4.75	4.75	4.75	4.75	4.75	4.75	4.75	4.75	5.00	5.00	5.00
5.00	5.00	5.00	5.00	5.00	5.00	5.00	5.00	5.00	5.00	5.00	5.00	5.00	5.00	5.25	5.25
5.25	5.25	5.25	5.25	5.25	5.25	5.25	5.25	5.25	5.25	5.25	5.25	5.25	5.25	5.25	5.25
5.25	5.25	5.25	5.50	5.50	5.50	5.50	5.50	5.50	5.50	5.50	5.50	5.50	5.50	5.50	5.50
5.75	5.75	5.75	5.75	5.75	5.75	5.75	5.75	5.75	5.75	5.75	6.00	6.00	6.00	6.00	6.00
6.00	6.00	6.00	6.00	6.00	6.00	6.00	6.00	6.00	6.00	6.00	6.00	6.25	6.25	6.25	6.25
6.25	6.25	6.25	6.50	6.50	6.50	6.50	6.50	6.50	6.50	6.75	6.75	6.75	6.75	6.75	6.75
6.75	6.75	7.00	7.00	7.25	7.25										

Table 8
Otis IQ Scores.

91	102	100	117	122	115	97	109	108	104	108	118	103
123	123	103	106	102	118	100	103	107	108	107	97	95
119	102	108	103	102	112	99	116	114	102	111	104	122
103	111	101	91	99	121	97	109	106	102	104	107	95

Table 9
Descriptive statistics.

Statistics	Real data sets	
	Times to infection for AIDS	Otis IQ Scores
Mean	4.16	106.70
Median	4.25	105.00
Skewness	−0.25	0.36
Kurtosis	−0.66	−0.67
Minimum	0.25	91.00
Maximum	7.25	123.00

Since the expected information matrix is not available, the standard errors of the MLEs are given by square roots of the elements in the diagonal of the inverse observed information matrix evaluated at these estimates.

The first data set represents the times to infection for AIDS of a random sample of 295 patients [42]. The data are given in Table 7.

The second data set given in [43] and [44] refers to the Otis IQ Scores for 52 non-White males hired by a large insurance company in 1971. The data are listed in Table 8. Table 9 provides some basic statistics for both data sets.

In many applications there is a qualitative information about the failure rate shape, which can help in selecting a specified model. In this context, a device called the total time on test (TTT) plot [45] is useful. The TTT plot is drawn by plotting $T(i/n) = \left[\sum_{k=1}^{i} y_{k:n} + (n-i)y_{i:n}\right] / \sum_{k=1}^{n} y_{k:n}$ against i/n, where $i = 1, \ldots, n$ and $y_{k:n}(k = 1, \ldots, n)$ are the order statistics of the sample. It is a straight diagonal for constant failure rates, it is convex for decreasing failure rates and concave for increasing failure rates. It is first convex and then concave if the failure rate is bathtub-shaped. It is first concave and then convex if the failure rate is upside-down bathtub. The TTT plots for the current data sets are displayed in Fig. 6, which reveal increasing hrfs in both cases. Therefore, these plots indicate the appropriateness of the new family to fit these data, since its special models can present increasing, decreasing, bathtub and upside-down bathtub hrfs.

For the sake of comparison, we consider the $\mathcal{GG}$ distribution and two of its well-known extensions: the beta generalized gamma ($\mathcal{BGG}$) and exponentiated generalized gamma ($\mathcal{EGG}$) distributions. The $\mathcal{GG}$ distribution was briefly discussed in Section 1. The $\mathcal{BGG}$ and $\mathcal{EGG}$ densities (for $x > 0$) are given by

$$f_{\mathcal{EGG}}(x; k, \alpha, \lambda, \beta) = \frac{\lambda\beta}{\alpha\Gamma(k)} \left(\frac{x}{\alpha}\right)^{\beta k-1} \exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\} \left\{\gamma_1\left[k, \left(\frac{x}{\alpha}\right)^{\beta}\right]\right\}^{\lambda-1}$$

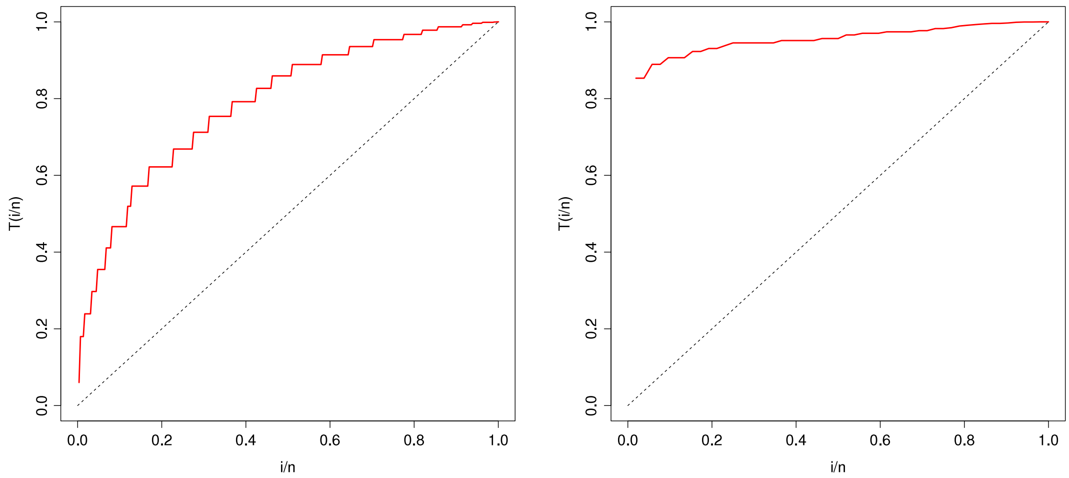


Fig. 6. TTT plots—times to infection for AIDS data (left panel); 52 Otis IQ Scores (right panel).

Table 10
MLEs and SEs (in parentheses) and the KS, W^* and A^* statistics for the AIDS infection data.

Distribution	Estimates					A^*	W^*	KS
$\mathcal{GGG}(\theta, \alpha, \beta, k)$	0.3527	10.7525	6.5696	0.2207		**0.4726**	**0.0772**	0.0464
	(0.2803)	(4.5514)	(0.3373)	(0.0814)		(0.2418)	(0.2241)	(0.5512)
$\mathcal{GGL}(\theta, \alpha, \beta, k)$	2.67×10^{-7}	7.7853	6.2394	0.2727		0.5035	0.0821	0.0471
	(0.0061)	(1.6905)	(0.2066)	(0.0748)		(0.2032)	(0.1936)	(0.5306)
$\mathcal{EGG}(\alpha, \beta, \lambda, k)$	6.3291	7.9821	2.4614	0.1079		0.5059	0.0825	0.0473
	(–)	(1.6105)	(–)	(–)		(0.2004)	(0.1916)	(0.5252)
$\mathcal{GG}(\alpha, \beta, k)$	7.7853	6.2394	0.2727			0.5035	0.0821	0.0471
	(1.6904)	(0.2066)	(0.0748)			(0.2032)	(0.1936)	(0.5306)
$\mathcal{BGG}(a, b, \lambda, c, \beta)$	0.0636	3.3855	0.2631	3.5215	10.0872	0.4741	0.0776	**0.0462**
	(0.0202)	(0.5660)	(0.0367)	(0.8204)	(–)	(0.2398)	(0.2221)	(0.5569)

Table 11
MLEs and SEs (in parentheses) and the KS, W^* and A^* statistics for the Otis IQ Score data.

Distribution	Estimates					A^*	W^*	KS
$\mathcal{GGG}(\theta, \alpha, \beta, k)$	0.7255	2.2793	23.9769	34.1847		0.4574	0.0723	**0.0930**
	(0.2781)	(0.1573)	(0.8003)	(1.2346)		(0.2548)	(0.2555)	(0.7395)
$\mathcal{GGL}(\theta, \alpha, \beta, k)$	0.9788	2.6661	31.2321	32.1675		**0.4247**	**0.0677**	0.1108
	(0.0349)	(0.0890)	(0.2590)	(1.566)		(0.3062)	(0.2940)	(0.5464)
$\mathcal{EGG}(\alpha, \beta, \lambda, k)$	24.9838	1.9854	8.4012	12.3869		0.4907	0.0785	0.1740
	(2.4704)	(0.0619)	(1.7231)	(–)		(0.2109)	(0.2121)	(0.0857)
$\mathcal{GG}(\alpha, \beta, k)$	2.5223	29.2167	26.5068			0.7635	0.1313	0.1863
	(2.4257)	(58.9362)	(50.9900)			(0.0441)	(0.0407)	(0.0541)
$\mathcal{BGG}(a, b, \lambda, c, \beta)$	20.1913	0.2099	0.0121	7.5063	0.8604	0.4706	0.0762	0.0947
	(–)	(0.0552)	(0.0001)	(0.2532)	(–)	(0.2366)	(0.2272)	(0.7397)

and

$$f_{\mathcal{BGG}}(x; a, b, \lambda, c, \beta) = \frac{c\lambda^{c\beta} x^{c\beta-1} \exp\{-(\lambda x)^c\} \, \gamma(\beta, (\lambda x)^c) \, \{\Gamma(\beta) - \gamma(\beta, (\lambda x)^c)\}^{b-1}}{B(a, b)\Gamma(\beta)^{a+b-1}}.$$

The $\mathcal{GG}$ distribution is obtained from the $\mathcal{EGG}$ model for $\lambda = 1$. Further, the $\mathcal{GG}$ distribution is a special case of the $\mathcal{BGG}$ model when $a = b = 1$. As stated by Proposition 1, the $\mathcal{GGPS}$ family has the $\mathcal{GG}$ distribution as an asymptotic case when $\theta \to 0$.

Table 10 lists the MLEs and the corresponding standard errors (SEs) (in parentheses) of the unknown parameters for the lifetime models fitted to the AIDS data, whereas Table 11 gives those values for the distributions fitted to the Otis IQ Score data. Further, Tables 10 and 11 provide the Cramér–von Mises (W^*), Anderson–Darling (A^*) and Kolmogorov–Smirnov (KS) statistics, which are formal goodness-of-fit tests to verify which distribution fits better to these data. These first two statistics

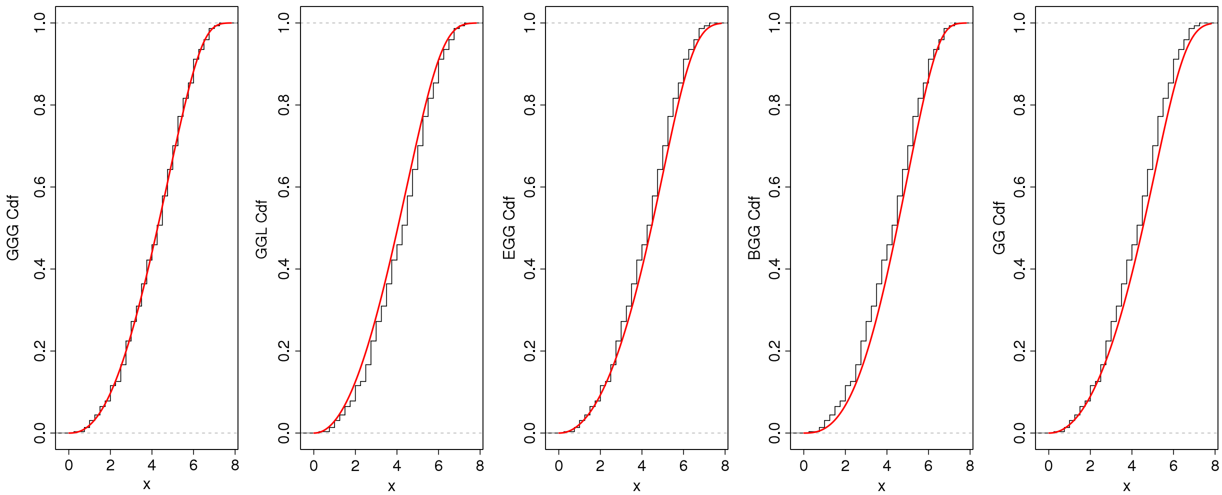


Fig. 7. Estimated cdfs for the $\mathcal{GGG}$, $\mathcal{GGL}$, $\mathcal{EGG}$, $\mathcal{BGG}$ and $\mathcal{GG}$ models for the AIDS data.

are described by Chen and Balakrishnan [46] and, in general, the smaller the values of them, the better the fit to the data. Further, let f and g (with same support Ω) be two absolutely continuous density functions with distribution functions $F(\cdot)$ and $G(\cdot)$, respectively. The KS statistic is defined by $KS(F, G) = \sup_{x\in\Omega} |F(x) - G(x)|$.

Roughly speaking, we conclude that all competing distributions can be used for modeling the times to infection for AIDS data. However, the W^*, A^* and KS statistics indicate that the $\mathcal{GGG}$ and $\mathcal{BGG}$ distributions provide better fits. It implies that any of them (that is, the $\mathcal{GGG}$ model or $\mathcal{BGG}$ model) could be chosen as the best distribution for modeling the first data set. However, the $\mathcal{GGG}$ distribution is the best choice because it has four parameters, whereas the $\mathcal{BGG}$ distribution has five parameters. In Fig. 8, we provide quantile–quantile (Q–Q) plots with confidence intervals to check graphically if the set of observations follows a particular competing model. The Q–Q plots also incorporate average lines that sketches a line passing through the two confidence bounds. The differences between the graphs are quite subtle. However, note that if the Q–Q plot points are really not mostly within the confidence interval, then the data do not come from the corresponding model. From a rapidly view of Fig. 8, we can confirm that all models in study are capable to fit the AIDS data set. This claim is supported by the inspection of the plots in Fig. 7, which display the estimated cumulative functions for the current data.

On the other hand, only the $\mathcal{GGL}$ and $\mathcal{GGG}$ distributions present the best fits for the Otis IQ Score data. In this case, the $\mathcal{GG}$ distribution provides the worst fit according to the W^*, A^* and KS statistics as we can verify in Figs. 9 and 10. In summary, the new four-parameter $\mathcal{GGPS}$ family may be an interesting alternative to well-known models available in the literature for modeling positive real data.

7. Concluding remarks

We define a new class of lifetime distributions called the *generalized gamma power series* ($\mathcal{GGPS}$) family, which generalizes the Weibull power series [14] and exponential power series [13] classes. We provide motivation and a mathematical treatment of the new family including expansions for the density function, ordinary and incomplete moments and generating function. The $\mathcal{GGPS}$ density function can be expressed as a mixture of exponentiated generalized gamma ($\mathcal{EGG}$) density functions, which is important to derive several of its properties. Maximum likelihood inference is implemented straightforwardly for estimating the model parameters. We fit some special $\mathcal{GGPS}$ distributions to two real data sets to illustrate the usefulness of these distributions. In conclusion, we define a general approach for generating new lifetime distributions, at least 76 distributions, some of them known and the great majority new ones. Further, we motivate the use of the new family in four different ways. We think these two facts combined may attract more complex applications in the literature of lifetime distributions. Note that the formulae derived are manageable by using modern computer resources with analytic and numerical capabilities. Finally, we remark that the $\mathcal{CGGPS}$ distributions are a very interesting family of distributions which are suitable for complementary risk problems as commented along the paper. However, the study of mathematical properties and the implementation of empirical examples for the $\mathcal{CGGPS}$ family would greatly increase the number of article pages. So, we let the $\mathcal{CGGPS}$ family as a recommendation for a future research.

Acknowledgments

We thank the editor and the reviewer again for the constructive comments.

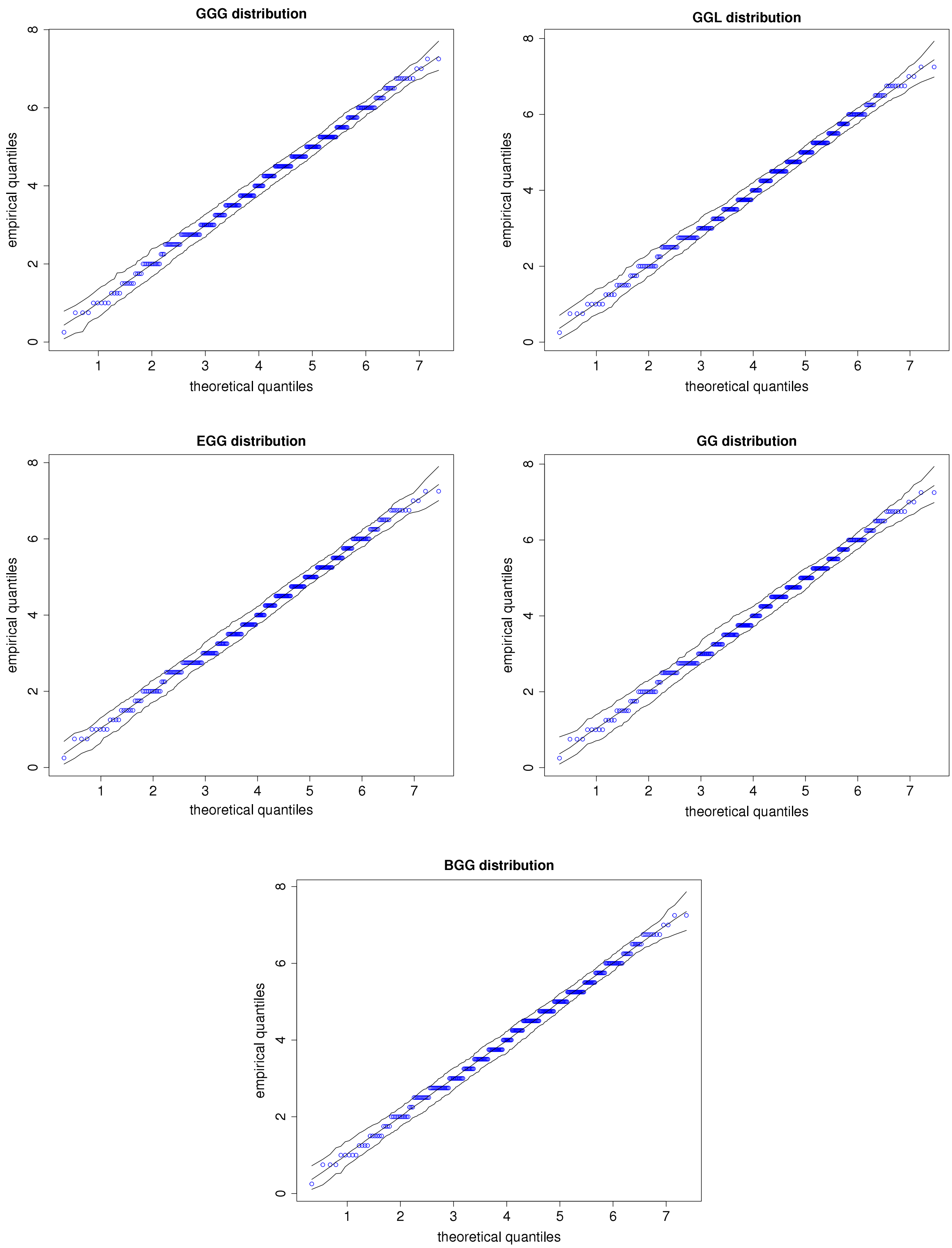


Fig. 8. Q–Q plots of the $\mathcal{GGG}$, $\mathcal{GGL}$, $\mathcal{EGG}$, $\mathcal{GG}$ and $\mathcal{BGG}$ distributions for AIDS data.

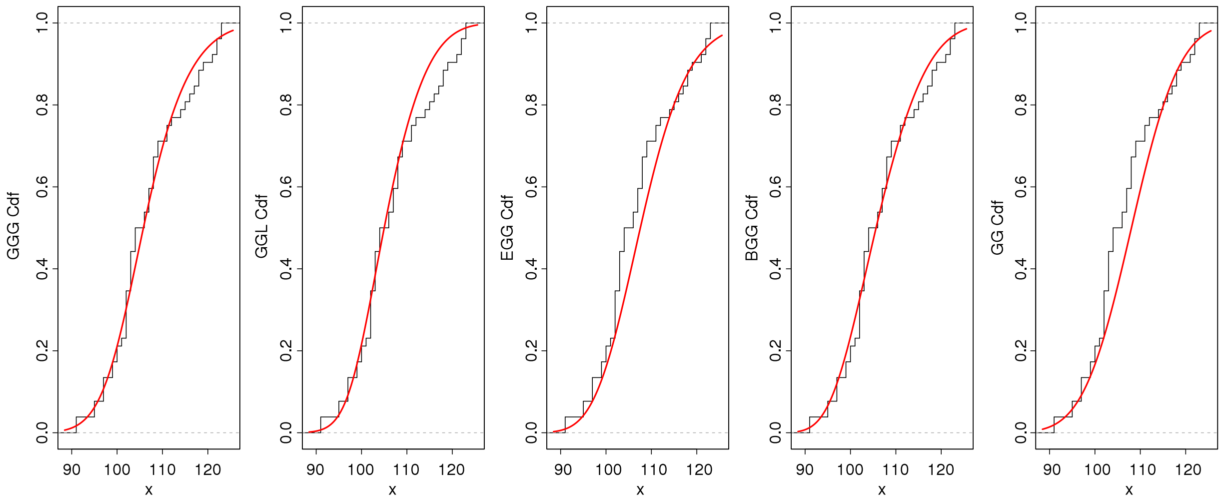


Fig. 9. Estimated cdfs for the $\mathcal{GGG}$, $\mathcal{GGL}$, $\mathcal{EGG}$, $\mathcal{BGG}$ and $\mathcal{GG}$ models for the 52 Otis IQ Scores.

Appendix

A.1. Proof of Proposition 1

For $x > 0$, we have

$$\lim_{\theta\to 0^+} F_{\mathcal{GGPS}}(x) = 1 - \lim_{\theta\to 0^+} \frac{\sum_{n=1}^{\infty} a_n \left[\theta\left(1-\gamma_1\left(k, \left(\frac{x}{\beta}\right)^{\alpha}\right)\right)\right]^n}{\sum_{n=1}^{\infty} a_n \theta^n}$$

$$= 1 - \lim_{\theta\to 0^+} \frac{1-\gamma_1\left(k, \left(\frac{x}{\beta}\right)^{\alpha}\right) + a_1^{-1}\sum_{n=2}^{\infty} a_n \theta^{n-1}\left(1-\gamma_1\left(k, \left(\frac{x}{\beta}\right)^{\alpha}\right)\right)^n}{1 + a_1^{-1}\sum_{n=2}^{\infty} a_n \theta^{n-1}} = \gamma_1\left(k, \left(\frac{x}{\beta}\right)^{\alpha}\right).$$

A.2. Proof of Theorem 1

(i) Let $c > 0$ and $Z = cX$. Then, $F_Z(z) = \text{P}(Z \le z) = \text{P}(cX \le z) = \text{P}(X \le z/c) = F_X(z/c; \theta, k, \alpha, \beta)$. Thus, $f_Z(z) = c^{-1} f_X(z/c; \theta, k, \alpha, \beta)$ is given by

$$f_Z(z) = c^{-1} f_X(z/c; \theta, k, \alpha, \beta) = \frac{\theta}{C(\theta)} f_{\mathcal{GG}}(z; k, \alpha, c\,\beta)\, C'\left\{\theta\left[1-\gamma_1\left(k, \left(\frac{z}{c\,\beta}\right)^{\alpha}\right)\right]\right\}$$
$$= f_{\mathcal{GGPS}}(z; \theta, k, \alpha, c\,\beta).$$

(ii) Let $Z = X^m$ for $m > 0$. Then, $F_Z(z; \theta, k, \alpha, \beta) = \text{P}(Z \le z) = \text{P}(X^m \le z) = \text{P}(X \le z^{1/m}) = F_X(z^{1/m}; \theta, k, \alpha, \beta)$. Thus, $f_Z(z) = m^{-1} z^{(1-m)/m} f_X(z^{1/m}; \theta, k, \alpha, \beta)$ is given by

$$f_Z(z) = z^{(1-m)/m} \frac{\theta}{m\,C(\theta)} f_{\mathcal{GG}}(z^{1/m}; k, \alpha, \beta)\, C'\left\{\theta\left[1-\gamma_1\left(k, \left(\frac{z}{\beta^m}\right)^{\alpha/m}\right)\right]\right\}$$
$$= f_{\mathcal{GGPS}}(z; \theta, k, \alpha/m, \beta^m).$$

(iii)–(iv) The proofs are similar to the proofs of (i) and (ii) and therefore are omitted.

A.3. Elements of the observed information matrix

The observed information matrix $J(\boldsymbol{\eta})$ for the parameters θ, α, β and k is given by

$$J(\boldsymbol{\eta}) = -\frac{\partial^2 \ell(\boldsymbol{\eta})}{\partial\boldsymbol{\eta}\partial\boldsymbol{\eta}^\top} = \begin{pmatrix} U_{\theta\theta} & U_{\theta\alpha} & U_{\theta\beta} & U_{\theta k} \\ \cdot & U_{\alpha\alpha} & U_{\alpha\beta} & U_{\alpha k} \\ \cdot & \cdot & U_{\beta\beta} & U_{\beta k} \\ \cdot & \cdot & \cdot & U_{kk} \end{pmatrix},$$

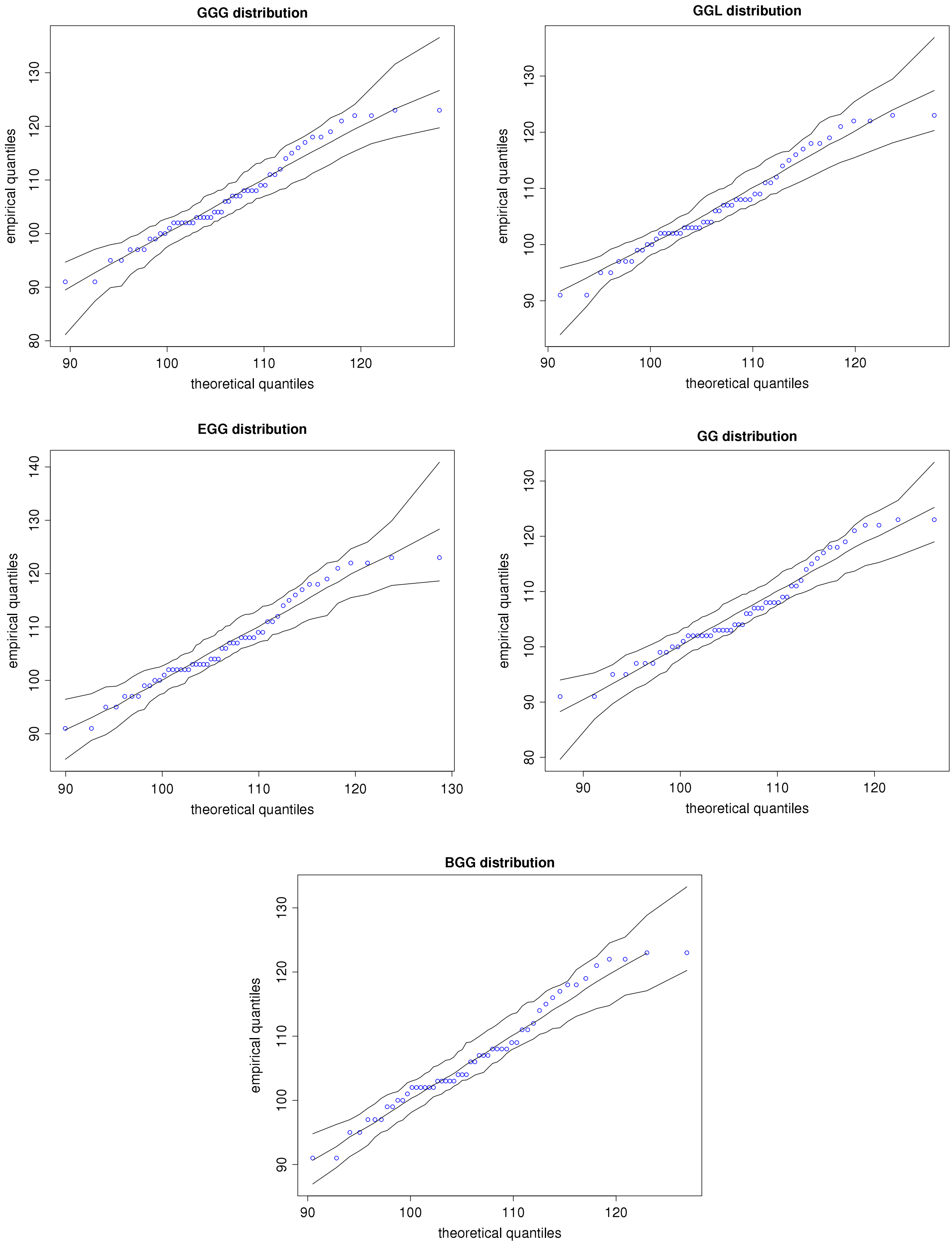


Fig. 10. Q–Q plots of the $\mathcal{GGG}$, $\mathcal{GGL}$, $\mathcal{EGG}$, $\mathcal{GG}$ and $\mathcal{BGG}$ distributions for the 52 Otis IQ Scores.

whose elements are

$$U_{\theta\theta} = -\frac{n}{\theta^2} - n\left[\frac{C''(\theta)}{C(\theta)} - \left(\frac{C'(\theta)}{C(\theta)}\right)^2\right] + \sum_{i=1}^{n} \delta_i\, \dot{\pi}_i^2, \qquad U_{\theta\alpha} = -\frac{1}{\alpha}\sum_{i=1}^{n} \lambda_i \left(\frac{C''(\theta\dot{\pi}_i)}{C'(\theta\dot{\pi}_i)} + \theta\, \delta_i\, \dot{\pi}_i\right),$$

$$U_{\theta\beta} = -\frac{\alpha}{\beta}\sum_{i=1}^{n}\left(\frac{C''(\theta\dot{\pi}_i)}{C'(\theta\dot{\pi}_i)} + \theta\,\delta_i\,\dot{\pi}_i\right)u_i^k\exp(-u_i), \qquad U_{\theta k} = -\sum_{i=1}^{n}\ddot{\pi}_i\left(\frac{C''(\theta\dot{\pi}_i)}{C'(\theta\dot{\pi}_i)} + \theta\,\delta_i\,\dot{\pi}_i\right),$$

$$U_{\alpha\alpha} = -\frac{n}{\alpha^2} + \frac{1}{\alpha^2\beta^\alpha}\sum_{i=1}^{n}x_i^\alpha[1+\alpha\log\beta - u_i - \log x_i^{1/\alpha}]\log u_i - \frac{\theta^2}{\alpha^2\Gamma(k)}\sum_{i=1}^{n}\delta_i\frac{u_i^2\exp(-2u_i)\log u_i}{\gamma_1(k,u_i)}$$
$$- \frac{\theta}{\alpha^2\Gamma(k)}\sum_{i=1}^{n}\frac{C''(\theta\dot{\pi}_i)}{C'(\theta\dot{\pi}_i)}\left[(k-u_i)\log u_i - \frac{\lambda_i}{\gamma_1(k,u_i)}\right]\frac{\lambda_i}{\gamma_1(k,u_i)},$$

$$U_{\alpha\beta} = -\frac{nk}{\beta} + \frac{(\beta-\alpha\log\beta)}{\beta^{\alpha+2}}\sum_{i=1}^{n}x_i^\alpha + \frac{\alpha}{\beta^{\alpha+1}}\sum_{i=1}^{n}x_i^\alpha\log x_i + \left(\frac{\theta}{\alpha}\right)^2\sum_{i=1}^{n}\delta_i\frac{[u_i^k\exp(-u_i)]^2\log u_i}{\gamma_1(k,u_i)}$$
$$+ \frac{\theta}{\alpha^2}\sum_{i=1}^{n}\frac{C''(\theta\dot{\pi}_i)}{C'(\theta\dot{\pi}_i)}\left(1+\frac{\lambda_i}{\gamma_1(k,u_i)}\right)\frac{u_i^k\exp(-u_i)}{\gamma_1(k,u_i)} - \frac{\theta}{\alpha^2}\sum_{i=1}^{n}\frac{C''(\theta\dot{\pi}_i)}{C'(\theta\dot{\pi}_i)}(k+u_i)\lambda_i,$$

$$U_{\alpha k} = \frac{1}{\alpha}\sum_{i=1}^{n}\log u_i + \frac{\theta\alpha}{\Gamma(k)}\sum_{i=1}^{n}\left[\frac{C''(\theta\dot{\pi}_i)}{C'(\theta\dot{\pi}_i)}\psi(k) + \frac{\theta}{\alpha^2}\delta_i\,\ddot{\pi}_i\right]\frac{\lambda_i}{\gamma_1(k,u_i)} - \frac{\theta\alpha}{\Gamma(k)}\sum_{i=1}^{n}\frac{C''(\theta\dot{\pi}_i)}{C'(\theta\dot{\pi}_i)}\frac{u_i^k\log u_i}{\gamma_1(k,u_i)}$$
$$+ \frac{\theta\alpha}{\Gamma(k)}\sum_{i=1}^{n}\frac{C''(\theta\dot{\pi}_i)}{C'(\theta\dot{\pi}_i)}\frac{u_i\,\ddot{\pi}_i}{\gamma_1(k,u_i)^2},$$

$$U_{\beta\beta} = \frac{n\,\alpha\,k}{\beta^2} - \frac{\alpha(\alpha+1)}{\beta^{\alpha+2}}\sum_{i=1}^{n}x_i^\alpha + \frac{\theta^2}{\beta}\sum_{i=1}^{n}\delta_i\frac{[u_i\exp(-u_i)]^2}{\gamma(k,u_i)}$$
$$- \frac{\theta}{\beta}\sum_{i=1}^{n}\frac{C''(\theta\dot{\pi}_i)}{C'(\theta\dot{\pi}_i)}\left[u_i^k(k+u_i)\exp(-u_i) + \left(\frac{u_i\exp(-u_i)}{\gamma_1(k,u_i)}\right)^2\right],$$

$$U_{\beta k} = -\frac{n\alpha}{\beta} + \frac{\theta^2}{\alpha}\sum_{i=1}^{n}\delta_i\,\ddot{\pi}_i\frac{u_i^k\exp(-u_i)}{\gamma_1(k,u_i)} - \frac{\theta}{\alpha}\sum_{i=1}^{n}\frac{C''(\theta\dot{\pi}_i)}{C'(\theta\dot{\pi}_i)}\left[\lambda_i - \frac{u_i\exp(-u_i)}{\gamma(k,u_i)}\ddot{\pi}_i\right],$$

$$U_{kk} = -n\psi(k)[\Gamma''(k) - \psi(k)] + \theta^2\sum_{i=1}^{n}\delta_i\,\ddot{\pi}_i^2$$
$$+ \theta\sum_{i=1}^{n}\frac{C''(\theta\dot{\pi}_i)}{C'(\theta\dot{\pi}_i)}\left\{\ddot{\gamma}_1(k,u_i) - \ddot{\pi}_i - \psi(k)\left[\dot{\gamma}_1(k,u_i) + \Gamma''(k)\gamma_1(k,u_i)\right]\right\},$$

where $u_i = u(\alpha,\beta) = (x_i/\beta)^\alpha$, $\dot{\pi}_i = \dot{\pi}(k,u_i) = 1-\gamma(k,u_i)$, $\ddot{\pi}_i = \ddot{\pi}_i(k,u_i) = \dot{\gamma}_1(k,u_i) - \psi(k)\gamma_1(k,u_i)$, $\lambda_i = \lambda(k,u_i) = u_i\exp(-u_i)\log(u_i)$,

$$\delta_i = \delta(\theta,k,u_i) = \left[\frac{C'''(\theta\dot{\pi}_i)}{C'(\theta\dot{\pi}_i)} - \left(\frac{C''(\theta\dot{\pi}_i)}{C'(\theta\dot{\pi}_i)}\right)^2\right] \quad\text{and}\quad \ddot{\gamma}_1(k,u_i) = \frac{1}{\Gamma(k)}\int_0^{u_i}\omega^{k-1}\mathrm{e}^{-\omega}(\log\omega)^2\mathrm{d}\omega.$$

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